Problem Set 1 Solutions, Math Problem Competition, Fall 2004, GSU

Problem 1. (MAA) A car holds 6 people (including the driver), 3 in the front seat and 3 in the back seat. How many different seating arrangements of the 6 people are possible if one person refuses to sit in the front and two different people (different from each other and different from the first) refuse to sit in the back? Assume all 6 are licensed drivers!

Solution: There are $\binom{3}{2}2! = 6$ possible arrangements for the two people who must sit in the front, and $\binom{3}{1} = 3$ for the one who must sit in the back. That leaves $3! = 6$ possibilities for the three remaing passengers. By the product rule of counting, there are a total of $6 \cdot 3 \cdot 6 = 108$ seating arrangements possible.

Problem 2. (S. Kersey) Let $x_0, \ldots, x_n$ be real numbers. Show that

$$V(x_0, x_1, \ldots, x_{n-1}, x_n) := \det \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} = \prod_{i>j}^{n} (x_i - x_j).$$

Solution: This matrix comes from polynomial interpolation, and is known as the Vandermonde. The following proof comes from *Interpolation and Approximation*, by P. J. Davis. Consider $V(x) := V(x_0, x_1, \ldots, x_{n-1}, x)$ with x_n replaced by the variable x. Note that $V(x)$ is a polynomial of degree n, and that $V(x_i) = 0$ for $i = 0, 1, \ldots, n-1$ because in each case there is repetition of two rows in the corresponding matrix. But then, since $V(x)$ is a polynomial of degree n with the n zeros $x_1, \ldots, x_{n-1}$, it can be written

$$V(x) = \alpha\,(x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

However, in the expansion of the determinant the coefficient on x^n is $V(x_0, \ldots, x_{n-1})$, and so

$$V(x) = (x - x_0)(x - x_1)\cdots(x - x_{n-1})\,V(x_0, \ldots, x_{n-1}).$$

Then, by induction,

$$\begin{aligned} V(x_n) &= (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1})\,V(x_0, \ldots, x_{n-1}) \\ &= (x_n - x_0)(x_n - x_1)\cdots(x_n - x_{n-1}) \prod_{i>j}^{n-1} (x_i - x_j) \\ &= \prod_{i>j}^{n} (x_i - x_j) \end{aligned}$$

Problem 3. (S. Kersey) We say that x is a *fixed point* of f if $f(x) = x$. Show that:
(a) A continuous function $f : [0,1] \to [0,1]$ has a fixed point.
(b) A differential function on $(-\infty, \infty)$ with $f'(x) \neq 1$ for all x has at most one fixed point.

Proof. (a) Let $g(x) := x - f(x)$. Then, since $0 \leq f(x) \leq 1$, it follows that $g(0) = 0 - f(0) \leq 0$ and $g(1) = 1 - f(1) \geq 0$. Now if $g(0) = 0$ then $f(0) = 0$ and if $g(1) = 0$ then $f(1) = 1$, which is a fixed point in both cases. Otherwise, $g(0) < 0$ and $g(1) > 0$. Since $g(x)$ is a continuous function, it follows by the intermediate value theorem that $g(x) = 0$ for some x in the open interval $(0,1)$. And so $f(x) = x$ for this x.

(b) Suppose that $f(x)$ has two fixed points, a and b. Then, $f(a)$ and $f(b)$ both lie on the line $y = x$ in the x-y plane. In particular, the slope of the secant line between them is 1. By the mean value theorem, there exists a c, $a < c < b$, at which $f'(x) = 1$. But this contradicts the hypothesis. □