Set 1 Solutions

1. Find all integers n, $1 \leq n \leq 6$, such that $2006^n + n^{2006}$ is divisible by 5.

Solution: Let $\mathrm{Rem}(n, m)$ be the remainder of n divided by m. Then, $\mathrm{Rem}(2006^n, 5) = 1$ because 2006^n ends in the number 6 for all n. Therefore, we need that $\mathrm{Rem}(n^{2006}, 5) = 4$. We can eliminate 1 easily since $\mathrm{Rem}(1^{2006}, 5) = \mathrm{Rem}(1, 5) = 1$, and we can eliminate 5 because 5^{2006} always ends in 5, hence leaves a zero remainder when divided by 5. Likewise, we can eliminate 6 since $\mathrm{Rem}(6^{2006}, 5) = 1$ for reasons given above. Therefore, only 2, 3, and 4 are possibilities. For these numbers, observe the following powers n^k:

$k =$	2	3	4	5	6	7	8	9	10
2	4	8	16	32	64	128	256	512	1024
3	9	27	81	243	729	2187	6561	19683	59049
4	16	64	256	1024	4096	16384	65536	262144	1048576

Only the numbers ending in 4 or 9 leave a remainder of 4 when divided by 5. This applies to $2^2, 2^6, 2^{10}, \ldots, 2^{2006}$ and $3^2, 3^6, 3^{10}, \ldots, 3^{2006}$, and so 2 and 3 solve the problem. However, 4 fails because 4^{2006} ends in a 6.

2. In a square, $2n$ distinct points are randomly marked. Is it always possible to cut the square along a straight line so that each piece contains exactly n points? Justify your answer!

Solution: Yes, it is always possible. One can construct such a line as follows.

For any two points from the given set, draw the straight line containing them. There will be a finite number (at most $n(2n-1)$) of these (say, "red") lines. Fix a point P outside the square that does not belong to any of them. Draw now a line (say, "green") not intersecting the square and rotate it about P. Eventually, it will start crossing the given points, and it will *never* contain any two of them at the same time. (Otherwise, the "green" must be one of the "reds" – there is just one line containing given two points, but this is impossible because the "green" contains P and the "reds" do not.) Thus, just count points crossed by the "green", one by one, stop the rotation somewhere after n-th but before $(n+1)$-st crossings.

3. In a polynomial $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, all the coefficients ($a$, b, c, d, and e) are different numbers chosen from the set $\{1, -4, 5, 6, -8\}$. Show that, regardless of the choice, $P(x)$ has a rational root.

Solution: For any polynomial $P(x)$, $P(1)$ is just the sum of its coefficients. In our case,

$$P(1) = a + b + c + d + e = 1 + (-4) + 5 + 6 + (-8) = 0$$

since $\{a, b, c, d, e\}$ is just a permutation of $\{1, -4, 5, 6, -8\}$. Thus, $x = 1$ is a root of $P(x)$ no matter how coefficients are chosen.