Set 1 Solutions

1. (V. Maymeskul) Let A, B, and C denote the angles in a triangle. Prove that

$$\tan(A-B)+\tan(B-C)+\tan(C-A)=0$$

if and only if the triangle is isosceles.

Solution 1: Let $\tan A = a$, $\tan B = b$, and $\tan C = c$. Applying the difference formula for the tangent function yields

$$\tan(A-B)+\tan(B-C)+\tan(C-A)=\frac{a-b}{1+ab}+\frac{b-c}{1+bc}+\frac{c-a}{1+ca}=0\,.$$

Adding these three fractions, after some algebra we obtain

$$\frac{a-b}{1+ab}\cdot\frac{b-c}{1+bc}\cdot\frac{c-a}{1+ca}=0\,.$$

Thus, one of the numerators must be zero. Since $0 < A, B, C < \pi$, it follows that at least two of these three angles are equal.

Solution 2: Let $A-B=\alpha$, $B-C=\beta$. Then $C-A=-(\alpha+\beta)$. Thus we have

$$\begin{aligned}\tan(\alpha)+\tan(\beta)-\tan(\alpha+\beta) &= \tan(\alpha)+\tan(\beta)-\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\\ &= \frac{-[\tan(\alpha)+\tan(\beta)]\tan(\alpha)\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}=0.\end{aligned}$$

Thus, one of the factors in the numerator must be zero.

(a) If $\tan(\alpha)+\tan(\beta)=0$, since $-180^\circ < \alpha+\beta < 180^\circ$, we have $\alpha=-\beta$ or $A=C$.

(b) If $\tan(\alpha)=0$, since $-180^\circ < \alpha < 180^\circ$, we have $\alpha=0$ or $A=B$.

(c) If $\tan(\beta)=0$, since $-180^\circ < \beta < 180^\circ$, we have $\beta=0$ or $B=C$.

Remark. We assume that all the terms in the above formulas exist. If $F(x)$ does not exist, under $F(x)$ we understand the limit of $F(r)$ at $r=x$. For example, in Solution 1, if $A=90^\circ$, then we cannot use the difference formula because $\tan(90^\circ)$ does not exist. So, we consider $(a-b)/(1+ab)$ as the limit of $(t-b)/(1+tb)$ at ∞. Namely,

$$\tan(90^\circ-B)=\lim_{D\to 90^\circ}\frac{\tan D-b}{1+(\tan D)b}=\lim_{t\to\infty}\frac{t-b}{1+tb}=\frac{1}{b}\,,$$

where $t=\tan D$.

2. (S. Kersey) Solve the equation

$$2\sqrt{x^2-2x+4}=3x^2-6x+4\,.$$

Solution: Let $y^2=x^2-2x+4$ (which is necessarily positive – graph the parabola). Then, $2\sqrt{x^2-2x+4}=2y$ and $3x^2-6x+4=3(x^2-2x)+4=3(y^2-4)+4=3y^2-8$. And so, $2y=3y^2-8$. Solve this quadratic for $y=2,-4/3$. Then, from $y^2=x^2-2x+4$ one gets $4=x^2-2x+4$ and $16/9=x^2-2x+4$. The second equation has no real solutions, and from the first equation $x^2-2x=0$. Therefore, $x=0$ or 2.

3. (Y. Wu) Evaluate the limit

$$\lim_{x \to a} \frac{\sqrt[2n]{x} - \sqrt[2n]{a}}{x - a}, \quad n = 1, 2, 3, \ldots$$

without using L'Hospital's Rule.

Solution: (Y. Wu). Let $\sqrt[2n]{x} = t$ so that $x = t^{2n}$, and let $b = \sqrt[2n]{a}$. Then $\lim_{x \to a} t = b$. Making this substitution in the limit yields

$$\begin{aligned} \lim_{x \to a} \frac{\sqrt[2n]{x} - \sqrt[2n]{a}}{x - a} &= \lim_{t \to b} \frac{t - b}{t^{2n} - b^{2n}} \\ &= \lim_{t \to b} \frac{1}{t^{2n-1} + t^{2n-2}b + \cdots + tb^{2n-2} + b^{2n-1}} = \frac{1}{2nb^{2n-1}} = \frac{1}{2na^{1-1/(2n)}}. \end{aligned}$$

Remark. The factorization $t^m - b^m = (t - b)(t^{m-1} + t^{m-2}b + \cdots + tb^{m-2} + b^{m-1})$ can be obtained, say, by using long or synthetic division.