Spring 2005 Mathematics Problem Competition Georgia Southern University
Problem Set 1 Due Date: Tuesday, February 8, 2005

Instructions: Do any or all of these problems, and submit to the Mathematics main office by 5:00 pm on the due date. All GSU undergraduate students are eligible. Please include your name and e-mail address with your solutions. Have fun!

Solutions:

1. Let $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2\,n}$.

(a) Show that $1/2 \le S_n < 1$ for all positive integers n. (Use only elementary arithmetic.)

Solution: S. Ngai proposed the problem, and gives the following proof:

$$S_n \le \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} = n(\frac{1}{n+1}) = \frac{n}{n+1} < 1;$$

$$S_n > \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = n(\frac{1}{2n}) = \frac{n}{2n} = \frac{1}{2}.$$

(b) Show that $\lim_{n\to\infty} S_n = \ln(2)$. (Here, Calculus may help)

Solution: By integral comparison with $f(x) = \frac{1}{x}$ we have

$$\ln(2 - \frac{1}{n+1}) = \int_{n+1}^{2n+1} \frac{1}{x}\,\mathrm{d}x < S_n < \int_{n}^{2n} \frac{1}{x}\,\mathrm{d}x = \ln(2).$$

By the "squeeze theorem," $S_n \to \ln(2)$.

2. (a) How many different four letter "words" can you form by rearranging the letters in the word "math"? Here, nonsense words, such as "mhat" are valid.

Solution: The answer is 24. To see this, consider four empty slots in which to place the letters. There are 4 choices for the first letter, while only 3 for the second since one letter is used, then 2 remaining for the third, which fixes the last letter. By the *multiplication principle*, the number of permutations is $4 \times 3 \times 2 \times 1 = 4! = 24$.

(b) How many different 8 letter words can you form by rearranging the letters in "calculus"?

Solution: Here, we consider combinations. The answer is

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1} = 28 \times 15 \times 6 \times 2 \times 1 = 5040.$$

That is, one places the c's in two of the 8 slots such that order does not matter, then the two l's in 2 of the remaining 6 slots, again order does not matter. The two u's are placed in two of the remaining 4 slots, and finally the 'a' and 's' are placed.

3. A polynomial of degree n is a function of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ with coefficients a_k.

(a) Find a polynomial $y = p(x)$ of degree 1 that passes through the points $(-1, 2)$ and $(2, 1)$.

Solution: The slope is $m = \frac{1-2}{2+1} = \frac{-1}{3}$, hence, in point slope form,

$$y = y_1 + m(x - x_1) = 1 + \frac{-1}{3}(x - 2) = \frac{-1}{3}x + \frac{5}{3}.$$

(b) Find a polynomial $y = p(x)$ of degree 2 that passes through the points $(-1, 2)$, $(0, 3)$ and $(2, 1)$.

Solution: The standard solution is by Lagrange polynomials. This gives,

$$\begin{aligned} p(x) &= \frac{(x-0)(x-2)}{(-1-0)(-1-2)}\,2 + \frac{(x+1)(x-2)}{(0+1)(0-2)}\,3 + \frac{(x+1)(x-0)}{(2+1)(2-0)}\,1 \\ &= \frac{2}{3}x(x-2) - \frac{3}{2}(x+1)(x-2) + \frac{1}{6}x(x+1) \\ &= -\frac{2}{3}x^2 + \frac{1}{3}x + 3. \end{aligned}$$

Alternatively, one could use the monomial form $p(x) = a_0 + a_1 x + a_2 x^2$ where the coefficients satisfy the linear system:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

which gives $a_0 = 3$, $a_1 = 1/3$ and $a_2 = -2/3$. The matrix is called the Vandermonde.

More on Problem 1a: F. Ziegler points out that, whereas S_n is an increasing sequence, the sequence $\frac{1}{n} + S_n$ is decreasing with the same limit, so that, with $S := \lim_{n\to\infty} S_n$, we can get arbitrarily tight (rational) estimates

$$S_n < S < \frac{1}{n} + S_n$$

to S. For example, one gets $.69065343 < S < .70065343$ for $n = 100$, while we know from part (b) that $S = \ln(2) \approx .693147$. And in particular, one can get upper bounds $S_n < B$ for all n by choosing $B := \frac{1}{m} + S_m$ for large m. This sharpens the results $S_n < 3/4$ and $S_n < 17/24$ by S. Damelin and V. Maymeskul, respectively.

More on Problem 1b: • Y. Hu proves it by Riemann sums, similar to that given above.

• S. Damelin uses the identity

$$\sum_{j=1}^{n} \frac{1}{j} - \ln(n) = \gamma + \sigma_n$$

with "Euler's constant" $\gamma \approx .57721566$ and $\sigma_n \to 0$. From this,

$$\begin{aligned} S_n &= \sum_{j=1}^{2n} \frac{1}{j} - \sum_{j=1}^{n} \frac{1}{j} \\ &= \ln(2n) - \log(n) + \sigma_{2n} - \sigma_n \\ &= \ln(2) + \sigma_{2n} - \sigma_n \longrightarrow \ln(2). \end{aligned}$$

• M. Ransom showed that $S_n = T_{2n}(2)$ where $T_{2n}(x)$ is the Taylor polyonomial approximation

$$T_{2n}(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots - \frac{(x-1)^{2n}}{2n}$$

to $f(x) = \ln(x)$. That is,

$$S_n = T_{2n}(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n},$$

which, as an alternating series with decreasing terms, converges. Moreover, this gives the error estimate

$$|S_n - \ln(2)| \le \frac{1}{2n+1} < 2\,\frac{1}{n},$$

which goes to zero as order $O(\frac{1}{n})$, as we already can see from the proof on the first page. On grouping successive terms, one sees that

$$S_n = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots + (\frac{1}{2n-1} - \frac{1}{2n}) \quad \text{is increasing, and}$$

$$\frac{1}{n} + S_n = 1 - (\frac{1}{2} - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{5}) + \cdots + (\frac{1}{2n-2} - \frac{1}{2n-1}) + \frac{1}{2n} \quad \text{is decreasing.}$$

• Another proof, given by F. Ziegler, uses the identity

$$f(\frac{1}{n}) + f(\frac{1}{n+1}) + f(\frac{1}{n+2}) + \cdots + f(\frac{1}{2n}) \longrightarrow f'(0) \lim_{n\to\infty} S_n$$

applied to the function $f(x) = \ln(1+x)$.