Set 1 Solutions

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1. (G. Lesaja) Find all real functions of one real variable that satisfy the following functional equation

$$f(x) + 2f\left(\frac{1}{x}\right) = x. \tag{1}$$

Solution. Replacing x by $1/x$ and multiplying by 2 yields

$$2f\left(\frac{1}{x}\right) + 4f(x) = \frac{2}{x}. \tag{2}$$

And so, substracting equation (1) from (2) gives the following:

$$3f(x) = \frac{2}{x} - x \quad \Rightarrow \quad f(x) = \frac{1}{3}\left(\frac{2}{x} - x\right).$$

2. (V. Maymeskul) Solve the following nonlinear system of equations (for ALL possible solutions):

$$\begin{aligned} &a + b + c = 7 \\ &a^2 + b^2 + c^2 = 21 \\ &abc = 8 \end{aligned}$$

Hints: Note that $x = a$, b and c are the roots of the cubic $f(x) = (x-a)(x-b)(x-c)$. Also consider $(a+b+c)^2$.

Solution. Since $(a+b+c)^2 = (a^2+b^2+c^2) + 2(ab+ac+bc)$, we have

$$ab + ac + bc = \frac{(a+b+c)^2 - (a^2+b^2+c^2)}{2} = \frac{7^2 - 21}{2} = 14.$$

Therefore,

$$f(x) = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = x^3 - 7x^2 + 14x - 8.$$

Factoring yields $x^3 - 7x^2 + 14x - 8 = (x-1)(x-2)(x-4)$. Thus, the roots of $f(x)$ are 1, 2, and 4, and so the system has $3! = 6$ solutions: $(a, b, c) = (1, 2, 4)$ and all permutations of this triple.

3. (S. Kersey) Prove that $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin(x)}{x}\,\mathrm{d}x < \frac{\sqrt{2}}{2}$. (Justify all steps.)

Solution. Let $f(x) := \sin(x)/x$. Then $f'(x) = g(x)/x^2$ with $g(x) := x\cos(x) - \sin(x)$. But since $g(0) = 0$ and $g'(x) = -x\sin(x) < 0$ on $(0, \pi/2)$, it follows that $g(x) < 0$, and so $f'(x) < 0$. Therefore, $f(x)$ is strictly decreasing on $(0, \pi/2)$. In particular, $f(x) < f(\pi/4)$ for $\pi/4 \le x \le \pi/2$, and so

$$\int_{\pi/4}^{\pi/2} \frac{\sin(x)}{x}\,\mathrm{d}x < \int_{\pi/4}^{\pi/2} \frac{\sin(\pi/4)}{\pi/4}\,\mathrm{d}x = \frac{2\sqrt{2}}{\pi}\int_{\pi/4}^{\pi/2} \mathrm{d}x = \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$