Solutions to Set 3, Fall 2008

1. (F. Ziegler) Five mathematicians of different nationalities parked their cars in a row and had different drinks. Here are facts that police observed:

1) The statistician drinks cognac
2) The topologist is next to the tequila drinker
3) The analyst is in the grey car
4) The geometer is just left of the logician
5) The logician is Polish
6) The Jeep is next to the purple car
7) The Swiss is in the red car
8) The vodka drinker is in the olive car
9) The French drinks whisky
10) The tequila drinker is in the leftmost car
11) The BMW is black
12) The Ford is next to the grey car
13) The Nigerian is in the middle
14) The gin drinker is in the Subaru
15) There is an American

Who drives the Toyota?

Solution. Conditions 2), 10), and 13) give the following possibilities to begin with:

	1	2	3	4	5
Specialty	AGLS	T	AGLS	AGLS	AGLS
Drink	T	CG VW	CG VW	CG VW	CG VW
Nationality	AF PS	AF PS	N	AF PS	AF PS
Car Color	BGOPR	BGOPR	BGOPR	BGOPR	BGOPR
Car Make	BFJST	BFJST	BFJST	BFJST	BFJST

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(We represent each attribute by its initial.) Now, the other conditions exclude many single attributes: for instance, 1) excludes Statistician from column 1 and Cognac from column 2; etc. Cycling over conditions until all such elimination is done (i.e., using 1), 3), 4), 5), 7), 8), 9), 12), 14), 3), 7), 11), 12), 14), 15), and 11)), we arrive at

	1	2	3	4	5
Specialty	A	T	G S	GLS	LS
Drink	T	VW	CG V	CG VW	CG VW
Nationality	A	F S	N	F PS	F PS
Car Color	G	OPR	B OP	B OPR	B OPR
Car Make	J T	F	B JST	B JST	B JST

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Next, consider attribute pairs. Column 2 contains either French/Whisky 9) or Swiss/Red 7). The latter contradicts Vodka/Olive 8), so it's the former:

	1	2	3	4	5
Specialty	A	T	G S	GLS	LS
Drink	T	W	CG V	CG V	CG V
Nationality	A	F	N	PS	PS
Car Color	G	P	B OP	B OPR	B OPR
Car Make	J T	F	B JST	B JST	B JST

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Likewise, consider the Black BMW 11). It goes neither with Gin 14) nor with Vodka 8). Therefore it goes with Cognac, and belongs to the Statistician 1). Now the Statistician cannot be in columns 4 or 5, for then 5) and 7) would exclude all nationalities for him. So he goes in column 3:

	1	2	3	4	5
Specialty	A	T	S	GL	L
Drink	T	W	C	G V	G V
Nationality	A	F	N	PS	PS
Car Color	G	P	B	OPR	OPR
Car Make	J T	F	B	JST	JST

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At this point, cycling once more over the remaining conditions (4), 5), 6), 7), 8), 14)) suffices to eliminate all attributes except the following:

	1	2	3	4	5
Specialty	Analyst	Topologist	Statistician	Geometer	Logician
Drink	Tequila	Whisky	Cognac	Gin	Vodka
Nationality	American	French	Nigerian	Swiss	Polish
Car Color	Grey	Purple	Black	Red	Olive
Car Make	Jeep	Ford	BMW	Subaru	Toyota

Thus, the *answer* is: the Polish logician drove an olive Toyota going to a meeting with his friends-mathematicians having his favorite vodka in the trunk (just in case).

Remark. This is a variant of the classic "zebra puzzle", whose 15 conditions result if we map the above table to a one at HTTP://EN.WIKIPEDIA.ORG/WIKI/ZEBRA_PUZZLE#SOLUTION

2. (V. Maymeskul) The Great Fermat's Theorem states, in particular, that there are no three positive integers a, b, and d such that $a^3 + b^3 = d^3$. Are there any four consecutive positive integers that satisfy $a^3 + b^3 + c^3 = d^3$? List all such sets if any.

Solution. Let the smallest number (if exists) is x. Then we have the equation

$$x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3 \,.$$

Simplifying yields

$$x^3 - 6x - 9 = 0.$$

One can now involve the Rational Root theorem to check this equation for integer roots. Alternatively, factoring gives

$$x^3 - 6x - 9 = \left(x^3 - 9x\right) + (3x - 9) = x(x-3)(x+3) + 3(x-3) = (x-3)\left(x^2 + 3x + 3\right).$$

Since the quadratic factor has complex roots, the only real solution is $x = 3$, and so the numbers 3, 4, 5, and 6 are the only numbers that satisfy the given condition, i.e.,

$$3^3 + 4^3 + 5^3 = 6^3 \,.$$

3. (S. Kersey) Let $f(x) = (x+1)\ln(x+1)$. Find the largest number a so that $f(x) \geq ax$ for $x \geq 0$.

Solution. Let $g(x) := (x+1)\ln(x+1) - ax$. We need the largest value of a so that $g(x) \geq 0$ for $x \geq 0$. Any $a \leq 1$ satisfies this since $g(0) = 0$ and

$$g'(x) \;=\; 1 + \ln(x+1) - a \;\geq\; \ln(x+1) \;\geq\; 0$$

for $x \geq 0$ (hence $g(x)$ increases away from 0 in this case). However, if $a > 1$ then $g'(0) < 0$, in which case it follows that $g(x) < 0$ for small positive x. And so, the answer is $a = 1$.