Solutions to Set 4, Fall 2008

1. (G. Lesaja) If $f(n+1) = (-1)^{n+1}n - 3f(n)$ for $n = 1, 2, 3, \ldots$, and $f(1) = f(2009)$, compute the sum $S = f(1) + f(2) + \cdots + f(2008)$. (3 pts.)

Solution. Using the recurrence relation and the "initial condition", we get

$$\begin{aligned} f(2008) &= \quad 2007 - 3f(2007); \\ f(2007) &= -2006 - 3f(2006); \\ &\vdots \\ f(2) &= \quad 1 - 3f(1); \\ f(1) &= \quad f(2009) = -2008 - 3f(2008). \end{aligned}$$

Summing these equations yields

$$S = (-2008 + 2007) + (-2006 + 2005) + \cdots + (-2 + 1) - 3S \quad \Rightarrow \quad 4S = -\frac{2008}{2} = -1004\,.$$

Thus, $S = -251$.

2. (Y. Wu) Let A and B be $n \times m$ and $m \times n$ matrices, respectively. Given that $I_m - BA$ is invertible (where I_m is the $m \times m$ multiplicative identity matrix), show that $I_n - AB$ is invertible. Find an expression for $(I_n - AB)^{-1}$. (3 pts.)

Solution. One has

$$\begin{aligned} AB &= A\left[(I_m - BA)(I_m - BA)^{-1}\right] B \\ &= (A - ABA)(I_m - BA)^{-1} B = (I_n - AB)\, A\, (I_m - BA)^{-1} B \end{aligned}$$

so that

$$I_n = (I_n - AB) + (I_n - AB)\, A\, (I_m - BA)^{-1} B = (I_n - AB)\left[I_n + A\,(I_m - BA)^{-1} B\right].$$

It follows then that $\det(I_n - AB) \neq 0$ and, therefore, it is invertible. Finally,

$$(I_n - AB)^{-1} = (I_n - AB)^{-1} I_n = I_n + A\,(I_m - BA)^{-1} B\,.$$

3. (S. Kersey, V. Maymeskul) There are various ways to define the "distance" between continuous functions on a closed interval, say $[0, 1]$. Two of the more popular ones are:

$$d_2(f,g) = \sqrt{\int_0^1 |f(x) - g(x)|^2\, \mathrm{d}x} \quad \text{and} \quad d_\infty(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)|.$$

Among all linear functions $g(x) = ax + b$, find the one that minimizes the distance to $f(x) = x^2$ using d_2, and find that distance. Repeat using d_∞. (3 pts each.)

Solution(1). Let $F(a,b) := d_2(f,g)^2 = \int_0^1 (ax + b - x^2)^2\, \mathrm{d}x$. For the first distance, we find the critical points of F by setting the partials F_a and F_b to zero. That is,

$$\begin{aligned} F_a(a,b) &= \frac{\partial}{\partial a}\int_0^1 (ax + b - x^2)^2\, \mathrm{d}x = \int_0^1 \frac{\partial}{\partial a}(ax + b - x^2)^2\, \mathrm{d}x \\ &= \int_0^1 2(ax + b - x^2)(x)\, \mathrm{d}x = 2\int_0^1 (ax^2 + bx - x^3)\, \mathrm{d}x = 2\left(\frac{a}{3} + \frac{b}{2} - \frac{1}{4}\right) = 0, \end{aligned}$$

and likewise

$$F_b(a,b) = \frac{\partial}{\partial b}\int_0^1 (ax+b-x^2)^2\,\mathrm{d}x = \int_0^1 2(ax+b-x^2)\,\mathrm{d}x = 2\left(\frac{a}{2}+b-\frac{1}{3}\right) = 0.$$

Hence, we require that $4a+6b-3=0$ and $3a+6b-2=0$, giving $a=1$ and $b=-1/6$. One can argue that this gives the minimum since F is positive definite or by a second derivative analysis. Hence, $g(x)=x-1/6$ with distance

$$d_2(f,g) = \sqrt{\int_0^1 (ax+b-x^2)^2\,\mathrm{d}x} = 1/\sqrt{180} = \sqrt{5}/30.$$

Let $G(x) := |ax+b-x^2|^2$. For the second distance, we first find that the maximum distance may occur when

$$\frac{\partial}{\partial x}G(x) = \frac{\partial}{\partial x}(ax+b-x^2)^2 = 2(ax+b-x^2)(a-2x) = 0,$$

giving either $ax+b-x^2=0$ or $a+2x=0$. Since the first implies $d_\infty(f,g)=0$, only the second can give a maximum. Hence, the maximum is achieved when either $x=a/2$ or at an end point $x=0$ or $x=1$. One can argue that the distance is moreover the same at those three points, and so $G(0)=G(a/2)=G(1)$. This gives the nonlinear equations $b^2=(a^2/4+b)^2=(a+b-1)^2$, with solution set $(a,b)=\{(0,1/2),(2,-1/2),(1,-1/8)\}$, which gives three possibilities for the distance: $G(0)=b^2=\{1/4,1/4,1/64\}$. And so the minimum distance is $d_\infty(f,g)=G(0)^{1/2}=1/8$ when $a=1$ and $b=-1/8$. That is, $g(x)=x-1/8$.

Solution(2). Let $F(t)$ be an even function on $[-1,1]$, $G(t)=at+b$ be a linear function. Looking for $G(t)$, closest to $F(t)$ (in both distances), one can look at even functions only, i.e., $H(t)=b$. Indeed, for d_∞,

$$|F(t)-H(t)| = \left|F(t)-\frac{G(t)+G(-t)}{2}\right| \le \frac{|F(t)-G(t)|+|F(-t)-G(-t)|}{2}$$

so that

$$\begin{aligned} d_\infty(F,H) &= \max_{|t|\le 1}|F(t)-H(t)| \\ &\le \frac{1}{2}\left(\max_{|t|\le 1}|F(t)-G(t)| + \max_{|t|\le 1}|F(-t)-G(-t)|\right) = (-t\to u) = d_\infty(F,G)\,. \end{aligned}$$

Similar arguments apply to d_2 with the u-substitution $u=-t$ in the integral.

Let now $F(t)=t^2$. For d_2,

$$\int_{-1}^{1}\left(t^2-b\right)^2\mathrm{d}t = 2\int_0^1\left(t^4-2t^2b+b^2\right)\mathrm{d}t = 2\left(\frac{1}{5}-\frac{2}{3}b+b^2\right),$$

which is a quadratic function with the minimum value of $8/45$ attained at $b=1/3$. Thus, the closest constant function to $F(t)$ is $H_2(t)=1/3$ with $d_2^2(t^2,1/3)=8/45$.

For d_∞, the closest constant function to $F(t)$ is clearly $H_\infty(t) = 1/2$, and $d_\infty(t^2, 1/2) = 1/2$.

Let now $f(x) = x^2$, $0 \le x \le 1$, and we look for the closest linear function $g(x) = ax + b$. Thinking of $x = (t+1)/2$, we need a linear function $G(t) = a(t+1)/2 + b$ closest to $F(t) = (t+1)^2/4 = (t^2 + 2t + 1)/4$ on $[-1, 1]$ which, in turn, is the closest linear function of the form $2(t+1)a + 4b - 2t - 1 = 2(a-1)t + 2a + 4b - 1$ to t^2. So, for both distances, $a = 1$. Solving for b's yields

$$d_2 : 4b + 1 = \frac{1}{3} \quad \Rightarrow \quad b = -\frac{1}{6}; \qquad d_\infty : 4b + 1 = \frac{1}{2} \quad \Rightarrow \quad b = -\frac{1}{8}.$$

Finally,

$$d_2\left(x^2, x - \frac{1}{6}\right) = \sqrt{\int_0^1 \left[x^2 - \left(x - \frac{1}{6}\right)\right]^2 \mathrm{d}x} = \sqrt{\frac{1}{32}\int_{-1}^1 \left(t^2 - \frac{1}{3}\right)^2 \mathrm{d}t} = \sqrt{\frac{1}{32} \cdot \frac{8}{45}} = \frac{\sqrt{5}}{30};$$

$$d_\infty\left(x^2, x - \frac{1}{8}\right) = \max_{0 \le x \le 1} \left|x^2 - \left(x - \frac{1}{8}\right)\right| = \frac{1}{4} \max_{|t| \le 1} \left|t^2 - \frac{1}{2}\right| = \frac{1}{8}.$$

Remark. Similar arguments, applied to $f(x)$ on an interval $[\alpha, \beta]$, give the following "closest" linear functions

$$g_2(x) = (\alpha + \beta)x - \frac{\alpha^2 + 4\alpha\beta + \beta^2}{6} \quad \text{and} \quad g_\infty(x) = (\alpha + \beta)x - \frac{\alpha^2 + 6\alpha\beta + \beta^2}{8}$$

(use $x = (\beta - \alpha)t/2 + (\beta + \alpha)/2$, $-1 \le t \le 1$). The "distances" can be also easily computed:

$$d_2(f, g_2) = \sqrt{\left(\frac{\beta - \alpha}{2}\right)^5} d_2\left(t^2, 1/3\right) = \frac{(\beta - \alpha)^2 \sqrt{5(\beta - \alpha)}}{30},$$

$$d_\infty(f, g_\infty) = \left(\frac{\beta - \alpha}{2}\right)^2 d_\infty\left(t^2, 1/2\right) = \frac{(\beta - \alpha)^2}{8}.$$