

1. (S. Kersey) Prove that  $\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$  for all real numbers  $x$  and  $y$ . (Hint: you may use the triangle inequality  $|x+y| \leq |x|+|y|$ ).

*Solution.* The standard proof to this problem is to observe that since the function  $f(t) := t/(1+t)$  is an increasing function and  $|x+y| \leq |x|+|y|$ , it follows that  $f(|x+y|) \leq f(|x|+|y|)$ . Therefore we have,

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|} = \frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

Alternatively, the first inequality above can be gotten more directly without considering  $f(t)$  as follows:

$$\frac{|x+y|}{1+|x+y|} = 1 - \frac{1}{1+|x+y|} \leq 1 - \frac{1}{1+|x|+|y|} = \frac{|x|+|y|}{1+|x|+|y|}.$$

But a common mistake is to assume that

$$\frac{1}{1+|x+y|} \leq \frac{1}{1+|x|+|y|},$$

which is of course false. There is also a geometric proof valid when  $x$  and  $y$  are of the same sign. Let  $A(t) = t/(1+t)$  be the area of the lower rectangle under  $g(t) = 1/(1+t)$  from 0 to some non-negative  $t$ . Since  $g(t)$  is decreasing and  $|x+y| = |x|+|y|$  when  $x$  and  $y$  are the same sign, it follows that  $A(|x+y|) \leq A(|x|) + A(|y|)$ , which is exactly the inequality we need. It seems this approach doesn't directly apply when  $x$  and  $y$  are of opposite signs.

2. (V. Maymeskul) Two farmers collected  $x$  bushels of oranges on their joint farm (equally owned) and sold them for  $x$  dollars each. They spent all the money that they gained for buying turkeys, \$10 a head, and one hen. Splitting the purchase, one of them got an extra turkey while the other one took the hen. Since the hen cost less, they used cash to get even. How much cash was paid?

*Solution.* Let  $x = 10a + b$ , where  $a$  and  $b$  are nonnegative integers and  $0 \leq b \leq 9$ , denote the number of bushels sold. Then the farmers' profit was

$$x^2 = (10a + b)^2 = 100a^2 + 10(2ab) + b^2.$$

Since the number of turkeys bought is odd (\$10 each) and  $2ab$  is even,  $b^2$  must be a two-digit number having an odd leading digit. Checking

$$4^2 = 16, \quad 5^2 = 25, \quad 6^2 = 36, \quad 7^2 = 49, \quad 8^2 = 64, \quad 9^2 = 81,$$

we see that the only possible choices for  $b$  are 4 and 6. In both cases, the last digit in  $x^2$  is 6 so that the hen's price was \$6. Thus, one farmer paid to the other a half of the difference, that is \$2.

3. Five mathematicians,  $M_1, M_2, M_3, M_4, M_5$ , take turns flipping a coin.  $M_1$  flips first, followed by  $M_2$ , then  $M_3$ , then  $M_4$ , then  $M_5$  followed by  $M_1$  again, etc. The game continues until a mathematician has flipped a head, at which time the game is stopped and that mathematician wins the game. Find the probability that some mathematician other than  $M_1$  wins. (Additionally, can you guess what this probability is when the number of mathematicians is infinity?)

*Solution.* Let  $N$  be the number of flips needed until a head shows up (including the flip at which the head shows up). Then  $M_1$  wins if and only if  $N$  is congruent to 1 modulo 5, i.e.,  $N$  is of the form  $5k + 1$ ,  $k = 0, 1, \dots$ . Now,  $N = n$  for a given value  $n$  if and only no head showed up in the first  $n - 1$  flips and a head appears in the  $n$ -th flip; the probability for this event is  $p_n = (\frac{1}{2})^{n-1}(\frac{1}{2})$ . Hence the probability that  $M_1$  wins is

$$\sum_{k=0}^{\infty} p_{5k+1} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{5k+1-1} \left(\frac{1}{2}\right) = \frac{1}{2} \sum_{k=0}^{\infty} \left(\left(\frac{1}{2}\right)^5\right)^k = \frac{1}{2(1 - (\frac{1}{2})^5)} = \frac{16}{31}.$$

Therefore the probability that some other mathematician wins is  $1 - \frac{16}{31} = \frac{15}{31}$ .

It is not hard to see from the solution that this probability approaches  $\frac{1}{2}$  when the number of mathematicians gets very large.