

1. (Proposed by V. Maymeskul) Some time after noon, a secretary of the GSU Department of Mathematical Sciences left the office for lunch. When she came back, she noticed that the minute and hour hands of the clock switched. Assuming that the lunch did not last more than an hour, what time did the secretary leave the office and what time did she come back?

Solution. In total, the minute and hour hands made a complete revolution during the secretary's lunch time. Since the minute hand is 12 times faster than the hour hand, the minute hand made $12/13$ -th and the hour hand made $1/13$ -th of the complete revolution. Then it follows that the secretary was absent for $(12/13) \times 60 = 55\frac{5}{13}$ min. To determine the time, when the secretary left, let's denote by x the number of minutes passed after noon till the time when the secretary left for lunch. At this moment, the "distance" between hands was $x - x/12 = (11/12)x$ min. Thus, we got the equation

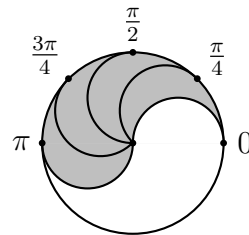
$$\frac{11}{12}x = \frac{1}{13} \cdot 60 \Leftrightarrow x = 5\frac{5}{143}.$$

Therefore, the secretary left the office at $5\frac{5}{143}$ min after noon and came back at

$$5\frac{5}{143} + 55\frac{5}{13} = 60\frac{60}{143} \text{ min}$$

after noon, or $60/143$ min after 1 p.m.

2. (Proposed by F. Ziegler) Can the shaded region, whose boundary consists of three half-circles, be cut into four pieces of equal area? Of equal area *and* shape?



Solution. (The problem is from a set by I. M. Gel'fand.) The answer is yes: see picture. Indeed it remains yes for any number of pieces (replace $\frac{\pi}{4}$ by $\frac{\pi}{n}$).

3. (Unknown) Suppose that M is an integer with the property that if x is randomly chosen from the set $\{1, 2, 3, \dots, 999, 1000\}$, the probability that x is a divisor of M is $1/100$. If $M \leq 1000$, determine the maximum possible value of M .

Solution. (This problem is from the Canadian Open Mathematics Challenge.)

We are not told that M must be a positive integer, but it makes sense to look for a positive integer M that satisfies these conditions, since we want the maximum possible value of M . Since there are 1000 numbers in the set $\{1, 2, 3, \dots, 999, 1000\}$ and the probability that an x chosen randomly from this set is a divisor of M is $1/100$, then M must have 10 divisors between 1 and 1000. Since we are told that $M \leq 1000$, then M must have exactly 10 positive divisors. Therefore, M must be of the form p^9 where p is a prime number, or p^4q where p and q are both primes. (Recall that to find the number of positive divisors of M , we find the

prime factorization of M and then take each of the exponents, add 1, and find the product of these numbers. For example, if $M = 48 = 2^4 3$, then the number of positive divisors is $(4 + 1)(1 + 1) = 10$.) Now, we want to determine the maximum M in each of these two forms.

Case (i) $M = p^9$ Since $M \leq 1000$, then we must have $p = 2$, i.e. $M = 512$. (If $p = 3$, then $p^9 = 19683$ is too large.)

Case (ii) $M = p^4 q$ Since $M \leq 1000$ and $5^4 = 625$, we must have $p = 2$ or $p = 3$. If $p = 2$, then the largest q can be so that q is prime and $M \leq 1000$ is 61, i.e. $M = (16)(61) = 976$. If $p = 3$, then the largest q can be so that q is prime and $M \leq 1000$ is 11, i.e. $M = (81)(11) = 891$. Therefore, the maximum possible value of M is 976.