Fall 2009 MPC – Set 3, Due by 5pm on Friday, Nov. 13.

Instructions: Welcome to the Fall'09 GSU Mathematics Problem Solving Competition! All GSU Undergrads are eligible. Submit your solutions to the Mathematics office. Please include your name and e-mail address. Have fun! For more information, go to:
http://math.georgiasouthern.edu/math/math_competition/math_competition.php

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1. (Proposed by F. Ziegler) Let D, E, F be three matrices such that, with the notation $[a, b] = ab - ba$,

$$[D, E] = 2E, \qquad [D, F] = -2F, \qquad [E, F] = D. \qquad \heartsuit$$

Show that the eigenvalues of D are integers. (Extra credit: Do all integers occur in this way?)

[Hint: If v is an eigenvector, show inductively that: (1) So are Ev, E^2v, etc. until one of them, say $E^{m+1}v$, is zero. (2) So are the $v_i := F^i E^m v$ up until, say, $v_{n+1} = 0$. (3) Ev_i is a multiple of v_{i-1}. Now consider (the trace of) the matrices representing D, E, F in the subspace with basis $v_0, \ldots, v_n$.]

2. (Proposed by H. Wang) Twenty five boys and twenty five girls sit around a table. Prove that there is always a person both of whose neighbors are girls.

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