Solutions to Set 1, Spring 2009

1. (H. Wang) In a religious village of 30 villagers, each of which has a dot (colored red or blue) on his/her forehead. It is believed that if one knows the color of his/her own dot, he/she has to commit suicide the next morning. So there is no mirror in the village and no one talks about the color. One day a visitor came, stayed for dinner with all 30 villagers, and the visitor said 'there is at least one red dot in this village' before leaving. Then on the 8th morning after the visitor left, the whole village is dead. The question is: how many red dots are there in this village? (This means that the last villager or the last group of villagers committed suicide on the 8th morning, there could be more than one suicide in one day, or none on the other. And if someone is dead, all others know on the same day. And of course, we assume all villagers to be very smart, meaning that they think.)

Solution: The answer is 7, or k if we replace 8th by $(k+1)$th in general ($k < n$, where n is the number of villagers). Consider the easiest case, if A is the only one in the village that has a red dot, then A realizes his/her own color after looking at every one else's color, and A will commit suicide on the 1st morning after the visitor left. Then every one else will know and think about how did A figure out his/her color, the only reason is that every one else has blue dot, so all others will commit suicide on the 2nd morning. A little more complicated, if there are two red dots, say A and B, then nothing would happen on the 1st morning for A is hoping that the visitor is talking about the dot of B, and similarly for B. But after the 1st morning, A and B see that they are both alive while every one else has blue dot, then A and B can figure out that they are the only two red dots in the village and will commit suicide on the 2nd morning. Then every one else will think about how did they know, and will figure out that they each has a blue dot, and commit suicide on the 3rd morning. Similarly, if the number of villagers is n and we have k red dots, $k < n$. Then the ones with red dots will commit suicide on the kth morning, the others on the $(k+1)$th morning. One special case is when $k = n$, in which case all committed suicide on the kth morning (instead of $k+1$).

2. (V. Maymeskul) There exist only two three-digit positive integers, say n, such that n^k ends with n for any positive integer k. Find these numbers and justify their uniqueness.

Solution: Let's denote a three digit number in question by n. First, note that n^k, $k \geq 2$, ends with n if n^2 ends with n. By induction: one has $n^{k+1} = (n^k)n = (1000r + n)n = 1000rn + n^2 = 1000rn + 1000s + n = 1000(rn + s) + n$. So, we focus on the fact that n^2 ends with n.

Since $n^2 - n = n(n-1) = 1000s = (125)(8)s$ and n and $n-1$ are mutually prime, one of the factors, either n or $n-1$ must be an odd multiple of 125 while the other must be divisible by 8. So, checking $n = 125m$ and $n = 125m + 1$, $m = 1, 3, 5, 7$, for this condition one gets $n = 625$ or $n = 376$, respectively.

3. (S. Kersey) Sketch the set $K + L := \{p + q : p \in K,\ q \in L\}$ with $K := B_2(-2, 0)$ and $L := B_1(1, 0)$, and verify your answer. Here, $B_r(h, k) := \{(x, y) : \sqrt{(x-h)^2 + (y-k)^2} \leq r\}$ is defined as the ball of radius r centered at (h, k).

Solution: Note that $L - \{(1, 0)\}$ is a ball of radius 1 centered at the origin, and adding this to another ball adds one to the radius of that ball. Therefore,

$$K + L = K + (L - \{(1, 0)\}) + \{(1, 0)\} = B_3(-2, 0) + \{(1, 0)\} = B_3(-1, 0).$$

We remark that the equality $A = (A - B) + B$ works when B is a single point as used above, but does not hold in general for arbitrary sets.