1. (D. Stone) Among all isosceles triangles with the area of 1 sq. ft., there are two triangles, for which the area of an inscribed square is largest possible. Find these triangles. What is the square's maximum area?

Solution. When a square is inscribed in a triangle, one of the sides of the triangle contains two vertices of the square (and, therefore, a side) while each of the other two sides of the triangle contain one vertex. See the figure on the left.

Let the $|AC| = b$ and the corresponding altitude $|BH| = h$. If the side of the square $DFIG$ is x, then from similarity of $\triangle ABC$ and $\triangle DBF$ we obtain

$$\frac{|BE|}{|BH|} = \frac{|DF|}{|AC|} \quad \Rightarrow \quad \frac{h-x}{h} = \frac{x}{b} \quad \Rightarrow \quad x = \frac{bh}{b+h} = \frac{1}{(b+h)/2}$$

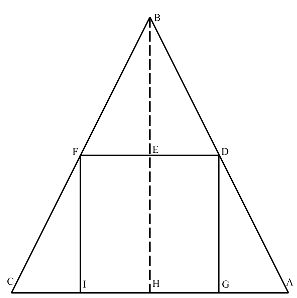
since the area of the triangle $bh/2 = 1$. Thus, to maximize x, we have to minimize the arithmetic mean of b and h given that their geometric mean $\sqrt{bh} = \sqrt{2}$. It is well-known that the minimum value is achieved when the numbers are equal so that $b = h = \sqrt{2}$. In this case, $x = 1/\sqrt{2}$, and the maximum area of the square is $x^2 = 1/2$.

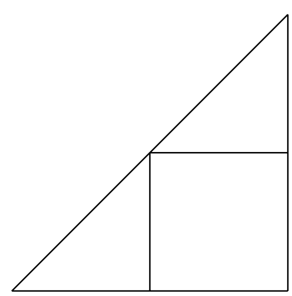
For isosceles triangles, we consider two possible configurations.

(a) The shortest side of the triangle contains a side of the square - the case shown on the left figure. In this case, two other sides of the triangle are

$$|AB| = |BC| = \sqrt{\left(\frac{b}{2}\right)^2 + h^2} = \sqrt{\frac{1}{2} + 2} = \sqrt{\frac{5}{2}}.$$

(b) A side of the square belongs to one of two equal sides of the triangle. Then, since the corresponding height is of the same length as the side, the triangle must be a right triangle with the legs of $\sqrt{2}$ and the hypotenuse of 2. See the figure on the right.





2. (V. Maymeskul) Give an example of a function that is differentiable on $(-1, 1)$ and whose derivative has a jump discontinuity at $x = 0$, or show that such a function does not exist. (We recall that a function has a "jump" discontinuity at $x = a$ if its one-sided limits at $x = a$ exist but are not equal.)

Solution. Assuming that $f(x)$ is continuous on $(-1, 1)$, differentiable on $(-1, 0) \cup (0, 1)$, and that its derivative has a jump discontinuity at $x = 0$, we show that $f(x)$ cannot be differentiable at $x = 0$. Indeed, let

$$\lim_{x\to 0^+} f'(x) = a \qquad \text{and} \qquad \lim_{x\to 0^-} f'(x) = b$$

with $a \neq b$. Then the Mean Value Theorem yields

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+} f'(\theta_h)=\lim_{x\to 0^+} f'(x)=a$$

since $0<\theta_h<h$. Similarly,

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-} f'(\theta_h)=\lim_{x\to 0^-} f'(x)=b$$

since $h<\theta_h<0$. Thus, $f'(0)$ does not exist.

3. (J. Kersey) Find a function $f(t)$ that satisfy $f\left(x+\dfrac{1}{x}\right)=x^2+\dfrac{1}{x^2}$, $x\neq 0$, and find the function $g(t)$ that satisfies $g\left(\dfrac{1+x}{x}\right)=\ln\left(\dfrac{2x}{1+2x}\right)$ for x in $(-\infty,-1/2)\cup(0,\infty)$.

Solution. For the first question, we see by completing the square that

$$f\left(x+\frac{1}{x}\right)=\left(x^2+\frac{1}{x^2}+2\right)-2=\left(x+\frac{1}{x}\right)^2-2,$$

so that, with $t=x+1/x$, $f(t)=t^2-2$, $|t|\geq 2$. For the second one, the substitution $t=(1+x)/x$ gives $x=1/(t-1)$ and

$$g(t)=\ln\left[\frac{2/(t-1)}{1+2/(t-1)}\right]=\ln\left(\frac{2}{t+1}\right),\quad t>-1,\, t\neq 1.$$