1. (S. Kersey) Suppose that M is a 3×3 matrix such that $M^3 = I_3$ (but $M \neq I_3$), where I_3 is the multiplicative identity. Find the eigenvalues of M, and give an example of such a matrix.

Solution. Since

$$1 - \lambda^3 = \det((1-\lambda^3)\, I_3) = \det\left(M^3 - \lambda^3 I_3\right) = \det\left(M - \lambda I_3\right) \cdot \det\left(M^2 + \lambda M + \lambda^2 I_3\right),$$

it follows that the only eigenvalue of M^3 is $\lambda^3 = 1$, and the eigenvalues of M satisfy $\lambda^3 = 1$. Therefore, $\lambda_k = e^{(2\pi i/3)k}$ for $k = -1, 0, 1$ are the only possible eigenvalues. One of the ways to find an example of such matrix, is to interpret M as a linear transformation of the space $\mathbb{R}^3$. From this point of view, the condition $M^3 = I_3$ means that, applying M consequently three times, we bring $\mathbb{R}^3$ back to the initial position. Clearly, say, counterclockwise rotation about the z-axis by $120°$, given by

$$M = \begin{bmatrix} \cos 120° & -\sin 120° & 0 \\ \sin 120° & \cos 120° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

satisfies this condition.

2. (V. Maymeskul) Consider the following "pool-like" game: a ball is placed a units off the center of a disk-shaped pool table of radius r. A player wins if the ball passes the original location for the first time after meeting the table's border three times. Find the winning hit (in terms of the angle between its direction and the diameter on which the ball is originally resting), and give a relation between a and r, which is required for success.

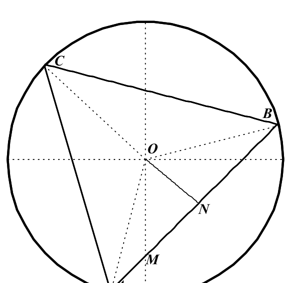
Solution. There are two solutions.

1. (F. Ziegler) On the left figure below, O is the center of the table, M indicates the original location of the ball so that $OM = a$, $\triangle ABC$ shows the route of the ball, and the interval ON is perpendicular to AB. $\angle OAN = \angle OBN$ since the $\triangle OAB$ is isosceles. By the Snell's law, $\angle OAN = \angle OAC$ and $\angle OBN = \angle OBC$. Applying similar arguments to $\triangle OAC$ and $\triangle OBC$, we conclude that $\triangle ABC$ is equilateral and $\angle OAN = 30°$. Therefore (from the right $\triangle OAN$), $|ON| = |OA| \sin(30°) = r/2$. Looking now at the right $\triangle OMN$, we conclude that

$$\sin(\angle AMN) = \frac{|ON|}{|OM|} = \frac{r}{2a} \quad \Rightarrow \quad \angle OMN = \sin^{-1}\left(\frac{r}{2a}\right).$$

This solution works only if $|OM| \geq |ON|$, i.e., $a \geq r/2$.

2. (V. Maymeskul) On the right figure, O is the center of the pool table, OA is a vertical radius, the point M indicates the initial position of the ball, and the quadrilateral shows the ball's route. We are given that $|OA| = r$ and $|OM| = a$. Let $|ON| = p$.



From the $\triangle OBA$, by the Law of Cosines,

$$|AB|^2 = |OB|^2 + |OA|^2 + 2|OA| \cdot |ON| = 2r^2 + 2rp\,. \tag{2}$$

Similarly, from $\triangle OBM$,

$$|MB|^2 = |OB|^2 + |OM|^2 - 2|OM| \cdot |ON| = r^2 + a^2 - 2ap\,. \tag{3}$$

Squaring (1) and substituting (2), (3) yields

$$\frac{r^2}{a^2} = \frac{|AB|^2}{|MB|^2} = \frac{2r^2 + 2rp}{r^2 + a^2 - 2ap}\,.$$

Solving now for p, we get

$$p = \frac{r(r-a)}{2a}\,.$$

On the picture, $p < a$ which is the case if $r/2 < a \leq r$. We have then

$$\tan \angle OMB = \frac{|NB|}{|NM|} = \frac{\sqrt{r^2 - p^2}}{a - p} \quad \Rightarrow \quad \angle OMB = \tan^{-1}\left(\frac{\sqrt{r^2 - p^2}}{a - p}\right).$$

If $p > a$ ($a < r/2$), $\angle OMB$ is an obtuse angle and the route of the ball is not convex, but similar arguments apply and give the same formula for p. (Check!) In this case,

$$\tan \angle NMB = \frac{|NB|}{|NM|} = \frac{\sqrt{r^2 - p^2}}{p - a} \quad \Rightarrow \quad \angle OMB = \pi - \tan^{-1}\left(\frac{\sqrt{r^2 - p^2}}{p - a}\right).$$

If $a = r/2$, then $p = a$ and $\angle OMB = \pi/2$.

This solution works for $p < r$, which yields $a > r/3$.

If $a \leq r/3$, there is no way to win the game.