

Set 2 Solutions

1. Construct a 2×2 matrix A with distinct entries chosen from $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\}$ so that $A^2 = A$ (such a matrix is called idempotent). Can you find all of them?

Solution: It would be helpful to find the eigenvalues (see Problem 2) of an idempotent matrix A . We have

$$Av = \lambda v \Rightarrow A^2v = \lambda Av = \lambda^2v \Rightarrow Av = \lambda^2v \Rightarrow \lambda^2v = \lambda v \Rightarrow \lambda = 0, 1.$$

Therefore, 0 and 1 are the only eigenvalues for an idempotent matrix and, to solve the problem, we have to construct a matrix A satisfying

- (i) the eigenvalues are 0 and 1;
- (ii) all the entries are distinct numbers from the given set.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc).$$

The condition (i) yields the system

$$\begin{cases} a + d = 1 \\ ad - bc = 0. \end{cases}$$

The rest of the solution is just elementary check of these two requirements for (distinct) numbers $a, b, c,$ and d from the given list. Say, $\{a, d\} = \{-1, 2\}$ yields $\{b, c\} = \{2, -1\}$, and so we have four solutions with these entries

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}.$$

Altogether, there exist 36 idempotent matrices with distinct entries chosen from the given set of integers.

2. Let $A = (a_{ij})_{n \times n}$ be a $n \times n$ matrix with $a_{ij} = 1$ for all i and j , i.e., all entries of A are 1's, and let I_n denote the unit $n \times n$ matrix. Show that the zeros of the characteristic polynomial $\chi(\lambda) = \det(A - \lambda I_n)$ are $\lambda = 0$ of multiplicity $n - 1$ and $\lambda = n$. (Zeros of the characteristic polynomial are called the eigenvalues of A).

Solution: Since the determinant of a matrix is invariant under elementary row operations, let us (1)– add all the rows, second through n -th, to the first one, (2) – factor $(n - \lambda)$ from the first row, (3) – subtract the first row from the others, and (4) – use the first column to evaluate the

determinant. Thus,

$$\chi(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1-\lambda \end{vmatrix} \stackrel{(1)}{=} \begin{vmatrix} n-\lambda & n-\lambda & n-\lambda & \cdots & n-\lambda \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1-\lambda \end{vmatrix} \\ \stackrel{(2)}{=} (n-\lambda) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & 1 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 1-\lambda \end{vmatrix} \stackrel{(3)}{=} (n-\lambda) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -\lambda & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & -\lambda \end{vmatrix} \stackrel{(4)}{=} (n-\lambda)(-\lambda)^{n-1}.$$

3. Given that a polynomial $P(x) = x^{20} - 20x^{19} + \boxed{\text{"black box"}} + 1$ has all real positive roots, recover the polynomial and find its roots.

Solution: Let x_1, x_2, \dots, x_{20} denote the roots of $P(x)$. Then factoring yields

$$P(x) = (x - x_1)(x - x_2) \cdots (x - x_{20}) = x^{20} - (x_1 + x_2 + \cdots + x_{20})x^{19} + \cdots + (x_1 x_2 \cdots x_{20}).$$

Hence

$$\frac{x_1 + x_2 + \cdots + x_{20}}{20} = 1 \quad \text{and} \quad \sqrt[20]{x_1 x_2 \cdots x_{20}} = 1.$$

Since, for any n positive real numbers x_1, x_2, \dots, x_n ,

$$\sqrt[n]{\prod_{k=1}^n x_k} \leq \frac{\sum_{k=1}^n x_k}{n}$$

and the equality holds if and only if all x_j 's are equal ("the geometric mean does not exceed the arithmetic mean"), we conclude that

$$x_1 = x_2 = \cdots = x_{20} = 1 \quad \text{and so} \quad P(x) = (x - 1)^{20}.$$