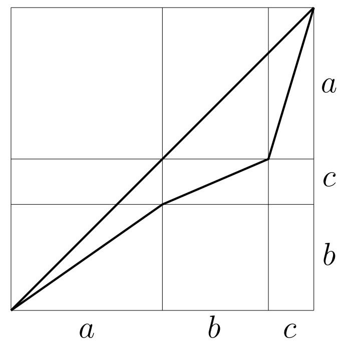
 # 1. (Dr. F. Ziegler) For real numbers a, b, and c, prove the inequality

$$\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2} \geq \sqrt{2}(a+b+c).$$

Solution: Without loss of generality, one can assume that a, b, $c \geq 0$. Then, see the picture below.



 # 2. (Dr. S. Kersey) Suppose that $f(1) = 1$ and $f(n) = 3f(n-1) + 2$ for integers $n > 1$. Find an explicit formula for $f(n)$.

Solution: We note that $f(n) + 1 = 3[f(n-1) + 1]$. Thus, $a_n := f(n) + 1$, $n = 1, 2, \ldots$, is a geometric sequence with $a_1 = f(1) + 1 = 2$. Thus, $a_n = a_1 3^{n-1} = 2 \cdot 3^{n-1}$ implying $f(n) = 2 \cdot 3^{n-1} - 1$.

 # 3. (Dr. Y. Wu) Prove that, for all integers $n > 1$,

$$\ln[(n-1)!] < 1 - n + n\ln(n) < \ln[n!].$$

Solution: Since $f(x) = \ln(x)$ is an increasing function, one has

$$\int_1^n \ln(x)\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} \ln(x)\,dx < \sum_{k=1}^{n-1} \max_{k \leq x \leq k+1} \ln(x) = \sum_{k=1}^{n-1} \ln(k+1) = \ln[n!].$$

On the other hand,

$$\int_1^n \ln(x)\,dx = \sum_{k=1}^{n-1} \int_k^{k+1} \ln(x)\,dx > \sum_{k=1}^{n-1} \min_{k \leq x \leq k+1} \ln(x) = \sum_{k=1}^{n-1} \ln(k) = \ln[(n-1)!].$$

All that remains is to note that (integration by parts)

$$\int_1^n \ln(x)\,dx = [x\ln(x) - x]\,|_1^n = n\ln(n) - n + 1.$$