Set 3 Solutions

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1. (G. Lesaja) Find $\max\{ABCD \,:\, A+2B+3C+4D=8\,, A,B,C,D>0\}$.

Solution 1: We use here the following well-known geometric/arithmetic means relation: for $a_j \geq 0$, $j=1,\ldots,n\,,$

$$\sqrt[n]{\prod_{j=1}^{n} a_j} \leq \frac{\sum_{j=1}^{n} a_j}{n}$$

where the equality sign holds if and only if all a_j's are equal. Thus,

$$\sqrt[4]{(A)(2B)(3C)(4D)} \leq \frac{A+2B+3C+4D}{4} = 2 \quad \Leftrightarrow \quad \sqrt[4]{24ABCD} \leq 2 \quad \Leftrightarrow \quad ABCD \leq \frac{2}{3}\,,$$

and the maximal value of $ABCD = 2/3$ attained when $A = 2B = 3C = 4D = 2$ or $A=2$, $B=1$, $C=2/3$, and $D=1/2$.

Solution 2: (S. Kersey – Calculus III approach.) We define functions $F(A,B,C,D) := ABCD$ and $G(A,B,C,D) := A+2B+3C+4D$. By the method of Lagrange multipliers,

$$\nabla F = \lambda\,\nabla G \qquad \Leftrightarrow \qquad \lambda = BCD, \quad 2\lambda = ACD, \quad 3\lambda = ABD, \quad 4\lambda = ABC$$

or, equivalently,

$$ABCD = \lambda A = 2\lambda B = 3\lambda C = 4\lambda D\,.$$

Therefore, $A = 2B = 3C = 4D$, and so $4A = A+2B+3C+4D = 8$, giving $A=2$, $B=1$, $C=2/3$, $D=1/2$, and $ABCD = 2/3$.

2. (S. Kersey) Let L=$[0,\frac{1}{2}]$ and R=$[\frac{1}{2},1]$ be left and right closed sub-intervals bisecting $[0,1]$. Then, let LL=$[0,\frac{1}{4}]$ and LR=$[\frac{1}{4},\frac{1}{2}]$ be left and right closed subintervals bisecting L, and let RL and RR left and right subintervals bisecting R. Continue in this way, e.g. LRL = $[\frac{2}{8},\frac{3}{8}]$. Find LRLRLRLR… , where this pattern is carried out to infinitely many steps.

Solution: Let $I_0 := [0,1]$, $I_1 = L$, $I_2 = LR$, $I_3 = LRL$, $I_4 = LRLR, \ldots$, and let $I := \bigcap_{n=0}^{\infty} I_n$. Since $(I_n)_0^\infty$ is a decreasing sequence (since $I_{n+1} \subset I_n$) of closed intervals, their intersection I is non-empty and, since $|I_n| \to 0$, the intersection is a point. Now, the midpoints of I_0, I_1, I_2, ... are $\frac{1}{2}$, $\frac{1}{2}-\frac{1}{4}$, $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}, \ldots$, respectively, which are partial sums of a geometric series with ratio $-\frac{1}{2}$. In the limit, the centers converge to $\frac{1}{2}\cdot\frac{1}{1+(1/2)} = \frac{1}{3}$.

3. (Y. Wu) Evaluate the integral

$$I := \int_0^{\pi/7} \frac{\sin(\theta)}{\sin[(\pi/7)-\theta]+\sin(\theta)}\,d\theta\,.$$

Solution: (Calculus I is a must.) Making a substitution $\phi = (\pi/7) - \theta$ so that $\theta = (\pi/7) - \phi$ and $d\theta = -d\phi$ yields

$$I = \int_0^{\pi/7} \frac{\sin(\theta)\, d\theta}{\sin[(\pi/7) - \theta] + \sin(\theta)} = \int_{\pi/7}^{0} \frac{\sin[(\pi/7) - \phi]\ (-d\phi)}{\sin(\phi) + \sin[(\pi/7) - \phi]} = \int_0^{\pi/7} \frac{\sin[(\pi/7) - \theta]\, d\theta}{\sin(\theta) + \sin[(\pi/7) - \theta]} \,,$$

where we have replaced, for convenience, the integration variable from ϕ to θ. Then, taking the sum of this integral with the original one yields

$$2I = \int_0^{\pi/7} d\theta = \frac{\pi}{7} \quad \Leftrightarrow \quad I = \int_0^{\pi/7} \frac{\sin(\theta)}{\sin[(\pi/7) - \theta] + \sin(\theta)}\, d\theta = \frac{\pi}{14}\,.$$