Spring 2008 MPC – Set 3 solutions.

 # 1. (Dr. G. Lesaja) Given a function f satisfying $f(x)+2f\left(\frac{1}{1-x}\right)=x$, find $f(2)$.

Solution: Substituting $x=2$, -1, and $1/2$ into the given relation leads to the system

$$\begin{aligned} f(2)+2f(-1)&=2\\ f(-1)+2f(1/2)&=-1\\ f(1/2)+2f(2)&=1/2\,. \end{aligned}$$

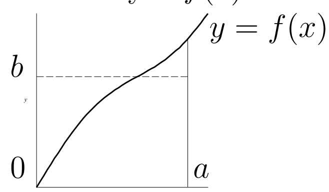
Solving yields $f(2)=2/3$.

 # 2. (Dr. V. Maymeskul) Let $f(x)$, $x\ge 0$, be a continuous strictly increasing function satisfying $f(0)=0$, and let $f^{-1}(x)$ denote its inverse. Show that, for any $a\ge 0$ and $b\ge 0$,

$$\int_0^a f(x)\,dx+\int_0^b f^{-1}(x)\,dx\ge ab,$$

where the equality holds if and only if $b=f(a)$. Use this result to compute $\int_0^{1/2}\arcsin(x)\,dx$.

Solution: WLOG we can assume $b\le f(a)$. From the figure below, the area under $y=f(x)$ on $[0,a]$ plus the area to the left of $x=f^{-1}(y)$ on $[0,b]$ is greater or equal to the area of the rectangle $[0,a]\times[0,b]$, with equality if and only if $f(a)=b$.



Since $\sin(\pi/6)=1/2$, we have

$$\int_0^{\pi/6}\sin(x)\,dx+\int_0^{1/2}\sin^{-1}(x)\,dx=\frac{\pi}{12}\ \Rightarrow\ \int_0^{1/2}\sin^{-1}(x)\,dx=\frac{\pi}{12}+\cos(x)\Big|_0^{\pi/6}=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$$

 # 3. (Dr. Y. Wu) Plot the polar curves $r=\cos(n\theta)$ for $n=1,\,2,\,3$, and 4, observing the number and position of distinct loops. Then prove that the graph of $r=\cos(n\theta)$, where n is a positive integer, has n distinct loops if n is odd and $2n$ distinct loops if n is even.

Solution: Firstly, note that two points with distinct polar coordinates (r_1,θ_1) and (r_2,θ_2) for $\theta_i\in\left[-\frac{\pi}{2n},\,2\pi-\frac{\pi}{2n}\right]$ and $r_i\ne 0$, have the same image on the xy-plane if and only if

(1) $$r_2=-r_1\quad\text{and}\quad\theta_2=\theta_1\pm\pi.$$

Secondly, since $r=\cos(n\theta)$ has exactly $n+1$ zeros $\theta_m:=\frac{\pi}{2}\cdot\frac{2m-1}{n}$, for $m=0,1,\ldots,n$, on the interval $\left[-\frac{\pi}{2n},\,\pi-\frac{\pi}{2n}\right]$ of length π, the polar curve $r=\cos(n\theta)$ has exactly n distinct loops on this range, one corresponding to each of the intervals $[\theta_{m-1},\,\theta_m]$. Since

$$r(\theta+\pi)=\cos\left[n(\theta+\pi)\right]=\begin{cases}\cos(n\theta), & n\text{ is even;}\\ -\cos(n\theta), & n\text{ is odd,}\end{cases}$$

it follows by (1) that the loops corresponding to $[\theta_{m-1}+\pi,\,\theta_m+\pi]$ and $[\theta_{m-1},\,\theta_m]$ are not distinct if n is odd, but n additional loops appear for $\theta\in\left[\pi-\frac{\pi}{2n},\,2\pi-\frac{\pi}{2n}\right]$ if n is even.