

Tips for Learning Algebra

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Abstract.

Algebraic skills are very important for students to study algebra, trigonometry, calculus, and other mathematical and scientific subjects. Many students are not good at math and science since they lack fundamental algebraic skills. In this talk, tips for learning algebra are presented. Interesting and challenge examples are given to explain how powerful these tips are. Mastering these tips will improve one's ability to learn mathematics.

Problems in algebra:

- Simplify expressions
- Evaluate expressions or functions
- Sketch the graphs of functions
- Find functions with certain conditions
- Solve equations
- Solve inequalities
- Prove identities or formulas
- Solve word problems

OUTLINE

1. Integers
2. Integers and polynomials
3. Formulas
4. Factoring
5. Functions and graphs
6. Rational expressions
7. Substitution and extension
8. Logical reasoning and math puzzles
9. Necessary and sufficient conditions
10. Mathematical induction

1. Integers

Perfect squares and square roots:

n	1	2	3	4	5	6	7	8	9	10	↑	\sqrt{a}
n^2	1	4	9	16	25	36	49	64	81	100		a

n	11	12	13	14	15	16	17	18	19	20	↑	\sqrt{a}
n^2	121	144	169	196	225	256	289	324	361	400		a

Powers of 2:

n	1	2	3	4	5	6	7	8	9	10	↑	$\log_2 a$
2^n	2	4	8	16	32	64	128	256	512	1024		a

Example 1.1. Find squares of the following integers:

n	15	25	35	45	55	65	75	85	95
n^2									

Is there an easy way to memorize the table?

Solution.

n	15	25	35	45	55	65	75	85	95	↑	\sqrt{a}
n^2	225	625	1225	2025	3025	4225	5625	7225	9025		a

Let $n = 10d + 5$. Then

$$n^2 = (10d + 5)^2 = 100d^2 + 100d + 25 = 100d(d + 1) + 25.$$

Example. $65^2 = 6 \cdot 7 \cdot 100 + 25 = 4225$.

Multiples of perfect squares

n^2	1	4	9	16	25	36	49	...
$2n^2$	2	8	18	32	50	72	98	
$3n^2$	3	12	27	48	75	108		
$5n^2$	5	20	45	80	125			

Formula

$$\sqrt{a^2b} = a\sqrt{b}, \text{ for } a, b > 0.$$

Example 1.2. Simplify $\sqrt{27} + \sqrt{32} - \sqrt{48} + \sqrt{72}$

Solution.

$$\begin{aligned}\sqrt{27} + \sqrt{32} - \sqrt{48} + \sqrt{72} &= \sqrt{9 \cdot 3} + \sqrt{16 \cdot 2} - \sqrt{16 \cdot 3} + \sqrt{36 \cdot 2} \\ &= 3\sqrt{3} + 4\sqrt{2} - 4\sqrt{3} + 6\sqrt{2} \\ &= 10\sqrt{2} - \sqrt{3}.\end{aligned}$$

Large integers

Example 1.3. Express the number $\underbrace{777 \cdots 7}_n$ in a compact form which is a function of n .

Example 1.4. Estimate 5^{20} ?

Example 1.3. Express the number $\underbrace{777 \cdots 7}_n$ in a compact form which is a function of n .

Solution. Let $a = \underbrace{777 \cdots 7}_n$. Then $\frac{a}{7} = \underbrace{1111 \cdots 1}_n$

$$\frac{9a}{7} = \underbrace{999 \cdots 9}_n = \underbrace{1000 \cdots 0}_n - 1 = 10^n - 1.$$

$$\text{So } a = \frac{7}{9}(10^n - 1).$$

Example 1.4. Estimate 5^{20} .

Solution. Let $y = 5^{20}$. Then

$$\log_{10} y = 20 \log_{10} 5 \approx 20 \cdot 0.699 = 13.98.$$

$$\text{So } y \approx 10^{13.98} \approx 10^{14}$$

2. Integers and Polynomials

There are many common properties between integers and real polynomials.

	Integers	Real polynomials
factor	$4 12$	$(x-2) (x^2+x-6)$
prime	5 is a prime number	x^2+2 is a irreducible polynomial
factoring	$48 = 2^2 \cdot 3$	$x^4 - 2x^3 + 3x^2 - 4x + 2 = (x-1)^2(x^2+2)$
division	$p = q \cdot g + r, 0 \leq r < g, g \geq 0$	$p(x) = q(x)g(x) + r(x), \deg(g) \geq 0, \deg(r) < \deg(g)$
gcd	$\gcd(12, 20) = 4$	$\gcd((x-1)(x+2)^2, (x+2)^3) = (x+2)^2$
lcm	$\text{lcm}(12, 20) = 60$	$\text{lcm}((x-1)(x+2)^2, (x+2)^3) = (x-1)(x+2)^3$
relative p.	$(12, 25) = 1$	$(p(x), g(x)) = 1$
	$a b, b c \implies a c$	$p(x) g(x), g(x) w(x) \implies p(x) w(x)$
	$a c, b c \implies \text{lcm}(a, b) c$	$p(x) w(x), g(x) w(x) \implies \text{lcm}(p(x), g(x)) w(x)$
	$a c, b c, (a, b) = 1 \implies (ab) c$	$p(x) w(x), g(x) w(x), (p, g) = 1 \implies (p(x)g(x)) w(x)$

Prime numbers

$\{2, 3, 5, 7, 11, 13, 17, \dots\}$

Theorem (Fundamental Theorem of Arithmetic). Each positive integer a with $a > 1$ is uniquely factored as

$$a = p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_t^{n_t},$$

where p_i 's are prime numbers and n_i 's are positive integers.

For example. $1800 = 2 \cdot 9 \cdot 100 = 2^3 \cdot 3^2 \cdot 5^2$.

Example 2.1. Let x be a positive integer less than 105. Suppose that as x is divided by 3, 5, 7, the remainders are 1, 2, 3, respectively. Find x .

Solution. Denote $r_3 = 1$, $r_5 = 2$, $r_7 = 3$.

(1) Let

$$\begin{aligned}y &= 70r_3 + 21r_5 + 15r_7 \\ &= 70 \cdot 1 + 21 \cdot 2 + 15 \cdot 3 = 157.\end{aligned}$$

(2) Divide y by 105,

$$y = 157 = 105 \cdot 1 + 52.$$

(3) The remainder 52 is the value of x .

Proof. (by division theorem) Let

$$x = 3q_3 + r_3$$

$$70x = 210q_3 + 70r_3$$

$$x = 5q_5 + r_5 \quad \text{Then}$$

$$21x = 105q_5 + 21r_5$$

$$x = 7q_7 + r_7.$$

$$15x = 105q_7 + 15r_7.$$

So

$$\begin{aligned} 106x &= 105(2q_3 + q_5 + q_7) + 70r_3 + 21r_5 + 15r_7 \\ &= 105(2q_3 + q_5 + q_7) + y, \end{aligned}$$

which implies that

$$y = 105(2q_3 + q_5 + q_7 - x) + x.$$

Thus, x is the remainder when y is divided by 105.

A prime in the form of $2^p - 1$, where p is a prime, is called a *Mason prime*.

For example, $2^2 - 1$, $2^3 - 1$, $2^5 - 1$.

More Mason primes

1772	$2^{31} - 1$	10 digits	the 8th Mason prime
1963	$2^{11213} - 1$	3376 digits	the 23rd Mason prime
2006	$2^{32582657} - 1$	9808358 digits	the 44th Mason prime

3. formulas

It is very useful to understand and memorize formulas in two directions.

$$\textit{Expression 1} = \textit{Expression 2}$$

is equivalent to

$$\textit{Expression 2} = \textit{Expression 1}.$$

A multiplication formula and a complete square formula,

$$(x \pm a)^2 = x^2 \pm 2ax + a^2 \quad \text{and} \quad x^2 \pm 2ax + a^2 = (x \pm a)^2.$$

Let $a, b > 0$, and let m, n , and p be positive real numbers.

$$\begin{aligned} a^{-n} &= \frac{1}{a^n} & \text{and} & & \frac{1}{a^n} &= a^{-n}, \\ \sqrt[p]{a^m} &= a^{\frac{m}{p}} & \text{and} & & a^{\frac{m}{p}} &= \sqrt[p]{a^m} = (\sqrt[p]{a})^m, \\ a^m a^n &= a^{m+n} & \text{and} & & a^{m+n} &= a^m a^n, \\ \frac{a^m}{a^n} &= a^{m-n} & \text{and} & & a^{m-n} &= \frac{a^m}{a^n}, \\ a^n b^n &= (ab)^n & \text{and} & & (ab)^n &= a^n b^n, \\ \frac{a^n}{b^n} &= \left(\frac{a}{b}\right)^n & \text{and} & & \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n}, \\ (a^m)^n &= a^{mn} & \text{and} & & a^{mn} &= (a^m)^n = (a^n)^m. \end{aligned}$$

$$\log(ab) = \log a + \log b \quad \text{and} \quad \log a + \log b = \log(ab).$$

Example 3.1. Determine whether each of the following expressions is a complete square or not:

(a) $x^2 + 8x + 64$

(b) $x^2 + 10x - 25$

(c) $x^2 - 12xy + 36y^2$

(d) $x^2 + 20xy + 40y^2$

Example 3.2. Rewrite $-x^2 + 6x + 13$ as a sum or difference of a constant and a complete square.

Solution.

$$\begin{aligned} -x^2 + 6x + 3 &= -(x^2 - 6x - 3) = -(x^2 - 6x + 3^2 - 3 - 9) \\ &= -((x - 3)^2 - 12) = 12 - (x - 3)^2. \end{aligned}$$

Example 3.3 Let $x > 0$. Simplify $\sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}$

Solution.

$$\begin{aligned}\sqrt{1 + \left(2x - \frac{1}{8x}\right)^2} &= \sqrt{1 + (2x)^2 - \frac{1}{2} + \left(\frac{1}{8x}\right)^2} \\ &= \sqrt{(2x)^2 + \frac{1}{2} + \left(\frac{1}{8x}\right)^2} \\ &= \sqrt{\left(2x + \frac{1}{8x}\right)^2} = 2x + \frac{1}{8x}.\end{aligned}$$

4. Factoring

Factoring is one of most important skills in algebra.

Factoring formulas

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

Example 4.1 Factor $2x^2 + 3xy - 2y^2 - 5x + 5y - 3$.

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Solution.

$$2x^2 + 3xy - 2y^2 = (x + 2y)(2x - y).$$

So we may write

$$2x^2 + 3xy - 2y^2 - 5x + 5y - 3 = (x + 2y + a)(2x - y + b).$$

Assume that $y = 0$. We have

$$2x^2 - 5x - 3 = (x - 3)(2x + 1).$$

Assume that $x = 0$. We have

$$-2y^2 + 5y - 3 = (2y - 3)(-y + 1).$$

Thus, we select $a = -3$ and $b = 1$.

$$2x^2 + 3xy - 2y^2 - 5x + 5y - 3 = (x + 2y - 3)(2x - y + 1).$$

Theorem (Fundamental Theorem of Algebra) Every complex polynomial of degree $n \geq 1$ has at least one complex zero.

Theorem 4.1. Let f be a polynomial of degree $n \geq 1$. If $x - c$ is a factor of $f(x)$ then c is a zero of f .

Theorem 4.2. Let f be a polynomial of degree $n \geq 1$. If $f(c) = 0$ then $x - c$ is a factor of $f(x)$.

Example 4.2 Let a , b and c be nonzero real numbers. Show that if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c} \quad (1)$$

then for any odd integer n ,

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{(a + b + c)^n} .$$

Solution.

$$\begin{aligned}\frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{a+b+c} \implies \frac{bc+ac+ab}{abc} = \frac{1}{a+b+c} \\ &\implies (a+b+c)(ab+bc+ac) - abc = 0.\end{aligned}$$

Denote

$$F = (a+b+c)(ab+bc+ac) - abc.$$

If $a = -b$ then $F = 0$. By Theorem 4.2, $a+b$ is a factor of F . So are $b+c$ and $a+c$. Thus

$$0 = F = (a+b)(b+c)(a+c),$$

We have $a = -b$, $b = -c$ or $c = -a$. In each case, for an odd integer n ,

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{(a+b+c)^n}.$$

5. Functions and graphs

A function is a relation of two sets called domain and range such that for each element of the domain there exists a unique corresponding element of the range.

The Properties of a function are reflected in its graph.

The graphs of fundamental functions such as

$$y = mx + b,$$

$$y = |x|,$$

$$y = ax^2 + bx + c \ (a \neq 0),$$

$$y = \frac{1}{x},$$

$$y = \sqrt{x},$$

$$y = a^x \ (a > 0 \text{ and } a \neq 1),$$

$$y = \log_a x \ (a > 0 \text{ and } a \neq 1).$$

An point (a, b) is on the graph of a function f if and only if $f(a) = b$.

Relationship among function, equation, and inequality.

- Evaluating $f(a)$ to find a point (a, b) on the graph.
- Solving equation $f(x) = 0$ to find x -intercepts.
- Solving inequality to determine the domain.
- Solving $f(x) > 0$ to find intervals for which the graph is above the x -axis.
- Finding the limit to determine the asymptotes of the graph.

Quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} \\ &= a(x - x_1)(x - x_2) \quad \text{if } b^2 - 4ac \geq 0. \end{aligned}$$

- the y -intercept c .
- the axis, $x = -\frac{b}{2a}$.
- If $a > 0$ ($a < 0$) then f has minimum (maximum) value $\frac{4ac - b^2}{4a}$ when $x = -\frac{b}{2a}$.
- f has two real zeros x_1 and x_2 if $b^2 - 4ac \geq 0$.

Three of the following conditions to determine a quadratic function

(i) a zero

(ii) the y -intercept

(iii) axis

(iv) minimum/maximum value

(v) a point on the graph

Note. The coordinates of the vertex are equivalent to conditions (iii) and (iv).

Example 5.1. Determine a quadratic function f such that

(a) 1 is a zero of f (b) the y -intercept is 6 (c) maximum value is 8.

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Solution 1. Let $f(x) = a(x - 1)(x - x_2)$.

By (b) $ax_2 = 6$. So $f(x) = ax^2 - (a + 6)x + 6$.

By (c) $a^2 + 20a + 36 = 0$, Solve for a . So $a = -2$ or -18 . Then

$$f(x) = -2x^2 - 4x + 6 \quad \text{or} \quad -18x^2 + 12x + 6.$$

Solution 2. Let $f(x) = ax^2 + bx + 6$.

By (a) and (c),
$$\begin{cases} a + b + 6 = 0, \\ \frac{4a \cdot 6 - b^2}{4a} = 8. \end{cases} \quad \text{Solve for } a \text{ and } b.$$

Solution 3. Let $f(x) = a(x + h)^2 + 8$.

By (a) and (c),
$$\begin{cases} a(1 + h)^2 + 8 = 0, \\ ah^2 + 8 = 6. \end{cases} \quad \text{Solve for } a \text{ and } h.$$

6. Rational expressions

Basic property,

$$\frac{a}{b} = \frac{ac}{bc}, \quad \text{where } c \text{ is a nonzero expression.}$$

Formulas:

$$\begin{aligned} \frac{a}{b} \pm \frac{c}{d} &= \frac{ad \pm bc}{bd}, & a \pm \frac{c}{d} &= \frac{ad \pm c}{d}, \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \\ \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{c} \end{aligned}$$

Example 6.1.

$$\begin{aligned} \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{a}}}} &= \frac{1}{1 + \frac{1}{1 + \frac{1}{a+1}}} = \frac{1}{1 + \frac{1}{1 + \frac{a}{a+1}}} \\ &= \frac{1}{1 + \frac{a+1}{2a+1}} = \frac{1}{\frac{3a+2}{2a+1}} = \frac{2a+1}{3a+2}. \end{aligned}$$

Example 6.2. Determine whether the following equations are correct or not.

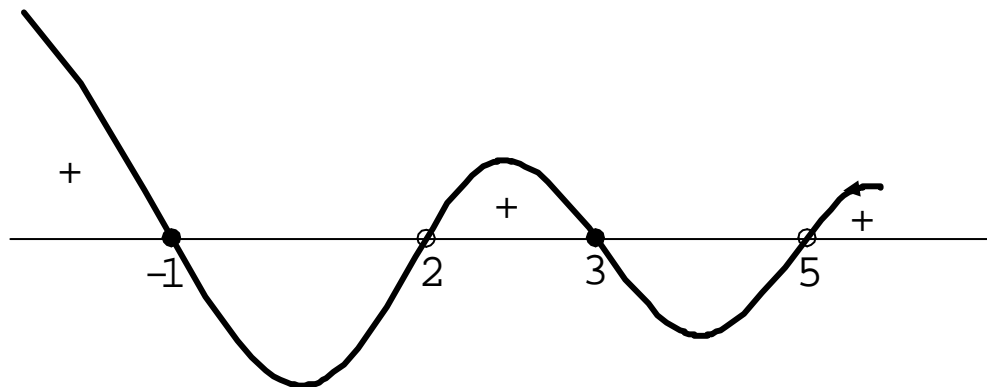
$$\begin{aligned} \frac{a + |a|}{a} &= \frac{a + a}{a} = 2, & \text{for } a \neq 0, \\ \frac{a + b}{a} &= \frac{1 + b}{1} = 1 + b, & \text{for } a \neq 0. \end{aligned}$$

Example 6.3. Solve the inequality $\frac{x^2 - 2x - 3}{x^2 - 7x + 10} \geq 0$.

Solution. By factoring, we solve $\frac{(x - 3)(x + 1)}{(x - 2)(x - 5)} \geq 0$.

Solve $(x - 3)(x + 1) = 0$. $x = -1, 3$. Plot $x = -1, 3$ with a dot.

Solve $(x - 2)(x - 5) = 0$. $x = 2, 5$. Plot $x = 2, 5$ with a small circle.



The solution set of the inequality is given by

$$(-\infty, -1] \cup (2, 3] \cup (5, \infty)$$

7. Substitution and extension

A symbol in a formula, identity or equation can be substituted by a meaningful expression for the symbol.

Start with a multiplication formula,

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Replace a and b by y ,

$$(x + y)^2 = x^2 + 2xy + y^2.$$

Replace y by $-y$,

$$(x - y)^2 = x^2 - 2xy + y^2.$$

Extend the exponent

$$(x \pm y)^3 = (x^2 \pm 2xy + y^2)(x \pm y) = x^3 \pm 3x^2y + 3xy^2 \pm y^3,$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n.$$

Extend 2 terms to 3 terms

$$\begin{aligned}(x + y + z)^2 &= x^2 + 2x(y + z) + (y + z)^2 \\ &= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.\end{aligned}$$

Extend 2 terms to n terms

$$\begin{aligned}(x_1 + x_2 + \cdots + x_n)^2 &= x_1^2 + x_2^2 + \cdots + x_n^2 + 2x_1x_2 + \cdots + 2x_1x_n + 2x_2x_3 + \cdots + 2x_{n-1}x_n \\ &= \sum_{i=1}^n x_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j\end{aligned}$$

Can we extend exponent 2 to $\frac{1}{2}$?

$$(1 + x)^{1/2} = ?$$

Applications of substitution:

Example 7.1 Show that for any n real numbers a_i 's,

$$|a_1| + |a_2| + \cdots + |a_n| \leq \sqrt{n} \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Solution. It is equivalent to show that

$$(|a_1| + |a_2| + \cdots + |a_n|)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

For any real numbers x and y , $(|x| - |y|)^2 = x^2 - 2|xy| + y^2 \geq 0$. Then

$$2|xy| \leq x^2 + y^2.$$

$$\begin{aligned} & (|a_1| + |a_2| + \cdots + |a_n|)^2 \\ &= \sum_{i=1}^n a_i^2 + \sum_{1 \leq i < j \leq n} 2|a_i a_j| \leq \sum_{i=1}^n a_i^2 + \sum_{1 \leq i < j \leq n} (a_i^2 + a_j^2) \\ &= \sum_{i=1}^n a_i^2 + [(n-1)a_1^2 + a_2^2 + \cdots + a_n^2] + [(n-2)a_2^2 + a_3^2 + \cdots + a_n^2] \\ &\quad + \cdots + [a_{n-1}^2 + a_n^2] \\ &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n (n-1)a_i^2 = n \sum_{i=1}^n a_i^2. \end{aligned}$$

8. Logical reasoning and math puzzles

It is critical to analyze and solve math problems with logical reasoning.

Example 8.1. Let f be a polynomial of degree $n \geq 1$. Show that if $f(c) = 0$ then $x - c$ is a factor of $f(x)$.

Proof. We want to show that $f(x) = (x - c)q(x)$ for some polynomial $q(x)$.

It follows from the division theorem that

$$f(x) = (x - c)q(x) + r,$$

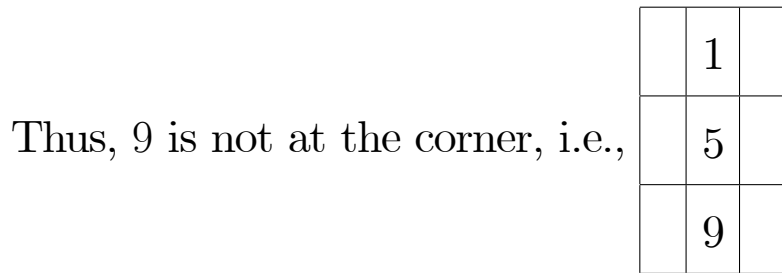
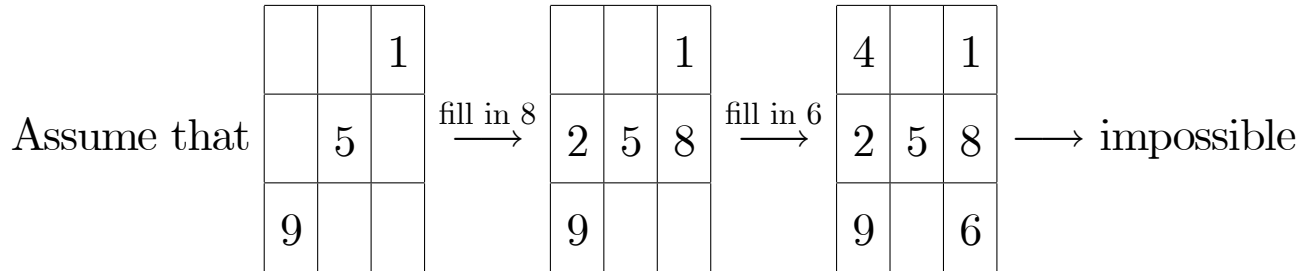
where $q(x)$ is a polynomial and r is a constant. Then $r = f(c) = 0$, which implies that

$$f(x) = (x - c)q(x).$$

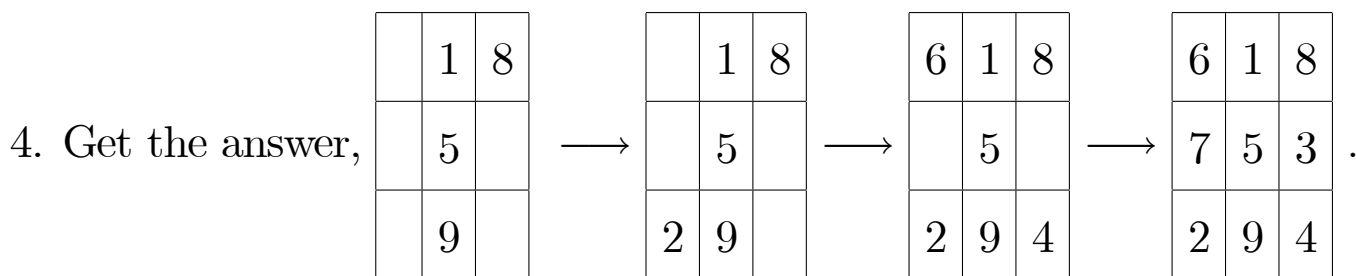
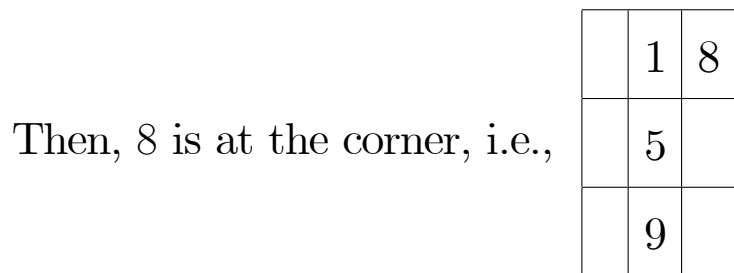
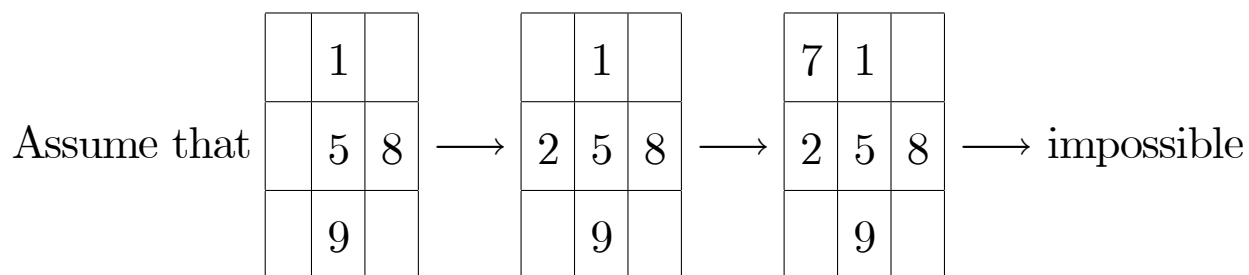
Example 8.2 Fill in a 3 by 3 matrix with the first 9 positive integers such that each row sum, each column sum, diagonal sum and anti-diagonal sum are all equal to 15.

Solution.

1. Select center 5. Pair numbers: (1,9), (2,8), (3,7), and (4,6).
2. Determine whether 9 is at the corner or not.



3. Determine whether 8 is at the corner or not.



Example 8.3. (Sudoku) Fill in the grid so that every row, every column, and every 3×3 box contains the digits 1 through 9.

4		3				2		6
							8	
	9			8			7	9
		8	7		5			
	5	2				1	3	
			6		3	5		
8	2			6				
	7						5	
9		1				8		3

Solution.

4	(8)	3				2		6
							8	(5)
	9			8			7	9
		8	7	5				
	5	2				1	3	
			6	3	5			(8)
8	2			6				
	7						5	
9		1				8		3

4	8	3	5	7	9	2	1	6
2	9	7	1	3	6	4	8	5
5	1	6	4	8	2	3	7	9
1	3	8	7	2	5	9	6	4
6	5	2	9	4	8	1	3	7
7	4	9	6	1	3	5	2	8
8	2	5	3	6	4	7	9	1
3	7	4	8	9	1	6	5	2
9	6	1	2	5	7	8	3	3

9. Sufficient and necessary conditions

It is very important to understand the difference between necessary and sufficient conditions.

Definition. Let A and B are two statements. If A is true implies that B is true, then A is a *sufficient condition* of B and B is a *necessary condition* of A , denoted by $A \implies B$.

Example 9.1 We know that

$$\frac{a}{b} = \frac{c}{d} \implies ad = bc.$$

So $\frac{a}{b} = \frac{c}{d}$ is a sufficient condition of $ad = bc$, and $ad = bc$ is a necessary condition of $\frac{a}{b} = \frac{c}{d}$.

Is it always true that $ad = bc \implies \frac{a}{b} = \frac{c}{d}$? No.

Is $ad = bc$ a sufficient condition of $\frac{a}{b} = \frac{c}{d}$? No.

If $b \neq 0$ and $d \neq 0$ then $ad = bc$ is a sufficient condition of $\frac{a}{b} = \frac{c}{d}$.

Definition. If B is both a necessary and sufficient condition of A (i.e., A is true *if and only if* B is true) then we say that two statements A and B are *equivalent*, denoted by $A \iff B$.

Example. 9.2. If $b \neq 0$ and $d \neq 0$ then $ad = bc \iff \frac{a}{b} = \frac{c}{d}$.

Meaning:

$$b \neq 0, d \neq 0, ad = bc \implies \frac{a}{b} = \frac{c}{d};$$

$$b \neq 0, d \neq 0, \frac{a}{b} = \frac{c}{d} \implies ad = bc.$$

Example 9.3. Let $r > 0$. Then

$$|x - a| < r \iff a - r < x < a + r.$$

Solve inequality $|5 - 2x| < 3$.

Solution.

$$\begin{aligned} |5 - 2x| < 3 &\iff |2x - 5| < 3 \\ &\iff 3 - 5 < 2x < 3 + 5 \\ &\iff -2 < 2x < 8 \\ &\iff -1 < x < 4. \end{aligned}$$

The solution set is $(-1, 4)$.

Example 9.4. Let $p(x)$ be a polynomial in x of degree $n \geq 1$. Then $x - c$ is a factor of $p(x)$ if and only if $p(c) = 0$.

Each definition is considered as a necessary and sufficient condition statement.

Definition. Let $a \geq 0$. A nonnegative real number x is called an *arithmetic square root* of a if $x^2 = a$, denoted as $x = \sqrt{a}$.

Meaning: Let $a \geq 0$ and $x \geq 0$. Then $x = \sqrt{a} \iff x^2 = a$.

Definition. $|a| = a$ if $a \geq 0$; otherwise, $|a| = -a$.

Example 9.5 Determine whether each of the following statements is true or false? If false, how to correct?

(i) For any $a \geq 0$, $(\sqrt{a})^2 = a$.

(ii) For any $a \geq 0$, $\sqrt{a^2} = a$.

(iii) For any real number a , $\sqrt{a^2} = a$.

(iv) For any real number a , $\sqrt{a^2} = |a|$.

(v) $\sqrt{x} = a \implies x = a^2$.

(vi) $\sqrt{x} = a \iff x = a^2$.

(vii) $|x| = a \iff x^2 = a^2$.

(viii) $|x| = |a| \iff x^2 = a^2$.

10. Mathematical induction

Suppose that a statement is true for $n = 1$. Suppose that the statement is true for $n = k \geq 1$ implies that the statement is also true for $n = k + 1$. Then the statement is true for every positive integer n .

Example 10.1. There are three poles and 9 different size rings on a pole such that each one is smaller than the one below it. You can only move one ring from a pole to another pole each time such that a smaller ring is always above a larger one. How many moves are required at least for you to move all 9 rings from one pole to another pole? How about n rings?

Solution. Let a_n be the numbers of moves required to move n rings from one pole to another pole.

n	1	2	3	4	5	6	...	n
a_n	1	$2 \cdot 1 + 1$	$2 \cdot 3 + 1$	$2 \cdot 7 + 1$				
a_n	1	$3 = 2^2 - 1$	$7 = 2^3 - 1$	$15 = 2^4 - 1$				$2^n - 1$ (guess)

So $a_1 = 1$, and $a_{n+1} = 2a_n + 1$. We show that $a_n = 2^n - 1$ for every positive integer n by induction.

(1) $a_1 = 1 = 2^1 - 1$. So the statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k \geq 1$, i.e., $a_k = 2^k - 1$. We must show the statement is true for $n = k + 1$. In fact,

$$a_{k+1} = 2a_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1.$$

By mathematical induction, proof is completed.

Example 10.2. Show that the sum of the first n terms of an arithmetic sequence,

$$a, a + d, a + 2d, \dots, a + (n - 1)d, \dots$$

is given by

$$s_n = na + \frac{n(n - 1)d}{2}, \quad \text{for } n \geq 1.$$

Example 10.3. Show that the sum of the first n terms of a geometric sequence,

$$a, ar, ar^2, \dots, ar^{n-1}, \dots \text{ where } r \neq 1,$$

is given by

$$s_n = \frac{a(1 - r^n)}{1 - r}.$$

Example 10.4. For any n positive numbers: $a_1, a_2, a_3, \dots, a_n$, show

$$\sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n} \leq \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}.$$