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by

L. Allison Jones
Department of Management Science
University of Miami, Coral Gables, FL 33124-8237
and
Charles W. Champ
Department of Mathematics and Computer Science
Georgia Southern University, Statesboro, GA 30460-8093

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L. Allison Jones*

University of Miami, Coral Gables, FL 33124-8237

Charles W. Champ†

Georgia Southern University, Statesboro, GA 30460-8093

Abstract

A count of the number of defects is often used to monitor the quality of a production process. When defects rarely occur in a process, it is often desirable to monitor the time between the occurrence of each defect rather than a count of the number of defects. An exponential distribution often provides a useful model of the time between defects. Phase I control charts for exponentially distributed processes are discussed. Methods for computing the control limits are given and the overall Type I error rates of these charts are evaluated.

1 Introduction

The exponential distribution is frequently used to model the lifetime of a component or a system. For example, if the occurrence of a discrete event, say a defect, in a process is modeled by a homogeneous Poisson distribution, the time between the occurrence of each defect is exponentially distributed. (Note that the terms *defect* and *nonconformity* will be used interchangeably throughout this paper.) When the defect rate in a process is low, substantial time may pass before a nonconformity is observed. Although this is desirable in terms of process quality, a long stretch of time may be required to gain information for further

*Dr. Jones is an Assistant Professor in the Department of Management Science. She is a Member of ASQ.

†Dr. Champ is a Professor in the Department of Mathematics and Computer Science. He is a Member of ASQ.

improvement activities. In this situation, Montgomery (2001, pp. 325-328) recommends monitoring the time between the occurrences of each defect. Several authors have discussed methods for monitoring the average time between defects. Lucas (1985) and Vardeman and Ray (1985) discuss a CUSUM control chart and Gan (1989, 1998) examined the EWMA chart.

Other authors have discussed control charts for monitoring the mean of an exponential distribution. Yegulalp (1975) developed control charts for monitoring exponentially distributed product life processes. Nelson (1994) suggests transforming the exponential random variables to symmetric Weibull random variables and monitoring the transformed variables with a traditional Shewhart-type control chart. Transforming the exponential data to approximately normal data for control charting purposes was also investigated by Kittlitz (1999). Each of these methods assumes that the in-control process parameter is either known or estimated from an in-control sample.

This paper addresses the early stage process improvement activities when the quality characteristic of interest is the mean time between nonconformities that occur according a homogeneous Poisson process. In the initial stages of process improvement, a control chart is used to define the state of statistical control. Hillier (1969) and Yang and Hillier (1970) refer to the initial stage of control charting as Stage I process improvement. Once statistical control is attained, they refer to the prospective monitoring of future observations as Stage II process improvement. Alt and Smith (1988) refer to a similar approach respectively as Phase I and Phase II control charting. For the remainder of this paper, the terms Phase I and Phase II are used to refer to retrospective and prospective control charts. Our discussion is limited to the development of Phase I control charts when independent samples of size $n = 1$ are periodically taken on the quality measurement which is assumed to follow an exponential distribution.

Two scenarios may occur in Phase I control charting. The process parameter may be given as a target, or it may be unknown. The latter, of course, is more likely in practice. Note that we refer only to the process mean due to the relationship between the mean and variance of an exponential random variable. If the process mean is known, a Phase I control chart can be used to determine if the process is in a state of statistical control. The Phase I chart is distinguished from the Phase II chart in the parameter known case by the choice of the false alarm probability. If control limits are chosen such that the false alarm rate is α and an initial sample of m independent observations is taken from an in-control process, the probability that at least one false signal occurs is $1 - (1 - \alpha)^m$. We refer to α as the individual false alarm rate, and the probability of at least one false alarm on a Phase I chart as the overall false alarm rate. Suppose the control limits are set such that $\alpha = 0.0027$, and $m = 50$. The probability of at least one false signal from an in-control process is 0.13 using this Phase I control chart. When the cost of investigating signals is high, this probability of a false alarm may be unacceptable. It may be necessary to adjust the individual false alarm rate used to determine the control limits in order to provide a reasonable overall false alarm rate.

If the process mean is unknown, the uses of a Phase I control chart are twofold. The chart is used simultaneously to estimate the mean, and to determine if the process is in-control. The traditional approach replaces the unknown mean with an unbiased estimate when computing the control limits. These limits are then used to judge if the process observations are in-control. If one or more points fall outside of the limits, these observations may be investigated and perhaps discarded. If so, the mean is recomputed, and the cycle is repeated. In retrospective Phase I control charting, this approach will inflate the overall false alarm rate not only because of the iterative nature of the method, but also because the control limits are dependent on the process observations. If one thinks of a Phase I control chart for m observations as a sequence of m hypothesis tests for the mean, the hypothesis tests are dependent when the mean is estimated from the sample. In order to achieve a fixed overall false alarm probability, the control limits must be based on the joint distribution of the control charting statistics.

Throughout this paper, a continuous production process with a constant production rate is assumed. It is further assumed that defects occur according to a homogeneous Poisson process. Under these assumptions, the times between defects are independent and identically distributed according to an exponential distribution. Let T_i denote the time of the occurrence of the i^{th} defect. The time between the $(i-1)^{th}$ and the i^{th} defect is expressed by $X_i = T_i - T_{i-1}$, $i = 1, 2, ..., m$. The times between defects, $X_1, ..., X_m$, are independent and follow an exponential distribution with means equal to $\mu_1, ..., \mu_m$, respectively. If at time i the process is in-control, then $\mu_i = \mu_0$. At some time, j, an assignable cause of variation may result in a value of μ_j that differs from μ_0. Since the variable of interest is the time between defects, a value of μ_j that is less than μ_0 implies that the process quality has deteriorated. Similarly, a value of μ_j that is greater than μ_0 implies that the process quality has improved.

This paper begins with a discussion of Phase I control charts for monitoring exponential random variables when the mean is known. The idea of an unbiased Phase I control chart is developed and a design procedure is recommended for unbiased Phase I charts when the mean is known. Next, the design of an exact one-sided

Phase I control chart for exponential observations is developed. Recommendations are made for designing approximate two-sided Phase I charts for exponential observations. Finally, the performance of each control chart for several out-of-control scenarios is discussed.

2 Exact Limits When the Process Mean is Given as a Target

When a standard μ_0 is given for μ, the center line and control limits for the Phase I chart are given by

$$\begin{aligned} LCL &= E[X_i|\mu_0] - k_L^* \sqrt{V[X_i|\mu_0]} = \mu_0 - k_L^* \mu_0 = k_L \mu_0 \\ CL &= \mu_0 \\ UCL &= E[X_i|\mu_0] + k_U^* \sqrt{V[X_i|\mu_0]} = \mu_0 + k_U^* \mu_0 = k_U \mu_0 \end{aligned}$$

where $0 < k_L = 1 - k_L^* < k_U = 1 + k_U^*$, $i = 1, ..., m$. It is important to set the LCL to a positive value since process deterioration is detected for small values of X_i.

Define α_0 to be the overall false alarm rate. For a given value of α_0, the values k_L and k_U are chosen such that

$$P[\bigcap_{i=1}^{m} \{k_L \mu_0 < X_i < k_U \mu_0\} | \mu_i = \mu_0] = 1 - \alpha_0. \tag{1}$$

When $0 < k_L < k_U < \infty$, there are infinite choices for k_L and k_U that will satisfy (1). One obvious solution may be to choose k_L and k_U such that

$$P(X_i < k_L \mu_0 | \mu_i = \mu_0) = P(X_i > k_U \mu_0 | \mu_i = \mu_0)$$

for all $i = 1, 2, ..., m$. This, however, may present an undesirable situation if $\mu_i \neq \mu_0$ for at least one i. If at least one $\mu_i \neq \mu_0$, some choices of k_L and k_U may allow that

$$P[\bigcap_{i=1}^{m} \{k_L \mu_0 < X_i < k_U \mu_0\} | \text{ at least one } \mu_i \neq \mu_0] \geq 1 - \alpha_0.$$

In other words, if at least one point is out-of-control, the probability of detecting a shift may be less than the overall false alarm rate. This is clearly an undesirable characteristic for a control chart. A similar situation occurs in two-tailed tests of statistical hypotheses when the distribution of the test statistic is asymmetric. A hypothesis test is referred to a biased hypothesis test if the power of the test when the null hypothesis is false is smaller than the Type I error probability of the test. Likewise, an unbiased hypothesis test is one in which the power of the test when the null hypothesis is false is at least as large as the Type I error probability. For details regarding unbiased hypothesis tests, the interested reader is referred to Lehmann (1986, pp. 134-181).

Unbiased control charts are discussed in the literature. Krumbholz (1992) discusses unbiased control charts based on the range and Champ (2001) gives a method for obtaining an ARL unbiased Phase II R chart. Champ and Lowry (1994) give a method for obtaining ARL unbiased Phase II S chart. Acosta-Mejia (2000) discusses ARL unbiased charts for monitoring process dispersion.

An unbiased Phase I control chart can be determined by satisfying equation (1) and simultaneously restricting the choices of k_L and k_U so that

$$P[\bigcap_{i=1}^{m}\{k_L\mu_0 < X_i < k_U\mu_0\}| \text{ at least one } \mu_i \neq \mu_0] \ \leq 1-\alpha_0 \tag{2}$$

for all $i = 1, 2, ..., m$. Stated another way, the constants k_L and k_U are chosen so that the probability of detecting a shift in an out-of-control process is always larger than the probability of falsely detecting a shift in an in-control process.

To determine the values $k_L\mu_0$ and $k_U\mu_0$ that satisfy these two conditions, we let $k_L\mu_0$ and $k_U\mu_0$ be the τth and the $(1-\alpha+\tau)$th percentage points, respectively, of an exponential distribution with mean μ_0 where $\alpha = 1-(1-\alpha_0)^{1/m}$ and $0 < \tau < \alpha$. It is easy to show that $k_L = -\ln(1-\tau)$ and $k_U = -\ln(\alpha-\tau)$. Now we define the function

$$\begin{aligned} g(\underline{\lambda}, \tau) &= P[\bigcap_{i=1}^{m}\{k_L\mu_0 < X_i < k_U\mu_0\}|\mu_1, \mu_2, ..., \mu_m] \\ &= \prod_{i=1}^{m}\int_{k_L\mu_0}^{k_U\mu_0}\frac{1}{\mu_i}e^{-x/\mu_i}dx = \prod_{i=1}^{m}\left[e^{\ln(1-\tau)/\lambda_i} - e^{\ln(\alpha-\tau)/\lambda_i}\right], \end{aligned} \tag{3}$$

where $\underline{\lambda}= [\lambda_1 \ \lambda_2 \ ... \ \lambda_m]^T$, and $\lambda_i = \mu_i/\mu_0$.

To determine the value of τ such that equations (1) and (2) are satisfied, we determine the value of τ such that $g(\underline{\lambda}, \tau)$ is maximized at $\lambda_i = 1$, for all $i = 1, 2, ..., m$. This value of τ is the solution to the equationx

$$\frac{\partial g(\underline{\lambda}, \tau)}{\partial \lambda_i}|_{\lambda_i=1} = 0. \tag{4}$$

Table 1: Exact unbiased exponential phase I control chart constants when the process mean is given as a target.

m		α_0 0.0010	0.0100	0.0250	0.0500	0.1000
	τ	0.000183	0.001796	0.004452	0.008872	0.017806
5	k_L	0.000183	0.001798	0.004462	0.008911	0.017966
	k_U	11.001945	8.458286	7.420900	6.619321	5.794045
	τ	0.000092	0.000906	0.002254	0.004509	0.009105
10	k_L	0.000092	0.000907	0.002257	0.004519	0.009147
	k_U	11.755929	9.228859	8.200346	7.406320	6.588874
	τ	0.000062	0.000607	0.001512	0.003029	0.006132
15	k_L	0.000062	0.000607	0.001513	0.003034	0.006151
	k_U	12.195222	9.676891	8.653097	7.863211	7.050337
	τ	0.000046	0.000457	0.001138	0.002283	0.004628
20	k_L	0.000046	0.000457	0.001139	0.002285	0.004639
	k_U	12.506182	9.993658	8.973006	8.185896	7.376166
	τ	0.000037	0.000366	0.000913	0.001833	0.003719
25	k_L	0.000037	0.000366	0.000914	0.001834	0.003726
	k_U	12.746989	10.238761	9.220430	8.435386	7.628012
	τ	0.000031	0.000306	0.000763	0.001531	0.003110
30	k_L	0.000031	0.000306	0.000763	0.001532	0.003114
	k_U	12.943496	10.438651	9.422144	8.638725	7.833222
	τ	0.000023	0.000230	0.000574	0.001153	0.002344
40	k_L	0.000023	0.000230	0.000574	0.001154	0.002346
	k_U	13.253135	10.753404	9.739645	8.958683	8.156022
	τ	0.000019	0.000184	0.000460	0.000925	0.001810
50	k_L	0.000019	0.000184	0.000460	0.000925	0.001883
	k_U	13.492959	10.997016	9.985290	9.206138	8.405591

Equation (4) is equivalent to

$$(1-\tau)\ln(1-\tau)-(\alpha-\tau)\ln(\alpha-\tau)=0. \tag{5}$$

The values of τ that are solutions to equation (5) for various values of α_0 and m are listed in Table 1. Also listed are the corresponding values of k_L and k_U.

3 Exact One-sided Limit When the Process Mean is Unknown

When using a Phase I control chart to monitor the time between defects in a process, a lower control limit is necessary to show a deterioration in quality. The lower control limit can be determined such that

$$P\left[\bigcap_{i=1}^{m}\{k\widehat{\mu}_0<X_i\}\,|\mu_i=\mu\right]=1-\alpha_0, \tag{6}$$

where $\widehat{\mu}_0=\frac{1}{m}\sum_{i=1}^{m}X_i$, μ is the unknown process mean, and α_0 is the overall false alarm rate. Because $\widehat{\mu}_0$ is a random variable, we can rewrite (6) as

$$P\left[\bigcap_{i=1}^{m}\{k<Y_i\}\,|\mu_i=\mu\right]=1-\alpha_0, \tag{7}$$

where $Y_i=X_i/\widehat{\mu}_0$. The joint probability distribution function of $Y_1,Y_2,...,Y_m$ is defined by

$$f(y_1,y_2,...,y_{m-1})=\frac{m!}{\left[\sum_{i=1}^{m}\left(\frac{y_i}{\mu_i}-\frac{y_i}{\mu_m}\right)+\frac{m}{\mu_m}\right]^m\prod_{i=1}^{m}\mu_i}, \tag{8}$$

when $0<y_1<m$, $0<y_2<m-y_1$, $0<y_3<m-y_1-y_2$, ..., and $0<y_{m-1}<m-y_1-y_2-...-y_{m-2}$, and takes on a value of 0 otherwise. Note that the joint distribution is only necessary for the first $m-1$ values of Y_i since Y_m can be rewritten as $m-Y_1-Y_2-...-Y_{m-1}$. The derivation of equation (8) appears in the Appendix. It is clear that when $\mu_i=\mu$ for all $i,j=1,2,...,m$, equation (8) simplifies to

$$f(y_1,y_2,...,y_{m-1})=\frac{m!}{m^m}, \tag{9}$$

when $0<y_1<m$, $0<y_2<m-y_1$, $0<y_3<m-y_1-y_2$, ..., and $0<y_{m-1}<m-y_1-y_2-...-y_{m-2}$, and takes on a value of 0 otherwise.

It is interesting to note that equation (9) is free from the unknown process mean, μ. This is a desirable property for the joint distribution of the statistics plotted on a Phase I control chart when the process mean is unknown. This property simply allows one to compute the control limits regardless of the unknown parameter. Provided the sample is drawn from populations with equal means, the chart behaves as if the process is in-control.

Using equation (9), equation (7) can be rewritten as

$$\int_{k}^{m-(m-1)k} \cdots \int_{k}^{-y_{m-3}-y_{m-4}-\ldots-y_1+m-2k} \int_{k}^{-y_{m-2}-y_{m-3}-\ldots-y_1+m-k} \frac{m!}{m^m} dy_{m-1} dy_{m-2} \ldots dy_2 dy_1 = 1-\alpha_0. \tag{10}$$

Equation (10), which is derived in the Appendix, simplifies to

$$k_L = 1-(1-\alpha_0)^{1/(m-1)}.$$

The control limit for the exact one-sided Phase I exponential chart is thus given by

$$LCL = \left[1-(1-\alpha_0)^{1/(m-1)}\right] \cdot \widehat{\mu}_0. \tag{11}$$

The center line or reference line of the chart is

$$CL = \widehat{\mu}_0.$$

4 Approximate Two-sided Limits When the Process Mean is Unknown

When monitoring the time between process defects, an especially large value of X_i denotes improvement in the process. Because detecting large values of X_i may give the practitioner insight into process improvement opportunities, one may wish to construct a two-sided Phase I control chart. When the process mean is estimated by $\widehat{\mu}_0 = \frac{1}{m}\sum_{i=1}^{m} X_i$, a general expression for exact control limits is difficult to attain. In this section, a method for attaining approximate control limits is discussed.

When the process is in-control, i.e., each of the m points is sampled from an identical exponential distribution, the random variable $Y_i = X_i/\widehat{\mu}_0$ is related to a well-known univariate probability distribution by the following expression:

$$Y_i = \frac{m}{1+(m-1)F_i}. \tag{12}$$

Here F_i follows an F-distribution with numerator and denominator degrees of freedom respectively equal to $2(m-1)$ and 2. Using equation (12), which is derived in the Appendix, it is easy to show that

$$P\left(\frac{m\widehat{\mu}_0}{1+(m-1)\,F_{2(m-1),2,1-\alpha+\tau}} < X_i < \frac{m\widehat{\mu}_0}{1+(m-1)\,F_{2(m-1),2,\tau}}\right) = 1-\alpha,$$

where $F_{2(m-1),2,\tau}$ and $F_{2(m-1),2,1-\alpha+\tau}$ are the τ^{th} and $(1-\alpha+\tau)^{th}$ percentage points, respectively, of the F-distribution with $2(m-1)$ and 2 degrees of freedom, and $0<\tau<\alpha$. Using Boole's well-known inequality it is shown in the Appendix that

$$P\left[\bigcap_{i=1}^{m}\left(\frac{m\widehat{\mu}_0}{1+(m-1)\,F_{2(m-1),2,1-\alpha+\tau}} < X_i < \frac{m\widehat{\mu}_0}{1+(m-1)\,F_{2(m-1),2,\tau}}\right)\right] \geq 1-\alpha_0, \tag{13}$$

when $\alpha=\alpha_0/m$. A two sided Phase I control chart can be defined so that the overall false alarm rate is at most α. The control limits for this chart are

$$LCL = \frac{m\widehat{\mu}_0}{1+(m-1)F_{2(m-1),2,1-\alpha/m+\tau}} \text{ and } UCL = \frac{m\widehat{\mu}_0}{1+(m-1)F_{2(m-1),2,\tau}}, \tag{14}$$

where $0<\tau<\alpha/m$. The center line is $CL=\widehat{\mu}_0$.

5 The Performance of Phase I Control Charts

The out-of-control performance of Phase I control charts can be evaluated in several ways. In the parameter known case, suppose that n of the m sampled points are from an exponential distribution with mean $\mu_0+c\mu_0$, and the remaining $m-n$ points originate from an exponential distribution with mean μ_0. The probability of at least one signal on the Phase I chart can be found by restating equation (3) as

$$\beta = 1-(1-\alpha)^{m-n}\cdot\left[e^{(1+c)\ln(1-\tau)} - e^{(1+c)\ln(\alpha-\tau)}\right]^n. \tag{15}$$

The out-of-control situations in Phase I control charts can be further generalized, but due to the multitude of cases possible, we will only consider this scenario. Solutions to equation (15) for several values of α_0, n, c, and samples of size $m=30$ appear in Table 2. Values are only given for $c<0$ since a small value of X_i is assumed to indicate process deterioration.

Table 2: Out-of-control performance of the exact unbiased phase I exponential control chart when the process mean is given as a target.

		c			
α_0	**n**	**−0.25**	**−0.5**	**−0.75**	**−0.99**
	1	0.0103	0.0151	0.0824	0.9999
	5	0.0114	0.0354	0.3227	1.0000
	10	0.0129	0.0603	0.5367	1.0000
0.0100	**15**	0.0143	0.0844	0.6831	1.0000
	20	0.0157	0.1080	0.7831	1.0000
	25	0.0171	0.1310	0.8516	1.0000
	30	0.0186	0.1534	0.8985	1.0000
	1	0.0509	0.0618	0.1586	0.9213
	5	0.0546	0.1074	0.4821	0.9999
	10	0.0593	0.1614	0.7177	1.0000
0.0500	**15**	0.0638	0.2121	0.8461	1.0000
	20	0.0684	0.2597	0.9161	1.0000
	25	0.0730	0.3045	0.9543	1.0000
	30	0.0775	0.3465	0.9751	1.0000
	1	0.1015	0.1162	0.2249	0.9320
	5	0.1074	0.1782	0.5737	0.9999
	10	0.1147	0.2496	0.7981	1.0000
0.1000	**15**	0.1219	0.3147	0.9043	1.0000
	20	0.1291	0.3743	0.9547	1.0000
	25	0.1362	0.4286	0.9785	1.0000
	30	0.1432	0.4783	0.9898	1.0000

As with traditional control charting, the sensitivity of the Phase I chart is inversely related to the overall false alarm rate. For example, when $n = 5$ points are from exponential distributions with means that differ from the remaining $m - n = 25$ points by a factor of $(1 + c) = 0.5$, the probability that the chart gives at least one signal is only $\beta = 0.035$ when the overall false alarm probability is $\alpha_0 = 0.01$. Under the same conditions, however, the chart signals with a probability of $\beta = 0.1782$ when the overall false alarm rate is increased to $\alpha_0 = 0.1$. Due to the joint consideration of m points, the overall false alarm rate should not be set too low or the chart may not achieve the desired level of sensitivity.

When the process mean is estimated from the sample, a Phase I control chart should designate the majority of points that are from distributions with the same mean as in-control points. Conversely the minority of points from distributions with a different mean should be designated as out-of-control points. As discussed in section 3, the joint distribution of the m statistics does not depend on the unknown parameter, and performance of the Phase I chart is the same provided all original sample points are identically distributed. If, however, n of those m points come from a different distribution, the control chart should signal with a reasonably high probability.

Simulation studies are used to evaluate the out-of-control performance of both the exact one-sided and the approximate two-sided Phase I charts for exponential data. For each chart, it is assumed that n of the m sampled points are from an exponential distribution with mean $\mu_0 + c\mu_0$, and the remaining $m - n$ points originate from an exponential distribution with mean μ_0. In the case of the exact one-sided chart, a sample is generated, and the control limit is constructed using equation (11). If at least one point signals out-of-control, a counter is incremented. A new sample is then generated, and the simulation is repeated many times. The proportion of charts with at least one out-of-control point, $\widehat{\beta}$, is computed. The simulation size is varied in order to obtain standard errors of $\widehat{\beta}$ that are at most 1% of the expected overall false alarm probability. A similar simulation is conducted for the approximate two-sided chart constructed according to equation (14). Values of $\widehat{\beta}$ for several values of α_0, n, c, and samples of size $m = 30$ appear in Table 3 for the exact one-sided control chart and Table 4 for the approximate two-sided control chart. Once again, values are only given for $c < 0$ since a small value of X_i indicates process deterioration.

By looking at the simulation results in Table 4, it is interesting to note that the approximate control limits constructed with an upper bound on the overall false alarm rate achieved an empirical false alarm rate very close to α_0. For instance, when the upper bound, α_0 is set to 0.01, and all $m = 30$ points are identically distributed, the simulated false alarm probability is 0.0099. This is a strong justification for the use of the approximate limits when a two-sided Phase I chart is desired.

Table 3: Out-of-control performance of the exact one-sided phase I exponential control chart when the process mean is estimated from the sample.

		c			
α_0	**n**	**−0.25**	**−0.5**	**−0.75**	**−0.99**
0.0100	**1**	0.0100	0.0100	0.0105	0.0417
	5	0.0102	0.0108	0.0133	0.1411
	10	0.0103	0.0113	0.0152	0.2095
	15	0.0103	0.0113	0.0158	0.2304
0.0500	**1**	0.0503	0.0509	0.0537	0.1958
	5	0.0505	0.0539	0.0662	0.5332
	10	0.0505	0.0549	0.0740	0.6910
	15	0.0510	0.0559	0.0780	0.7259
0.1000	**1**	0.1009	0.1023	0.1077	0.3608
	5	0.1011	0.1069	0.1300	0.7861
	10	0.1023	0.1108	0.1484	0.9045
	15	0.1023	0.1119	0.1529	0.9219

Table 4: Out-of-control performance of the approximate two-sided phase I exponential control chart when the process mean is estimated from the sample.

		c			
α_0	**n**	**−0.25**	**−0.5**	**−0.75**	**−0.99**
	0	0.0099	0.0099	0.0099	0.0099
	1	0.0100	0.0105	0.0112	0.0273
0.0100	**5**	0.0108	0.0135	0.0186	0.0897
	10	0.0118	0.0188	0.0363	0.1675
	15	0.0125	0.0253	0.0712	0.3069
	0	0.0485	0.0485	0.0485	0.0485
	1	0.0491	0.0537	0.0537	0.1283
0.0500	**5**	0.0524	0.0621	0.0812	0.3614
	10	0.0539	0.0768	0.1293	0.5408
	15	0.0578	0.0954	0.2087	0.7049
	0	0.0944	0.0944	0.0944	0.0944
	1	0.0955	0.0987	0.1044	0.2383
0.1000	**5**	0.1016	0.1181	0.1497	0.5822
	10	0.1054	0.1425	0.2239	0.7692
	15	0.1083	0.1672	0.3220	0.8805

6 An Example

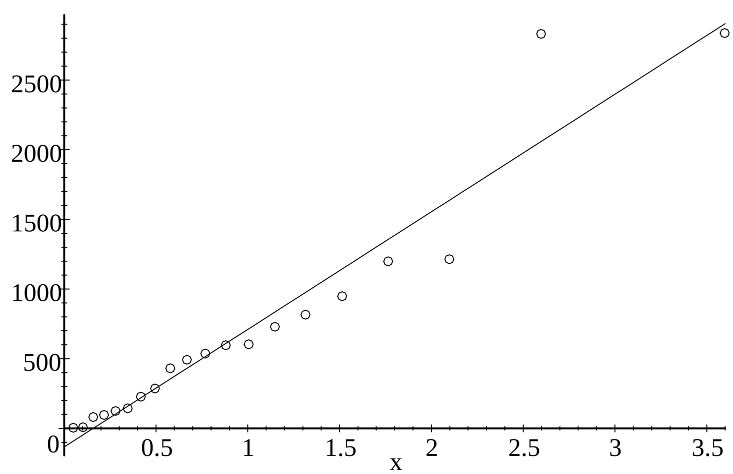
Montgomery (2001, pp. 326) gives an example in which a chemical engineer wishes to control the average time between failures of an important value. She has observed twenty times between failures for this value. These data are given in Table 5.

Table 5: Times between failure data.

Failure:	1	2	3	4	5	6	7	8	9	10
Time:	286	948	536	124	816	729	4	143	431	8
Failure:	11	12	13	14	15	16	17	18	19	20
Time:	2837	596	81	227	603	492	1199	1214	2831	96

An exponential probability plot of these data, see Figure 1, provides no evidence against the data being exponentially distributed.

Figure 1: Exponential probability plot of times between failures.



A Phase I control chart based on these times between events (failures) data has center line and control limits given by

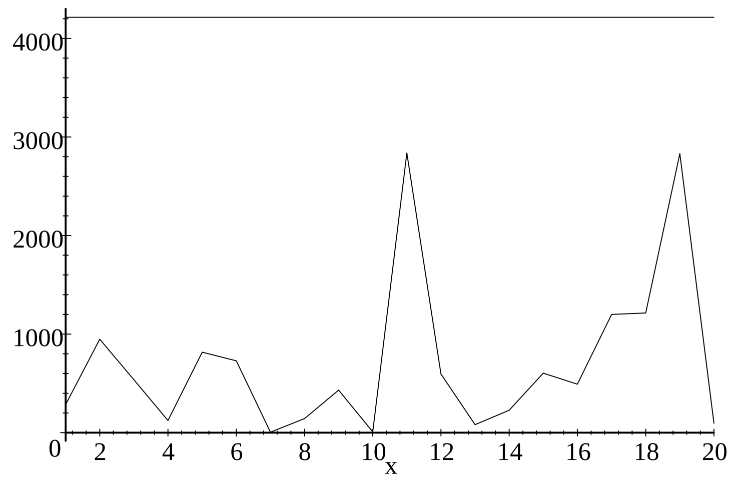
$$UCL = \frac{m\overline{x}}{1 + (m-1)\,F_{2(m-1),2,1-\alpha/(2m)}} = \frac{(20)\,(710.25)}{1 + (19)\,(0.1248)} = 4213.63$$

$$CL = \overline{x} = 710.25$$

$$LCL = \frac{m\overline{x}}{1 + (m-1)\,F_{2(m-1),2,\alpha/(2m)}} = \frac{(20)\,(710.25)}{1 + (19)\,(799.48)} = 0.9351,$$

where $m = 20$ and $\alpha = 0.05$. This Phase I control chart is given in Figure 2.

Figure 2: Phase I control chart for times between failures.



There are no values plotting outside the control limits and the data appear to be random. This would suggest to the practitioner that she can move to Phase II and begin monitoring the process using these data to estimate the in-control value of the mean.

7 Concluding Remarks

Phase I control charts for exponential processes have been discussed for the cases of the mean known and unknown. In the early stages of process control when the process parameter is known from historical data, a Phase I control chart can be used to determine if the existing process is in a state of statistical control. Methods have been given to design a Phase I control chart with a fixed overall false alarm probability in this case. Furthermore, unbiased Phase I control charts are defined, and unbiased control limits are recommended.

When the process parameter is unknown, a Phase I control chart is used to simultaneously estimate the process parameter and judge if the process is in-control. When the parameter is estimated from the data, the control charting statistics are dependent, and thus a joint determination must be made to give a bounded overall false alarm probability. In the case of a one-sided Phase I chart, a design method is recommended that gives an exact overall false alarm probability. For a two-sided Phase I chart, control limits are given that achieve a false alarm probability less than or equal to an upper bound.

A final point worth mentioning is that there are two methods found in the literature for constructing Phase I Shewhart-type control charts when the process parameters must be estimated from the data. In one method, each statistic of interest is plotted against the same center line and control limits. The center line and control limits are based on parameter estimates computed from all preliminary sample values. Chou (1994) and Chou and Champ (1995) refer to this procedure as the standard limits Phase I Shewhart control chart. A slightly different Phase I Shewhart-type control chart is suggested by Yang and Hillier (1970). In this method, each value of the statistic of interest is plotted against different center lines and sets of control limits. At some time, i, the center line and control limits are computed from the $(m-1)$ preliminary observations excluding the observation at time i. This method is referred to as the individual limits Phase I Shewhart control chart. In the case of exponentially distributed process observations, it is straightforward to show that both the standard limits and individuals limits Phase I Shewhart control charts are equivalent to the approximate two-sided chart developed in Section 4.

8 Appendix

Derivation of Equation (8)

Let $X_1, X_2, ..., X_m$ be independent random variables from $f(x_i) = \frac{1}{\mu_i} e^{-x_i/\mu_i}$. The joint probability distribution function is given by

$$f(x_1, x_2, ..., x_m) = \frac{1}{\prod_{i=1}^{m} \mu_i} e^{-\sum_{i=1}^{m}(x_i/\mu_i)} \prod_{i=1}^{m} I_{(0,\infty)} x_i.$$

Let $Y_i = mX_i / \sum_{i=1}^{m} X_i$ for $i = 1, 2, ..., m-1$,and let $W = \sum_{i=1}^{m} X_i$. It is clear that $X_i = WY_i/m$ for $i = 1, 2, ..., m-1$, and $X_m = W - W \sum_{i=1}^{m-1} Y_i/m$. Let

$$J = \begin{vmatrix} \frac{W}{m} & 0 & \dots & 0 & \frac{Y_1}{m} \\ 0 & \frac{W}{m} & \dots & 0 & \frac{Y_2}{m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{W}{m} & \frac{Y_{m-1}}{m} \\ -\frac{W}{m} & -\frac{W}{m} & \dots & -\frac{W}{m} & 1 - \sum_{i=1}^{m-1} \frac{Y_i}{m} \end{vmatrix} = \frac{W^{m-1}}{m^{m-1}}.$$

The joint probability distribution function of $Y_1, Y_2, ..., Y_{m-1}, W$ is

$$f(y_1, y_2, ..., y_{m-1}, w) = \frac{w^{m-1}}{m^{m-1} \prod_{i=1}^{m} \mu_i} e^{-(w/m)\left(\sum_{i=1}^{m-1}(y_i/\mu_i - y_i/\mu_m) + m/\mu_m\right)}$$

when $0 < W < \infty$, $0 < Y_1 < m$, $0 < Y_2 < m - Y_1$, $0 < Y_3 < m - Y_1 - Y_2$,...,and $0 < Y_{m-1} < m - Y_1 - Y_2 - ... - Y_{m-2}$, and takes on a value of 0 otherwise. In order to simplify the expression, it is restated as

$$\left(m^{m-1} \prod_{i=1}^{m} \mu_i\right) f(y_1, y_2, ..., y_{m-1}, w) = w^{m-1} e^{-w\frac{c}{m}}.$$

$$\begin{aligned} \int_0^\infty w^{m-1} e^{-cw/m} dw &= -\frac{m}{c} w^{m-1} e^{-cw/m} - \frac{m^2(m-1)}{c^2} w^{m-2} e^{-2w/m} - ... - \frac{m^m(m-1)(m-2)...1}{c^m} e^{-cw/m} \Big]_0^\infty \\ &= \frac{m^{m-1} m!}{c^m}. \end{aligned}$$

It is clear that

$$f(y_1, y_2, ..., y_{m-1}) = \frac{m!}{\left[\sum_{i=1}^{m-1}\left(\frac{y_i}{\mu_i} - \frac{y_i}{\mu_m}\right) + \frac{m}{\mu_m}\right]^m \prod_{i=1}^{m} \mu_i},$$

when $0 < Y_1 < m$, $0 < Y_2 < m - Y_1$, $0 < Y_3 < m - Y_1 - Y_2$,...,and $0 < Y_{m-1} < m - Y_1 - Y_2 - ... - Y_{m-2}$, and takes on a value of 0 otherwise.

Derivation of Equation (10)

We wish to find k_L such that

$$P\left[\bigcap_{i=1}^{m}\{k_L < Y_i\}|\mu_i = \mu_j, \forall i, j = 1, 2, ..., m\right] = 1 - \alpha_0.$$

This is equivalent to finding k_L such that

$$P\left[\left\{\bigcap_{i=1}^{m-1}\{k_L < Y_i\}\right\}\bigcap\left\{m - k_L - \sum_{i=1}^{m-2} Y_i > Y_{m-1}\right\}|\mu_i = \mu_j, \forall i, j = 1, 2, ..., m\right] = 1 - \alpha_0.$$

Integrating $f(y_1, y_2, ..., y_{m-1})$ with respect to $y_1, y_2, ..., y_{m-1}$ over the appropriate regions should provide the correct value of k_L. We begin by integrating with respect to y_{m-1} over the region $\left(k_L, m - k_L - \sum_{i=1}^{m-2} y_i\right)$. The $m-1$ dimensional hyperplane $y_{m-1} = m - k_L - \sum_{i=1}^{m-2} y_i$ intersects the hyperplane $y_{m-1} = k_L$ in the $m-2$ dimensional hyperplane $y_{m-2} = m - 2k_L - \sum_{i=1}^{m-3} y_i$. Thus we will next integrate with respect to y_{m-2} over the region $\left(k_L, m - 2k_L - \sum_{i=1}^{m-3} y_i\right)$. Again, the $m-2$ dimensional hyperplane $y_{m-2} = m - 2k_L - \sum_{i=1}^{m-3} y_i$ intersects the hyperplane $y_{m-2} = k_L$ in the $m-3$ dimensional hyperplane $y_{m-3} = m - 3k_L - \sum_{i=1}^{m-4} y_i$, and the next step is to integrate with respect to y_{m-3} over the region $\left(k_L, m - 3k_L - \sum_{i=1}^{m-4} y_i\right)$. The process continues similarly until we integrate with respect to y_1 over the region $(k_L, m - (m-1)k_L)$. Thus,

$$P\left[\bigcap_{i=1}^{m}\{k_L < Y_i\}|\mu_i = \mu_j, \forall i, j = 1, 2, ..., m\right]$$

is equivalent to

$$\int_{k_L}^{m-(m-1)k_L} ... \int_{k_L}^{-y_{m-3}-y_{m-4}-...-y_1+m-2k_L} \int_{k_L}^{-y_{m-2}-y_{m-3}-...-y_1+m-k_L} \frac{m!}{m^m} dy_{m-1} dy_{m-2} ... dy_2 dy_1.$$

Derivation of Equation (12)

Let $X_1, X_2, ..., X_m$ be i.i.d. random variables from $f(x) = \frac{1}{\mu}e^{-x/\mu}$. The joint probability distribution function is given by

$$f(x_1, x_2, ..., x_m) = \frac{1}{\mu^m} e^{-1\left(\sum_{i=1}^{m} x_i\right)/\mu} \prod_{i=1}^{m} I_{(0,\infty)} x_i.$$

Let $Y_i = X_i/\widehat{\mu}_0$, where $\widehat{\mu}_0 = \frac{1}{m}\sum_{i=1}^{m} X_i$.

$$\begin{aligned} Y_i &= \frac{mX_i}{X_i + \left(\sum_{j=1}^{m} X_j - X_i\right)} = \frac{m}{1 + \left(\sum_{j=1}^{m} X_j - X_i\right)/X_i} \\ &= \frac{m}{1 + (m-1)\left(\frac{2\left(\sum_{j=1}^{m} X_j - X_i\right)/(2(m-1)\mu_0)}{2X_i/2\mu_0}\right)} = \frac{m}{1 + (m-1)\frac{\chi^2_{2(m-1)}/(2(m-1))}{\chi^2_2/2}} \\ &= \frac{m}{1 + (m-1)F_{2(m-1),2}}. \end{aligned}$$

Derivation of Equation (13)

Let A_i be the event that $\left(\frac{m\widehat{\mu}_0}{1+(m-1)F_{1-\alpha+\tau}} < X_i < \frac{m\widehat{\mu}_0}{1+(m-1)F_\tau}\right)$, and $P(A_i) = \alpha$ for $i = 1, 2, ..., m$. Also let $\alpha_0 = m\alpha$.

$$P\left(A_1 \bigcup A_2 \bigcup ... \bigcup A_m\right) < P(A_1) + P(A_2) + ... + P(A_m)$$

which directly implies

$$P\left(A_1 \bigcap A_2 \bigcap ... \bigcap A_m\right) \geq 1 - \alpha_0$$

and

$$P\left[\bigcap_{i=1}^{m}\left(\frac{m\widehat{\mu}_0}{1 + (m-1)\,F_{1-\alpha+\tau}} < X_i < \frac{m\widehat{\mu}_0}{1 + (m-1)\,F_\tau}\right)\right] \geq 1 - \alpha_0.$$

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