DEPARTMENT OF MATHEMATICS
AND
COMPUTER SCIENCE

TECHNICAL REPORT SERIES

Computational Origami: Reexamining an Old Problem

by

Fleming R. Gibson and Jacalyn M. Huband
Department of Mathematics and Computer Science
Georgia Southern University, Statesboro, GA 30460-8093

Number 2002-010
Submitted: December 3, 2002

Computational Origami: Reexamining an Old Problem

Fleming R. Gibson

Georgia Southern University, Statesboro, GA 30460-9294

Jacalyn M. Huband

Georgia Southern University, Statesboro, GA 30460-8093

Abstract

This paper presents recent research that was conducted in the field of computational origami. First, it examines previous work done by Kan Chu Sen in 1721 on the idea of a fold-and-cut problem. Second, the paper will look at Kawasaki's Theorem, which states that the sum of the alternating interior angles of a given figure is 180°. Then the sum of the fold pattern of a folded origami figure explained in Maekawa's Theorem will be explored. By combining these three ideas a new problem arises: given multiple folds that bisect the vertex and making a complete straight cut of the vertex, can the sum of the alternating fold-cut angle be determined by just the number of folds of the paper alone? This paper will present and prove a formula for solving this problem.

AMS Subject Classification: 51N05, 97A20

Keywords: Paper Folding, Fold-and-Cut Problem, Computational Origami

1. Introduction

The idea of computational origami originated in the early 1990's when Robert Lang first coined the term. He wrote an algorithm call TreeMaker based on a set of mathematical conditions on mapping between the crease pattern and a tree graph representing the base[12]. Robert Lang's program is considered the foundational work in this field. Since that time many mathematicians have done work to further this field, but Thomas Hull and Erik D. Demaine are the most notable. As this field of study grows many people have realized that older work completed by Kan Chu Sen (in 1721) and other mathematicians actually fall into today's definition of computational origami even though when their work was published, the field of study had not yet originated.

In 1721 Kan Chu Sen, published a book called *Wakoku Chiyekurabe* (Mathematical Contest)[6] in which he first presented the idea of a fold-and-cut problem. Sen's idea was to create new origamic shapes by folding paper multiple times and then making one complete cut in the fold to produce a figure. Sen would then unfold the cut paper and look to see if the desired figure had been achieved. Sen's goal was to be able to make any figure by folding the paper in such a way that when one complete cut was made and unfolded, only the desired origamic figure remained in the non-removed section of the paper. This method is similar to when kids make paper dolls and snow flakes by folding and cutting paper. Houdini later used Sen's ideas in magic tricks around the turn of the 20th century. Houdini's paper folding tricks were often based on Sen's fold-and-cut problem.

In the late 20th century, Jun Maekawa studied flat folds in origami. A flat fold is defined as a mapping of the paper into another two-dimensional shape. This completed shape can be placed in a book and the book closed shut on the figure without the book changing the shape of the figure in any way by introducing new creases into the figure[9]. In 1983, Maekawa published a paper in which he proved that the difference between the mountain and valley folds in a flat vertex fold must always be 2. This is now called Maekawa's Theorem[7,8,9,10]. The two terms mountain and valley folds are completely dependent upon which side of the paper one is looking at. Mountain folds are convex creases in the paper, while valley folds are concave creases in the paper[9]. A crease is defined as a line segment that divides a polygon into regions[12].

Following Maekawa's work came the idea by Toshikazu Kawasaki in 1989, in which he proved that the sum of alternating interior angles of an origamic figure is 180°. This is now commonly referred to as Kawasaki's Theorem[1,7,8,9,10]. The alternating interior angles are the angles that are formed by the vertex. A vertex is a point formed by two or more folds such that all made edges meet. In other words a point at which all creases of the paper meet.

This paper will examines a new problem based on the works of Sen, Maekawa, and Kawasaki in determining if the sum of alternating fold-cut angles can be calculated by the number of folds alone?

This paper is organized in the following way. Section 2 discusses the problem and illustrates an example of the problem. Section 3 gives a statement based on the solution to the problem as well as a complete proof for this statement. Section 4 gives the statement and proof for a more general case of the specific problem.

2. The Problem

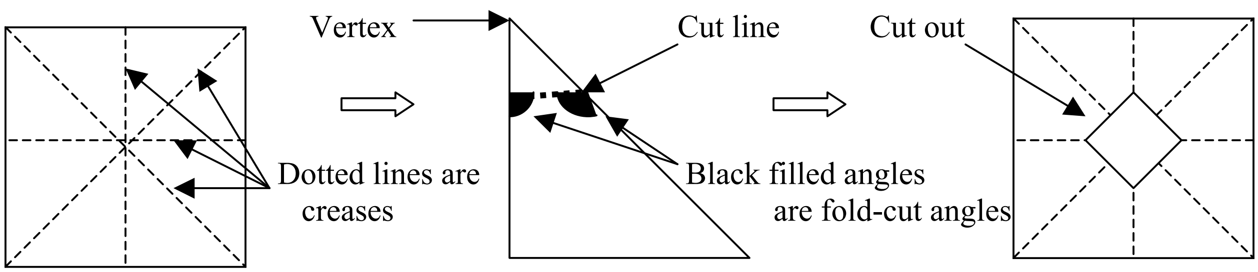
Consider making multiple flat-folds through a single vertex. The paper should be folded such that there are 2^n layers of paper. Then take from Sen's fold-and-cut problem the idea of cutting off the vertex. Examine the result from adding up all of the alternating fold-cut angles, similar to what Kawasaki did in his theorem by adding all of the alternating interior angles. Finally look at the number of folds and see if one can calculate the total degrees of the alternating fold-cut angles, this is similar to Maekawa's Theorem in that he only looked at the number of folds of the paper. Therefore the problem can be stated in the following:

1. Make multiple flat-folds through the vertex with each fold yielding 2^n layers of paper

2. Cut off the vertex of folded origamic figure at any angle

3. Determine the total number of degrees of the alternating fold-cut angles, by only looking at the number of times the paper has been folded to produce the origamic figure.

An example of the fold-cut problem is given in Figure 2.1.



(a) (b) (c)

Figure 2.1 An example of the problem. (a) The fold/crease pattern of the paper. (b) The flat vertex folded origamic figure. (c) The unfolded figure showing what part of the paper was removed by cutting off the vertex.

3. Solution and Proof

Statement 1: Let n be the number of times the paper was folded. If n flat-folds bisect a given vertex and a complete straight cut is made of the vertex then the sum of the alternating fold-cut angles is

$$(2^{n-1}+1)*180°$$

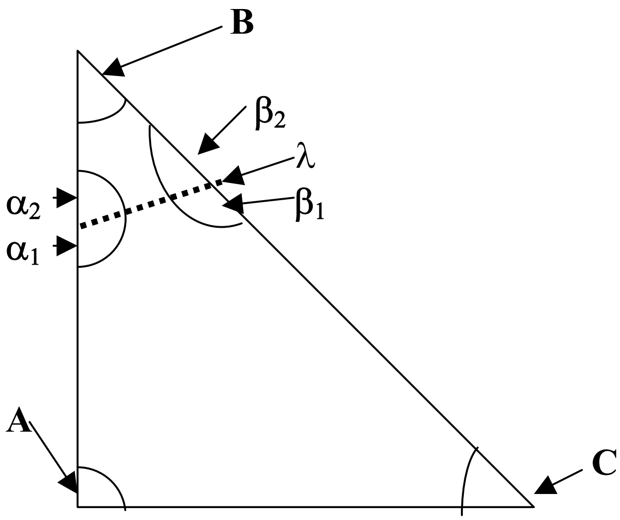


Figure 3.1 Diagram of triangle's angles

Proof:

Let ΔABC be a triangular flat fold formed by bisecting vertex B repeatedly. Let ℓ be an arbitrary cut that removes vertex B from the fold. From cut ℓ, angles α_2, B, and β_2 form a triangle and satisfy the equation

$$\alpha_2 + B + \beta_2 = 180° \tag{3.1}$$

Because α_1 and α_2, β_1 and β_2 are supplementary angles we known that

$$\alpha_2 = 180° - \alpha_1 \tag{3.2}$$

$$\beta_2 = 180° - \beta_1 \tag{3.3}$$

By substituting Equations 3.2 and 3.3 into Equation 3.1 we get the following:

$$(180° - \alpha_1) + (180° - \beta_1) + B = 180°$$
$$360° - \alpha_1 - \beta_1 + B = 180°$$

$$180° + B = \alpha_1 + \beta_1 \tag{3.4}$$

If n is the number of folds of the figure, then 2^n represents the number of regions/layers of the flat fold. Since B is bisected each time the paper is folded then for n-folds B is represented by

$$B = \frac{360°}{2^n} \tag{3.5}$$

By substituting Equation 3.5 into Equation 3.4, the following is obtained

$$\alpha_1 + \beta_1 = 180° + \frac{360°}{2^n} \tag{3.6}$$

Thus the sum of all fold-cut angles is given by

$$\sum_{i=1}^{2^n}(\alpha_1 + \beta_1) = \sum_{i=1}^{2^n}(180° + \frac{360°}{2^n}) \tag{3.7}$$

By the symmetry of angles across a crease, the sum of the alternating angles is half of the sum of the fold-cut angles:

$$\frac{1}{2}\sum_{i=1}^{2^n}(\alpha_1 + \beta_1) = \frac{1}{2}\sum_{i=1}^{2^n}(180° + \frac{360°}{2^n}) \tag{3.8}$$

Expanding the right-hand-side of Equation 3.8 yields

$$\sum_{i=1}^{2^n}(\frac{1}{2}(\alpha_1 + \beta_1) = \frac{1}{2}(180° * 2^n + 360°))$$

$$= \frac{1}{2}(2^n + 2) * 180°$$

$$= (2^{n-1} + 1) * 180°$$

Thus, obtaining the formula that was given in Statement 1.

4. Consideration of a More General Case

Given the same problem from the previous section is it possible to look at more general case in which vertex B does not have to be bisected. In other words if vertex B is not bisected, can one determine the sum of all of the fold-cut angles by only the number of times the paper is folded. The following lemma will be necessary for the more general case.

Lemma: Given an arbitrary number of flat folds at a vertex, but not necessarily bisecting the vertex, the sum of the alternating fold-cut angles is half the sum of all fold-cut angles.

Proof: Each fold will produce one or more creases (depending on the number of previous folds).

A cut at each crease will produce symmetric, adjacent fold-cut angles about the crease. By counting alternating angles, only one of each pair is counted.

Statement 2: Let n be the number of flat-folds at a given vertex and there should be 2^n layers/regions of paper in the flat-folded figure. If a complete straight cut of the vertex is made, then the sum of the alternating fold-cut angles is

$$(2^{n-1}+1)*180°$$

Proof:

Let n be the total number of flat folds. There are still 2^n layers/regions of paper in the flat-folded figure. Thus the center cut off piece has 2^n triangles ($\Delta\alpha_{2i}B_i\beta_{2i}$). Since all the angles of a triangles sum to 180° thus yielding Equation 4.1

$$2^n*180° = \sum_{i=1}^{2^n}(\alpha_{2i}+B_i+\beta_{2i}) \qquad (4.1)$$

Because $\sum_{i=1}^{2^n} B_i = 360°$ Equation 4.1 can be written as

$$2^n * 180° - \sum_{i=1}^{2^n} B_i = \sum_{i=1}^{2^n} (\alpha_{2i} + \beta_{2i})$$

$$2^n * 180° - 360° = \sum_{i=1}^{2^n} (\alpha_{2i} + \beta_{2i})$$

$$(2^n - 2) * 180° = \sum_{i=1}^{2^n} (\alpha_{2i} + \beta_{2i}) \tag{4.2}$$

Using the property of supplementary angles gives the following results

$$((\alpha_{1i} + \alpha_{2i}) + (\beta_{1i} + \beta_{2i})) = (180° + 180°)$$

$$\sum_{i=1}^{2^n} (\alpha_{1i} + \alpha_{2i} + \beta_{1i} + \beta_{2i}) = \sum_{i=1}^{2^n} 360°$$

$$\sum_{i=1}^{2^n} (\alpha_{1i} + \alpha_{2i} + \beta_{1i} + \beta_{2i}) = 360° * 2^n \tag{4.3}$$

By subtracting Equation 4.2 from Equation 4.3 produces the following:

$$\sum_{i=1}^{2^n} (\alpha_{1i} + \alpha_{2i} + \beta_{1i} + \beta_{2i}) - \sum_{i=1}^{2^n} (\alpha_{2i} + \beta_{2i}) = 360° * 2^n - (2^n - 2) * 180°$$

$$\sum_{i=1}^{2^n} (\alpha_{1i} + \beta_{1i}) = 2^n * 180° + 360°$$

$$\sum_{i=1}^{2^n} (\alpha_{1i} + \beta_{1i}) = (2^n + 2) * 180° \tag{4.4}$$

By the previous lemma, the sum of the alternating fold-cut angles is

$$\frac{1}{2} \sum_{i=1}^{2^n} (\alpha_{1i} + \beta_{1i}) = (2^{n-1} + 1) * 180° \tag{4.5}$$

Thus Equation 4.5 yields the formula given in Statement 2.

5. Conclusion

The most significant finding in this paper is that one can calculate the sum of the alternating fold-cut angles without knowing how the paper is cut or folded. Also note that the same formula was found regardless of whether one bisects the vertex on each fold or not. It is a convenient result, since all that is required is to determine any desired property is to look at the number of times the paper was folded. Future work that could be conducted in this area would be to consider cases where there is more than one vertex and determine if a formula for the sum of all alternating fold-cut angles can be calculated only by the number of folds. Another direction in which future research could be conducted is to determine whether a formula exists based on the paper not having to be folded with 2^n layers but instead being folded in a more general sense of 2n layers.

Acknowledgements: The writer wishes to thank Professor Jacalyn M. Huband for her countless hours she put into helping make this paper possible. Also the writer wishes to thank Professor Jimmy C. Woods of North Georgia College and State University for his technical advice on this paper.

References

[1] Anderson, Eric. *Origami & Math*. 31 Dec. 2001. 4 Mar 2002. <http://www.paperfolding.com/math/index.php>

[2] Bern, Marshall, Erik D. Demaine, David Eppstein, Eric Kuo, Andrea Mantler, and Jack Snoeyink. *Computational Geometry: Theory and Application.* Special issue of selected papers from the 1999 CGC Workshop on Computational Geometry.

[3] Cipra, Berry. "In the Fold: Origami Meets Mathematics." *SIAM News*. Oct. 2001.

[4] Demaine, Erik D., and Martin L. Demaine. “Resent Results in Computational Origami.” *The Proceedings of the Third International Meeting of Origami Science, Math and Education.* AK Peters, 2002. Monterey, California, Mar. 9-11, 2001.

[5] ---. “Computing Extreme Origami Bases.” Technical Report. CS-97-22, University of Waterloo, May 1997.

[6] Demaine, Erik D., Martin L. Demaine., and Anna Lubiw. “Folding and Cutting Paper.” In J. Akiyama, M. Kano, and M. Urabe, editors, *Revised Papers from the Japan Conference on Discrete and Computational Geometry,* Vol. 1763 of *Lecture Notes in Computer Science.* 104-17. Tokyo, Japan, Dec. 1998.

[7] Hull, Thomas. “On the Mathematics of Flat Origamis.” *Congressus Nimerantium*. Vol. 100. 1994. 215-24

[8] ---. *Tom’s Combinatorial Geometry Class.* 4 Mar. 2002. <http://www.web.merrimack.edu/~thull/combgeom/flatfold/flat.html>

[9] ---. “Counting Mountain-Valley Assignments for Flat Folds.” *Ars Combinatoria.* 2002.

[10] ---. “The Combinatorics of Flat Folds: A Survey.” *The Proceedings of the Third International Meeting of Origami Science, Math and Education.* AK Peters, 2002. Monterey, California, Mar. 9-11, 2001.

[11] Knowles, Josh and Sean Owen. *The Calculus of Origami*. 2 May 2001. 3 Feb. 2002. <http://community.corest.com/~gera/origami/index.html>

[12] Lang, Robert. “A Computational Algorithm for Origami Design.” *Proceeding of the 12th Annual Symposium on Computational Geometry*. 1996. 98-105.