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Fundamental Theory of Control of Systems Involving a Kronecker Product of Matrices

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Abstract

This paper presents a criterion for the equivalence relation of an n-th order nonhomogeneous differential equation to its companion vector equation, and then presents a necessary and sufficient condition for the equivalence of the general first-order nonhomogeneous equation to the scalar differential equation. Next a Kronecker product first-order system is formulated by embedding two companion vector equations of different dimensions, and the relationship between the scalar differential equations of different orders to their companion Kronecker product systems is obtained. A necessary and sufficient condition for their linear equivalence is presented. Finally a set of sufficient conditions is given for the controllability of the Kronecker product first-order system in terms of the fundamental matrix solutions. Several interesting examples will be presented that highlight the necessity of these investigations.

Keywords and phrases: Control theory, Kronecker product system, fundamental matrix solutions

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1 Introduction

Since it is a well-known fact that an n-th order scalar differential equation can always be recast in the form of a first-order system, and it is also a well-recognized fact that two scalar differential equations of different orders can always be embedded in the form of a first-order Kronecker product system, it is reasonable to investigate the fundamental control theory for Kronecker product companion vector equations and to present certain fundamental results on controllability of the system. More specifically, we consider two scalar non-homogeneous differential equations of different orders, say

(1.1) $$Lu := u^{(n)} + p_1 u^{(n-1)} + \cdots + p_n u = f(t)$$

and

(1.2) $$Lv := v^{(m)} + k_1 v^{(m-1)} + \cdots + k_m v = g(t)$$

where p_i $(i = 1, \ldots, n)$ and k_j $(j = 1, \ldots, m)$ are real constant coefficients, and where f and g are scalar functions defined on some interval I.

The companion vector equations associated with (1.1) and (1.2), respectively, are given by

$$y' = Ay + e_n f \tag{1.3}$$

and

$$z' = Bz + e_m g, \tag{1.4}$$

where $A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ p_n & -p_{n-1} & -p_{n-2} & \ldots & -p_1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ k_m & -k_{m-1} & -k_{m-2} & \ldots & -k_1 \end{bmatrix}$

are the respective nxn and mxm companion matrices, $y = \begin{bmatrix} y_1, & y_2, & \ldots, & y_n \end{bmatrix}^T$ and $z = \begin{bmatrix} z_1, & z_2, & \ldots, & z_m \end{bmatrix}^T$ are column matrices, and e_n and e_m are the standard basis vectors. There is an interesting relationship between the scalar differential equation of order n (1.1) and the companion vector equation (1.3) in the solution space. If $u_1, u_2, \ldots, u_n$ are any n linearly independent solutions of $Lu = 0$, then their Wronskian vectors form a fundamental matrix $K(u) = \begin{bmatrix} k(u_1), & k(u_2), & \ldots, & k(u_n) \end{bmatrix}$ of the companion vector equation (1.3), and conversely, if $K(u)$ is a fundamental matrix of the companion vector equation (1.3), then its first row forms a fundamental vector for $Lu = 0$. A similar result also holds for (1.2) and its companion vector system (1.4).

Before we present our basic results on controllability of the Kronecker product companion system, we present some of the relevant properties of the Kronecker product of two matrices. In fact, the Kronecker product is defined for any two matrices of arbitrary orders over any ring, although we are interested in the real field.

If $P \in R^{m \times n}$ and $Q \in R^{p \times q}$, then the Kronecker (or tensor) product of P and Q, which is denoted by $(P \otimes Q)$, is defined as

$$(P \otimes Q) = (p_{ij} Q) \in R^{mp \times nq} \ \ (i = 1, \ldots, m; j = 1, \ldots, n).$$

The Kronecker product as defined above has the following properties:

(*i*) $(P \otimes Q)(A \otimes B) = (PA \otimes QB)$ (provided PA and QB are defined);
(*ii*) $(P \otimes Q)^T = (P^T \otimes Q^T)$ (where P^T is the matrix transpose of P);
(*iii*) $(P \otimes Q)^{-1} = (P^{-1} \otimes Q^{-1})$ (provided P and Q are invertible matrices);
(*iv*) $\frac{d}{dt}(P \otimes Q) = (\frac{d}{dt}P \otimes Q) + (P \otimes \frac{d}{dt}Q)$.

The Kronecker product first-order system of differential equations is an interesting area of current research and has significant applications in systems engineering [1], method of lines [5], and mathematical biology [3]. The first result in the formulation of Kronecker product boundary value problems was presented by the authors in [2]. The importance of Kronecker products of matrices has gained attention because of its computational and notational advantages. The primary purpose of this paper is to introduce a set of ideas in mathematical control theory involving Kronecker products of matrices, and then to present an equivalence relation between the general Kronecker product first-order systems and the companion first-order Kronecker product systems. The paper is organized as follows. Section 2 presents the equivalence relation between (1.1) and its companion vector equation (1.3), and then presents a necessary and sufficient condition for the equivalence of the general system

$$x' = Mx + bf(t),$$

and the n-th order non-homogeneous scalar equation (1.1). In section 3, the Kronecker product vector companion system is formulated, and then it is shown by examples that it is not always possible to transform the given Kronecker product system into a companion Kronecker product system. A necessary and sufficient condition for linear equivalence is also presented in section 3. Controllability of the Kronecker product first-order system is discussed in section 4.

2 Basic Results on Linear Systems

We first consider the general first-order non-homogeneous equation

$$x' = Mx + bf(t), \tag{2.1}$$

where M is an nxn constant matrix and b is an nx1 vector, and show by examples that it is not always possible to transform the system in (2.1) into the companion system (1.3). Then we present a necessary and sufficient condition for the equivalence of these two systems. Let us consider the simple example

$$x' = Mx + bf(t),$$

where $M = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have

$$x_1' = -3x_1 + 3x_2 + f(t), \tag{2.2}$$

and

$$x_2' = x_1 - x_2. \tag{2.3}$$

Now we make a transformation $y = Tx$ where T is a 2×2 constant nonsingular matrix. Then

$$\begin{aligned} y' = Tx' &= TMx + Tbf(t) \\ &= (TMT^{-1})(Tx) + (Tb)f(t) \\ &= Ay + e_n f(t). \end{aligned} \tag{2.4}$$

The fundamental question that arises is the following. Will there always exist a constant non-singular matrix T such that $TMT^{-1} = A$ (the companion matrix) and $Tb = e_n = \begin{bmatrix} 0, & 0, & \ldots, & 1 \end{bmatrix}^T$? If so, is T unique? From (2.3), we have that

$$x_1 = x_2' + x_2.$$

Substituting this expression for x_1 in (2.2), we get

$$\begin{aligned} & x_2'' + x_2' = -3(x_2' + x_2) + 3x_2 + f(t), \\ \text{or} \quad & x_2'' = -4x_2' + f(t). \end{aligned}$$

Thus the companion vector equation associated with (2.1) is

$$y' = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} y + e_2 f(t).$$

In this example, there exists a 2×2 constant, nonsingular matrix $T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ that transforms (2.1) into the companion vector equation (2.4). Since the differential equation (1.1) is linear, we expect that the transformation $y = Tx$ is linear. On the other hand, if we consider

$$x' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x + bf(t), \tag{2.5}$$

where $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then

$$x_1' = 2x_1 + f(t)$$

and

$$x_2' = 2x_2.$$

The eigenvalues of the matrix M are 2 and 2, and we observe that there is no linear transformation $y = Tx$ such that $TMT^{-1} = A$ and $Tb = e_2$. Thus it is not always possible to transform a given general linear system (2.1) into the companion vector system (1.3). We therefore define the following equivalence relation.

Definition 1. The linear system (2.1) is *linearly equivalent* to the companion vector system (1.3) if there exists a constant $n \times n$ nonsingular matrix T such that

$$TMT^{-1} = A \quad \text{and} \quad Tb = e_n.$$

Definition 2. The vector x is a *cyclic vector* for the matrix M if the n vectors

$$\{x, \; Mx, \; M^2x, \; \ldots, \; M^{n-1}x\}$$

are linearly independent.

In (1.3) and (1.4), e_n is a cyclic vector for A and e_m is a cyclic vector for B. Explicitly, we can calculate that

$$\begin{bmatrix} e_n, & Ae_n, & A^2e_n, & \dots, & A^{n-2}e_n, & A^{n-1}e_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & * \\ 0 & 0 & 0 & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & * & \dots & * & * \\ 1 & * & * & \dots & * & * \end{bmatrix}$$

and

$$\begin{bmatrix} e_m, & Be_m, & B^2e_m, & \dots, & B^{m-2}e_m, & B^{m-1}e_m \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & * \\ 0 & 0 & 0 & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & * & \dots & * & * \\ 1 & * & * & \dots & * & * \end{bmatrix}.$$

Clearly each of these sets of vectors is linearly independent. If M is similar to A, with $TMT^{-1} = A$, then M has a cyclic vector given by $T^{-1}e_n$. Thus existence of a cyclic vector for a matrix is a similarity invariant. It may be noted that such a vector T exists if and only if

$$\text{(2.6)} \quad \operatorname{rank} \begin{bmatrix} e_n, & Ae_n, & A^2e_n, & \dots, & A^{n-1}e_n \end{bmatrix} = \operatorname{rank} \begin{bmatrix} b, & Mb, & M^2b, & \dots, & M^{n-1}b \end{bmatrix} = n.$$

For $M = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $\begin{bmatrix} b, & Mb \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$; but if we let $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} b, & Mb \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, whose rank is 2. Similarly for the diagonal matrix M in (2.5), we have $\begin{bmatrix} b, & Mb \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, which is of rank 1, and hence there does not exist a linear transformation $y = Tx$ such that $TMT^{-1} = A$.

Now to verify the rank condition (2.6), assume that there exists a nonsingular constant matrix T such that $TMT^{-1} = A$ and $Tb = e_n$. Then clearly $TMT^{-1}e_n = Ae_n$, $(TMT^{-1})^2e_n = TM^2T^{-1} = A^2e_n$, $\dots$, $(TMT^{-1})^ke_n = TM^kT^{-1} = A^ke_n$ for any $k \geq 1$. T being nonsingular implies

$$\operatorname{rank} \begin{bmatrix} e_n, & Me_n, & M^2e_n, & \dots, & M^{n-1}e_n \end{bmatrix} = \operatorname{rank} \begin{bmatrix} b, & Ab, & A^2b, & \dots, & A^{n-1}b \end{bmatrix} = n.$$

We conclude this section with the following result.

Result 1. There exists at most one nonsingular linear transformation $y = Tx$ taking (2.1) into (1.3), and further, such a nonsingular T exists if

$$\operatorname{rank} \begin{bmatrix} b, & Mb, & M^2b, & \dots, & M^{n-1}b \end{bmatrix} = n.$$

3 Kronecker Product Companion Systems and Equivalence

In this section, we shall be concerned with formulation of the Kronecker product companion vector system associated with (1.1) and (1.2), and we present certain interesting results on the solution space of the Kronecker product companion system and the scalar differential equation (1.1) and (1.2). Then we consider the general Kronecker product first-order system and show through examples that it is not always possible to transform the general Kronecker product system into the companion vector system. Finally we present a necessary and sufficient condition for their equivalence.

The vector systems associated with (1.1) and (1.2), respectively, are given by

$$L_1 y := y' - Ay = e_n f(t) \tag{3.1}$$

and

$$L_2 z := z' - Bz = e_m g(t). \tag{3.2}$$

Next define $\tilde{L_1}(y \otimes z)$ and $\tilde{L_2}(y \otimes z)$ by

$$\begin{aligned} \tilde{L_1}(y \otimes z) &:= L_1 y \otimes I_m z = (I_n y' - Ay) \otimes I_m z = e_n f(t) \otimes I_m z \\ &= (I_n \otimes I_m)(y' \otimes z) - (A \otimes I_m)(y \otimes z) \\ &= (e_n \otimes I_m)(f(t) \otimes z), \end{aligned}$$

and similarly,

$$\begin{aligned} \tilde{L_2}(y \otimes z) &:= I_n y \otimes L_2 z = (I_n \otimes I_m)(y \otimes z') - (B \otimes I_n)(y \otimes z) \\ &= (I_n \otimes e_m)(y \otimes g(t)). \end{aligned}$$

Now, let

$$L(y \otimes z) = \tilde{L_1}(y \otimes z) + \tilde{L_2}(y \otimes z).$$

then

$$\begin{aligned} L(y \otimes z) &:= (I_n \otimes I_m)(y \otimes z)' - (A \otimes I_m + I_n \otimes B)(y \otimes z) \\ &= (e_n \otimes I_m)(f(t) \otimes z) + (I_n \otimes e_m)(y \otimes g(t)). \end{aligned} \tag{3.3}$$

Let $F(t) = (e_n \otimes I_m)(F(t) \otimes z)$ and $G(t) = (I_n \otimes e_n)(y \otimes G(t))$. We observe that F and G are $nm \times 1$ vectors and $I_n \otimes I_m$ is a unit matrix of order nm. We now state the interesting relationship on the solution space of the Kronecker product system (3.3) and the scalar differential equations (1.1) and (1.2) in the next two theorems.

Theorem 1. *$Y \otimes Z$ is a fundamental matrix of*

$$L(y \otimes z) := (I_n \otimes I_m)(y \otimes z)' - (A \otimes I_m + I_n \otimes B)(y \otimes z) = 0 \tag{3.4}$$

if and only if Y and Z are fundamental matrix solutions of

$$y' = A(t)y \text{ and } z' = B(t)z,$$

respectively.

Proof. Suppose Y and Z are fundamental matrix solutions of $y' = A(t)y$ and $z' = B(t)z$, respectively, so that $Y' = A(t)Y$ and $Z' = B(t)Z$. Consider

$$\begin{aligned}(Y \otimes Z)' &= Y' \otimes Z + Y \otimes Z' \\ &= A(t)Y \otimes Z + Y \otimes B(t)Z \\ &= (A(t) \otimes I_m)(Y \otimes Z) + (I_n \otimes B(t))(Y \otimes Z).\end{aligned}$$

Thus $Y \otimes Z$ is a fundamental matrix of (3.4).
Conversely, suppose that $Y \otimes Z$ is a fundamental matrix of (3.4). Then

$$(Y \otimes Z)' = (A(t) \otimes I_m)(Y \otimes Z) + (I_n \otimes B(t))(Y \otimes Z)$$

or

$$Y' \otimes Z + Y \otimes Z' = (A(t) \otimes I_m)(Y \otimes Z) + (I_n \otimes B(t))(Y \otimes Z)$$

or

$$(Y' - A(t)Y) \otimes Z = -Y \otimes (Z' - B(t)Z).$$

Multiplying both sides by $(Y \otimes Z)^{-1} = Y^{-1} \otimes Z^{-1}$, we obtain

$$Y^{-1}(Y' - A(t)Y) \otimes I_m = -(I_n \otimes Z^{-1})(Z' - B(t)Z). \tag{3.5}$$

This result is true only if Y is a fundamental matrix of $y' = A(t)y$ and Z is a fundamental matrix of $z' = B(t)z$. □

Theorem 2. *If $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_m$ are linearly independent solutions of $Lu = 0$ and $Lv = 0$, respectively, then*

$$(K(u) \otimes K(v)) = (k(u_1), \ldots, k(u_n)) \otimes (k(v_1), \ldots, k(v_m))$$

is a fundamental matrix of the companion vector equation (3.4), and conversely, if $(K(u) \otimes K(v))$ is a fundamental matrix of (3.4), then the first row of $K(u)$ is a fundamental vector for $Lu = 0$ and the first row of $K(v)$ is a fundamental vector for $Lv = 0$.

Proof. Since $K(u)$ and $K(v)$ are fundamental matrix solutions of $y' = A(t)y$ and $z' = B(t)z$, respectively, it follows by Theorem 3.1 that $(K(u) \otimes K(v))$ is a fundamental matrix of (3.4). Conversely, suppose $(K(u) \otimes K(v))$ is a fundamental matrix of (3.4), then it follows that $K(u)$ is a fundamental matrix of $y' = Ay$ and $K(v)$ is a fundamental matrix of $z' = Bz$. Hence the first row of $K(u)$ is a fundamental vector for $Lu = 0$ and the first row of $K(v)$ is a fundamental vector for $Lv = 0$. □

Definition 3. The Kronecker product first-order system

$$\begin{aligned}L(x_1 \otimes x_2) &:= (I_n \otimes I_m)(x_1 \otimes x_2)' - (M \otimes I_m + I_n \otimes N)(x_1 \otimes x_2) \\ &= (f_1 \otimes f_2)(x_1 \otimes x_2),\end{aligned} \tag{3.6}$$

where M is an $n \times n$ constant matrix and N is an $m \times m$ constant matrix is *linearly equivalent* to the companion system (3.3) if and only if there exists an $m \times m$ constant nonsingular matrix T such that

$$T(M \otimes I_m + I_n \otimes N)T^{-1} := A \otimes I_m + I_n \otimes B \quad \text{and}$$
$$T(f_1 \otimes f_2) = e_n f + e_m g.$$

It may be noted that such a linear transformation may not always be possible. For example, consider

$$M(t) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } N(t) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then

$$(M(t) \otimes I_2 + I_2 \otimes N(t)) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad (f_1 \otimes f_2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then there exists no constant 4×4 matrix T such that

$$T(M(t) \otimes I_2 + I_2 \otimes N(t))T^{-1} = (A(t) \otimes I_2 + I_2 \otimes B(t)).$$

On the other hand, if we consider

$$M(t) = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad N(t) = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix},$$

we get

$$(M(t) \otimes I_2 + I_2 \otimes N(t)) = \begin{bmatrix} -5 & 2 & 3 & 0 \\ 1 & -4 & 0 & 3 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & -2 \end{bmatrix} \quad \text{and} \quad (f_1 \otimes f_2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then $M(t)$ is reduced to $A(t) = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$ and $N(t)$ is reduced to $B(t) = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$, and hence there exists a nonsingular, constant 4×4 matrix T such that

$$T(M(t) \otimes I_2 + I_2 \otimes N(t))T^{-1} = (A(t) \otimes I_2 + I_2 \otimes B(t)).$$

Definition 4. The vector $(x_1 \otimes x_2)$ is a *cyclic vector* for the matrix $(M(t) \otimes I_m + I_n \otimes N(t))$ if the nm vectors

$$\{(x_1 \otimes x_2),\ \ (M(t) \otimes I_m + I_n \otimes N(t))(x_1 \otimes x_2),\ \ \ldots,\ \ (M(t) \otimes I_m + I_n \otimes N(t))^{n-1}(x_1 \otimes x_2)\}$$

are linearly independent.

Note that existence of a cyclic vector for a matrix is a similarity invariant.

Result 2. There exists at most one nonsingular linear transformation $(y \otimes z) = T(x_1 \otimes x_2)$ taking (3.6) into (3.3), and further, such a transformation exists if

$$\text{rank}[(x_1 \otimes x_2), (M(t) \otimes I_m + I_n \otimes N(t))(x_1 \otimes x_2), \ldots, (M(t) \otimes I_m + I_n \otimes N(t))^{n-1}(x_1 \otimes x_2)] = nm.$$

Note that in such a linear transformation, T is nonsingular.

4 Controllability of the Kronecker Product Systems

In this section we shall present sufficient conditions for the complete controllability of the Kronecker product first-order system

$$\begin{aligned} L(x_1 \otimes x_2) &= (I_n \otimes I_m)(x_1 \otimes x_2)' - (M \otimes I_m + I_n \otimes N)(x_1 \otimes x_2) \\ &= (f_1 \otimes f_2)(u_1 \otimes u_2), \end{aligned}$$

which can be rewritten as

(4.1) $$(x_1 \otimes x_2)' = (M \otimes I_m + I_n \otimes N)(x_1 \otimes x_2) + (f_1 \otimes f_2)(u_1 \otimes u_2),$$

where M and N are $n \times n$ and $m \times m$ continuous matrices on an interval $J = [t_0, t_f]$ and $(f_1 \otimes f_2)(u_1 \otimes u_2)$ is an $nm \times lk$ matrix with $(u_1 \otimes u_2)$ continuous on J. For the controllability of linear systems, we refer to [1] and [5].

Definition 5. The system (4.1) is *completely controllable from* (t_0, x_0) *to* (t_f, x_f) if, for some $(u_1 \otimes u_2)(t)$ on J, the solution of (4.1) with $(x_1 \otimes x_2)(t_0) = x_0$ is such that $(x_1 \otimes x_2)(t_f) = x_f$, where t_f and x_f are preassigned terminal time and state, respectively. If (4.1) is controllable for all x_0 at $t = t_0$ and for all x_f at $t = t_f$, then the system (4.1) is said to be *completely controllable* (c.c.).

We use the following notation: Y stands for a fundamental matrix of $x_1' = Mx_1$, Z stands for a fundamental matrix of $x_2' = Nx_2$, $\Phi(t, t_0) = Y(t)Y^{-1}(t_0)$, and $\Psi(t, t_0) = Z(t)Z^{-1}(t_0)$.

Theorem 3. *Any solution of the Kronecker product system (4.1) satisfying* $(x_1 \otimes x_2)(t_0) = x_0$ *is given by*

$$(x_1 \otimes x_2)(t) = [\Phi(t, t_0) \otimes \Psi(t, t_0)]\big\{x_0 + \int_{t_0}^{t} [\Phi(s, t_0) \otimes \Psi(s, t_0)](f_1 \otimes f_2)(s)(u_1 \otimes u_2)(s)ds\big\}.$$

Proof. Any solution of (4.1) is given by

(4.2) $$\begin{aligned} (x_1 \otimes x_2)(t) &= (Y \otimes Z)(t)(\sigma_1 \otimes \sigma_2) \\ &\quad + (Y(t) \otimes Z(t)) \int_{t_0}^{t} (Y \otimes Z)^{-1}(s)(f_1 \otimes f_2)(s)(u_1 \otimes u_2)(s)ds, \end{aligned}$$

where σ_1 and σ_2 are constant matrices. Now

$$(x_1 \otimes x_2)(t_0) = (Y \otimes Z)(t_0)(\sigma_1 \otimes \sigma_2) = x_0,$$

or

$$(\sigma_1 \otimes \sigma_2) = (Y \otimes Z)^{-1}(t_0)x_0.$$

Substituting this value for $(\sigma_1 \otimes \sigma_2)$in (4.2), and noting that

$$\Phi(t, t_0) = Y(t)Y^{-1}(t_0) \text{ and } \Psi(t, t_0) = Z(t)Z^{-1}(t_0),$$

we get

$$(x_1 \otimes x_2)(t) = [\Phi(t, t_0) \otimes \Psi(t, t_0)]\big\{x_0 + \int_{t_0}^{t} [\Phi(s, t_0) \otimes \Psi(s, t_0)](f_1 \otimes f_2)(s)(u_1 \otimes u_2)(s)\big\}ds.$$

□

Theorem 4. *The Kronecker product system (4.1) is completely controllable if and only if the $nm \times nm$ symmetric controllability matrix*

$$v(t_0, t) = \int_{t_0}^{t} [\Phi(s, t_0) \otimes \Psi(s, t_0)](f_1 \otimes f_2)(s)(f_1 \otimes f_2)^T(s) \times [\Phi^T(t_0, s) \otimes \Psi^T(t_0, s)]ds, \tag{4.3}$$

where $\Phi(t, t_0)$ and $\Psi(t, t_0)$ are fundamental matrix solutions of $x_1' = Mx_1$ and $x_2' = Nx_2$ such that $\Phi(t_0, t_0) = I$ and $\Psi(t_0, t_0) = I$ are nonsingular; i.e., for some positive constant c, $det(v(t_0, t)) \geq c$. The control function

$$(u_1 \otimes u_2)(t) = -(f_1 \otimes f_2)^T(t)\big(\Phi^T(t_0, t) \otimes \Psi^T(t_0, t)\big)v^{-1}(t_0, t) \times [x_0 - \big(\Phi^T(t_0, t) \otimes \Psi^T(t_0, t)\big)]x_f \tag{4.4}$$

defined on J transfers $(x_1 \otimes x_2)(t_0) = x_0$ to $(x_1 \otimes x_2)(t_f) = x_f$.

Proof. First assume that $v(t_0, t_f)$ as given by (4.3) is nonsingular. Then $(u_1 \otimes u_2)$ as given by (4.4) is well-defined. Any solution of (4.2) satisfying $(x_1 \otimes x_2)(t_0) = x_0$ is given by

$$(x_1 \otimes x_2)(t) = [\Phi(t, t_0) \otimes \Psi(t, t_0)]\big\{x_0 + \int_{t_0}^{t} [\Phi(s, t_0) \otimes \Psi(s, t_0)](f_1 \otimes f_2)(s)(u_1 \otimes u_2)(s)ds\big\}.$$

Substitution into this equation of the expression for the control function $(u_1 \otimes u_2)(t_f)$ given by (4.4) yields

$$\begin{aligned}(x_1 \otimes x_2)(t_f) = [\Phi(t_f, t_0) \otimes \Psi(t_f, t_0)]\big\{x_0 + \int_{t_0}^{t_f} &[\Phi(s, t_0) \otimes \Psi(s, t_0)](f_1 \otimes f_2)(s) \\ &\times \{-(f_1 \otimes f_2)^T(t_f)\Phi^T(t_0, t_f) \otimes \Psi^T(t_0, t_f)v^{-1}(t_0, t_f) \\ &\times [x_0 - \Phi^T(t_0, t_f) \otimes \Psi^T(t_0, t_f)]x_f\}ds\big\} \\ &= [\Phi(t_f, t_0) \otimes \Psi(t_f, t_0)]\{x_0 - x_0 + \Phi^T(t_0, t_f) \otimes \Psi^T(t_0, t_f)\}x_f \\ &= x_f.\end{aligned}$$

Hence the system is c.c. Now, to prove the converse, suppose that the system (4.1) is c.c., and that there exists an arbitrary column vector α such that

$$\begin{aligned}\alpha^T v(t_0, t_f)\alpha &= \int_{t_0}^{t_f} (u_1 \otimes u_2)^T(s)(\Phi^T(t_0, s) \otimes \Psi^T(t_0, s))(\Phi(t_0, s) \otimes \Psi(t_0, s))(u_1 \otimes u_2)(s)ds \\ &= \int_{t_0}^{t_f} K^T(t_0, s)K(t_0, s)ds,\end{aligned}$$

where $K(t_0, s) = (u_1 \otimes u_2)(s)(\Phi(t_0, s) \otimes \Psi(t_0, s))$. Then $\alpha^T v(t_0, t_f)\alpha \geq 0$, which implies that $v(t_0, t_f)$ is positive semi-definite. We now prove that $\alpha^T v(t_0, t_f)\alpha \neq 0$. To the contrary, suppose that there exists a vector $\tilde{\alpha} \neq 0$ such that

$$\tilde{\alpha}^T v(t_0, t_f)\tilde{\alpha} = 0. \quad \text{Then } \int_{t_0}^{t_f} K^T(t_0, s)K(t_0, s)ds = 0,$$

i.e.

$$\int_{t_0}^{t_f} |K(t_0,s)|^2 ds = 0. \quad \text{This implies that } K(t_0,s) \equiv 0 \text{ on } [t_0,t_f].$$

Since (4.1) is c.c., we must have that $|\tilde{\alpha}|^2 = \tilde{\alpha}^T\tilde{\alpha} = 0$. Hence $\tilde{\alpha} = 0$, which is a contradiction. Therefore $v(t_0,t_f)$ is positive definite and hence nonsingular. □

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