# DEPARTMENT OF MATHEMATICS
AND
# COMPUTER SCIENCE

# TECHNICAL REPORT SERIES

## On Reptiles and Self-Affine Tiles in $\mathbf{R}^2$ with Connected Interiors

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Number 2002-005
Submitted: March 19, 2002

# ON REPTILES AND SELF-AFFINE TILES IN $\mathbb{R}^2$ WITH CONNECTED INTERIORS

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ABSTRACT. We prove that if the interior of a reptile or a self-affine tile in the plane is connected, then the tile must be a topological disk. In the process of establishing this result, we study arcwise connectedness, simply connectedness, and other related topological properties of such reptiles and self-affine tiles. In particular, we answer a question raised by Grünbaum concerning the existence of an open connected set whose closure is a reptile which is not simply connected (see [CFG]).

## 1. INTRODUCTION

A self-affine tile in $\mathbb{R}^d$ is a compact set $T \subseteq \mathbb{R}^d$ of positive Lebesgue measure satisfying the following property. There is an expanding $d$ by $d$ matrix $A$ and a digit set $\mathcal{D} \subseteq \mathbb{R}^d$ such that $T = T(A, \mathcal{D})$ satisfies

$$A(T) = T + \mathcal{D} := \bigcup_{v \in \mathcal{D}} (T + v), \tag{1.1}$$

or

$$T = \bigcup_{v \in \mathcal{D}} A^{-1}(T + v), \tag{1.2}$$

where, for distinct $v, v' \in \mathcal{D}$, $T + v$ and $T + v'$ are assumed to be *essentially disjoint*, i.e., $(T + v) \cap (T + v')$ has zero Lebesgue measure. A necessary condition for $T$ to have positive Lebesgue measure together with the essentially disjoint property is that $|\det(A)| = \#\mathcal{D}$, the cardinality of $\mathcal{D}$. A matrix $A$ is said to be *expanding* if its eigenvalues $\lambda_i$ satisfy $|\lambda_i| > 1$. A *digit set* $\mathcal{D}$ in $\mathbb{R}^d$ is just a finite set of vectors in $\mathbb{R}^d$. We refer the reader to Lagarias and Wang [LW] and Kenyon [K] for some basic results in the theory of self-affine tilings.

An *n-reptile* (or *n-rep tile*) $T$ in $\mathbb{R}^d$ is a compact set with nonempty interior that can be tiled by $n$ congruent tiles, each similar to $T$ (see [GS], [CFG]). We assume in

*Date*: March 18, 2002.

1991 *Mathematics Subject Classification.* Primary 52C20; Secondary 28A80.

*Key words and phrases.* Reptile, self-affine tile, tiling, fractal, iterated function system, self-similar set, connected, arcwise connected, simply connected, topological disk.

The first author is supported in part by an HKRGC grant and the following grants from Georgia Southern University: a Faculty Research Grant, an Academic Excellence Award from the College of Science and Technology, and a Foundation Grant from the Department of Mathematics and Computer Science.

addition, as in the literature, that $T$ is the closure of its interior. If the number of pieces $n$ is irrelevant in the discussion, we will simply call $T$ a *reptile*.

Reptiles form a special class of self-similar sets. Let $\{\phi_i\}_{i=1}^n$ be an *iterated function system* of contractive similitudes on $\mathbb{R}^d$ defined as

$$\phi_i(x) = \frac{1}{\sqrt[d]{n}} R_i x + d_i, \quad i = 1, \dots, n,$$

where $R_i$ is an orthogonal transformation, $d_i \in \mathbb{R}^d$, and the factor $1/\sqrt[d]{n}$ is the *contraction ratio* of $\phi_i$. Then there exists a unique compact set $T$ satisfying

$$T = \bigcup_{i=1}^{n} \phi_i(T). \tag{1.3}$$

(See [H], [F].) $T$ is called the *self-similar set* (or *n-repset*, or *attractor*) defined by $\{\phi_i\}_{i=1}^n$. It follows from (1.3) and the uniqueness of $T$ that $T$ is the closure of its interior. If the interior of $T$ is nonempty, then it follows from (1.3) that $T$ is an $n$-reptile. The assumption $T$ has a nonempty interior is equivalent to that $\{\phi_i\}_{i=1}^n$ satisfies the *open set condition* (see [S]). Thus, there is a one-to-one correspondence between an $n$-reptile and a self-similar set defined by an iterated function system consisting of $n$ similitudes having the same contraction ratio $1/\sqrt[d]{n}$ and satisfying the open set condition.

Identity (1.3) is equivalent to

$$\sqrt[d]{n} T = \bigcup_{i=1}^{n} (R_i T + \sqrt[d]{n} d_i), \tag{1.4}$$

an analogue of (1.1). If $T$ is a self-affine tile or a reptile, then it follows from a standard blow-up argument that $\mathbb{R}^d$ can be tiled by essentially disjoint congruent copies of $T$ (see [LW, Theorem 1.2]). Let $\{T_i\}_{i=0}^\infty$ be a sequence of essentially disjoint congruent copies of $T$, with $T_0 = T$, such that $\cup_{i=0}^\infty T_i = \mathbb{R}^d$. $\{T_i\}_{i=0}^\infty$ is called a *tiling* of $\mathbb{R}^d$.

This paper mainly focuses on reptiles and self-affine tiles $T$ in $\mathbb{R}^2$ with the interior of $T$ being connected. This study is motivated by a question raised by Grünbaum, as well as by some recent studies on characterizing self-affine tiles in the plane that are topological disks.

Answering a question of John Conway, Grünbaum gave an example of a 36-reptile in $\mathbb{R}^2$ which is not simply connected (see [CFG, Figure C17]). Grünbaum then suggested the following question (see [CFG, C17]): Is there a connected open set in the plane whose closure is a reptile that is not simply connected? The following result answers this question in the negative. The conclusion also holds for self-affine tiles.

**Theorem 1.1.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$. If the interior of $T$ is connected then $T$ must be simply connected.*

The main purpose of this paper is to establish a stronger conclusion than that of Theorem 1.1, namely, that $T$ must be a topological disk. An open subset $E \subseteq \mathbb{R}^2$ is called an *open topological disk* if it is homeomorphic to the open unit disk; a closed subset $E \subseteq \mathbb{R}^2$ is called a *(closed) topological disk* if it is homeomorphic to the closed unit disk. The main result of this paper is

**Theorem 1.2.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$. If the interior of $T$ is connected then $T$ is a topological disk.*

In view of Theorems 1.1 and 1.2, it should be pointed out that there exists an open topological disk $E$ whose closure is simply connected but is not a topological disk. Example 5.1 shows such a set. Also, Theorem 1.2 fails for reptiles in $\mathbb{R}^d$, $d > 2$. We will provide a counter-example in Section 5.

The characterization of reptiles and self-affine tiles in the plane that are topological disks has been an important part of the theory of such tiles (see Bandt and Gelbrich [BG], Gelbrich [G]). Recently, Song and Kang [SK], and Bandt and Wang [BW] have obtained conditions for the class of integral self-affine tiles in the plane with standard digit sets to be topological disks. These conditions are algebraic and geometric in nature. Theorem 1.2 provides a quite general topological condition.

We will only prove Theorem 1.2, which yields Theorem 1.1 immediately. In Section 2, we will show that if a tile $T$ is connected, then each component of its interior is an open topological disk (Theorem 2.1). Under the stronger hypothesis that the interior of $T$ is connected we show that $T$ is arcwise connected (Theorem 2.2). The self-similarity (or self-affinity) of $T$ plays a crucial role in the proof. We then show, under this stronger hypothesis, that the complement of $T$ must be connected (Proposition 2.6). These results form the basis for the proof of the main theorem.

To show that $T$ is a topological disk, we will show that each boundary point of $T$ is simple; we use some ideas in [BW]. This is the more difficult part of the proof. The main idea is as follows. From the proof for the arcwise connectedness of $T$ we can actually show that every two points on $T$ can be connected by a simple curve with interior lying in the interior of $T$ (Proposition 2.3). Combining this fact with the Jordan Curve Theorem provides an essential tool for analyzing the local structure of any boundary point of $T$.

The Jordan Curve Theorem is used frequently in the proof of Theorem 1.2. In Section 3, we will also use it to study the intersections of a tile with its neighboring tiles in a tiling. We call a boundary point of a tile $T$ a *vertex point* if each of its neighborhoods has nonempty intersection with at least two neighboring tiles of $T$. We show that under the hypotheses of Theorem 1.2, $T$ has at most a finite number of vertex points (Theorem 3.2). Although this result is not needed in the proof of the main theorems, it is of interest in its own right.

This paper is organized as follows. In Section 2, we establish some basic topological results for reptiles and self-affine tiles in the plane. In Section 3, we show that under the assumption of Theorem 1.2, $T$ has only a finite number of vertex points. Section

4 is devoted to the proof of Theorem 1.2. Section 5 concludes the paper with some remarks and counter-examples.

## 2. Connectedness and arcwise connectedness

We will use the following standard notation and terminology throughout this paper. A *region* is a nonempty open connect subset of $\mathbb{R}^d$. For $E \subseteq \mathbb{R}^d$, let $E^\circ$, $\overline{E}$, $\partial E$, and $E^c$ denote, respectively, the interior, closure, boundary, and complement of $E$. Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$. For $r > 0$, let $B_r(x) := \{y \in \mathbb{R}^d : \|y - x\| \le r\}$ be the closed ball in $\mathbb{R}^d$ with radius $r$ and centered at $x$. For subsets $E, F \subseteq \mathbb{R}^d$, let $\operatorname{diam}(E)$ denote the *diameter* of $E$, and let $\operatorname{dist}(E, F)$ denote the *distance* between $E$ and $F$. They are defined as

$$\operatorname{diam}(E) := \sup\big\{\|x - y\| : x, y \in E\big\},$$
$$\operatorname{dist}(E, F) := \inf\big\{\|x - y\| : x \in E, y \in F\big\}.$$

We first prove that if $T$ is connected, then the components of $T^\circ$ do not have holes. It is known that all 2-reptiles in $\mathbb{R}^d$ are connected; in fact, all 2-repsets in $\mathbb{R}^d$ are connected (see [HSV], [KL], [NSVW]).

**Theorem 2.1.** *Let $T$ be a connected reptile or self-affine tile in $\mathbb{R}^2$ and let $U$ be a component of $T^\circ$. Then $U^c$, the complement of $U$, is connected. Consequently, $U$ is an open topological disk. In particular, if $T^\circ$ is connected, $T^\circ$ is an open topological disk.*

*Proof.* Suppose $U^c$ is not connected, and hence can be written as a disjoint union of maximal connected subsets $\cup_\alpha V_\alpha$, with only one component, say $V_1$, being unbounded. Notice that each $V_\alpha$ is closed in $\mathbb{R}^2$, since $U^c$ is closed in $\mathbb{R}^2$ and $V_\alpha$ is closed in $U^c$. Let $V := \cup_\alpha V_\alpha \setminus V_1$. Then $\mathbb{R}^2 = U \cup V \cup V_1$. If $V = \emptyset$, then $U^c = V_1$ is connected. Hence it suffices to assume $V \neq \emptyset$.

We first show that the interior of $V$ is nonempty. Suppose $V^\circ = \emptyset$. Let $x$ be an arbitrary point of $V$. Then $x \in V_2$ for some component $V_2 \neq V_1$. As the $V_\alpha$ are maximal connected, $V_1$ and $V_2$ are separated. There are disjoint open sets $O_1$ and $O_2$ such that $V_1 \subseteq O_1$ and $V_2 \subseteq O_2$. In particular, $O_2 \cap V_1 = \emptyset$. It follows that for $r > 0$ sufficiently small we have $B_r(x) \subseteq O_2 \subseteq U \cup V$. Hence,

$$B_r(x) = \big(B_r(x) \cap U\big) \cup \big(B_r(x) \cap V\big). \tag{2.1}$$

The assumption $V^\circ = \emptyset$ implies that $B_r(x) \cap U \neq \emptyset$. So $x \in \partial U \subseteq \overline{U}$. Consequently, $V \subseteq \overline{U}$. But then (2.1) would imply that $B_r(x) \subseteq \overline{U}$, and therefore $x \in (\overline{U})^\circ \subseteq T^\circ$. Since $x \in U^c$, $x$ must belong to another component $U'$ of $T^\circ$. This contradicts the conclusion $x \in \partial U$ obtained above. This contradiction shows that $V^\circ \neq \emptyset$.

Now, let $\{T_i\}_{i=0}^\infty$ be a tiling of $\mathbb{R}^2$ consisting of congruent copies of $T$, with $T_0 = T$. Let $V_2$ be a bounded maximal connected subset of $U^c$ that has an interior point. Let

$O_1, O_2$ be open sets as defined above. Then $V_2 \subseteq O_2 \subseteq O_1^c \subseteq V_1^c$ and therefore

$$\mathrm{diam}(V_2) < \mathrm{diam}(O_2) \leq \mathrm{diam}(O_1^c) < \mathrm{diam}(V_1^c).$$

The first and last inequalities are strict because $V_2$ and $O_1^c$ are compact, while $O_2$ and $V_1^c$ are open.

Notice that for any bounded set $S$, $\mathrm{diam}(S) = \mathrm{diam}(\partial S)$. Next, we observe that $\partial V_1^c \subseteq \partial U$. To see this, let $x \in \partial V_1^c$ and therefore $x \in \partial V_1$. Suppose $x \notin \partial U$. Then for $r > 0$ sufficiently small, the ball $B_r(x)$ would satisfy

$$B_r(x) \cap U = \emptyset, \quad V_1 \subsetneqq B_r(x) \cup V_1 \subseteq U^c, \quad B_r(x) \cup V_1 \text{ is connected,}$$

contradicting that $V_1$ is a component of $U^c$. We conclude from these observations that $\mathrm{diam}(V_1^c) \leq \mathrm{diam}(U)$. Therefore $\mathrm{diam}(V_2) < \mathrm{diam}(U)$.

Since $V_2^\circ \neq \emptyset$, there is an $i \neq 0$ such that $T_i \cap V_2 \neq \emptyset$. Because $\mathrm{diam}(V_2) < \mathrm{diam}(U)$, $T_i \setminus V_2 \neq \emptyset$. So $T_i \cap V_\alpha \neq \emptyset$ for some $V_\alpha \neq V_2$. Hence $T_i$ is disconnected, contradictory to the hypothesis of the theorem. Therefore $U^c$ is connected. That $U$ is an open topological disk follows as a standard result (see e.g. [R]). ☐

A self-affine tile $T = T(A, \mathcal{D})$ satisfies $A(T) = T + \mathcal{D}$ (see (1.1)). Iterating this equality gives

$$A^k(T) = T + \mathcal{D}_k, \quad \text{where} \quad \mathcal{D}_k := \mathcal{D} + A\mathcal{D} + \cdots + A^{k-1}\mathcal{D}.$$

That is,

$$T = \bigcup_{v \in \mathcal{D}_k} A^{-k}(T + v).$$

We call $A^{-k}(T + d)$, $d \in \mathcal{D}_k$, the *$k$-th level pieces* of $T$.

Similarly we can define $k$-th level pieces for reptiles. Iterating the self-similar identity (1.3) $k$ times yields

$$T = \bigcup_{i_1, \dots, i_k = 1}^{n} \phi_{i_1} \circ \cdots \circ \phi_{i_k}(T), \tag{2.2}$$

with the union being essentially disjoint. We call each $\phi_{i_1} \circ \cdots \circ \phi_{i_k}(T)$ a *$k$-th level piece* of $T$. More generally let $T$ be a self-affine tile or a reptile in $\mathbb{R}^d$ and let $\{T_i\}_{i=0}^{\infty}$ be a tiling of $\mathbb{R}^2$ consisting of congruent copies of $T$. For each $i$, fix an isometry $\tau_i : T \to T_i$. Then we define the *$k$-th level pieces* of $T_i$ to be the images of the $k$-th level pieces of $T$ under $\tau_i$. It is obvious that each $x \in T$ can intersect only a bounded number of $k$-th level pieces with a bound independent of $k$.

Recall that $E \subseteq \mathbb{R}^d$ is *arcwise connected* if for any two points $a, b \in E$ there exists a continuous function $\gamma : [0,1] \to E$ such that $\gamma(0) = a$ and $\gamma(1) = b$. $\gamma$ is called a *curve*, and if $\gamma(0) \neq \gamma(1)$ then the image of $[0,1]$ under $\gamma$ is called an *arc* joining $a$ and $b$. With a slight abuse of notation, we will often call a curve an arc, and identify a curve $\gamma$ with its image $\gamma[0,1]$. A curve is *simple* if it does not cross itself except possibly at the end-points. A curve is *closed* if the end-points coincide: $\gamma(0) = \gamma(1)$.

Any open connected set is arcwise connected, but the closure of an arcwise connected set is not necessarily arcwise connected (the topologist's sine curve is a counterexample). Nevertheless, for reptiles with connected interior we have the following

**Theorem 2.2.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$ with $T^\circ$ being connected. Then $T$ is arcwise connected. In fact, for any $x \in \partial T$ and $y \in T^\circ$, there exists a curve $\gamma$ with $\gamma(0) = y$, $\gamma(1) = x$ such that $\gamma[0,1) \subset T^\circ$.*

*Proof.* Since $T^\circ$ is arcwise connected, it suffices to show that for each $x \in \partial T$ and $y \in T^\circ$, there exists an arc joining $y$ and $x$.

Let $R_1$ be a first level piece of $T$ containing $x$ as a boundary point and let $y_1 \in R_1^\circ$. Since $y, y_1 \in T^\circ$, we can join $y$ and $y_1$ by a curve $\gamma_1$ such that $\gamma_1(0) = y$, $\gamma_1(1) = y_1$, and $\gamma_1[0,1] \subseteq T^\circ$. Now let $R_2$ be a second level piece of $T$ contained in $R_1$ and containing $x$ as a boundary point. Again, let $y_2 \in R_2^\circ \subseteq R_1^\circ$. Since $R_1^\circ$ is arcwise connected, we can join $y_1$ and $y_2$ by a curve $\gamma_2$ in $R_1^\circ$. Continue this process. We get a sequence of curves $\{\gamma_k\}_{k=1}^\infty$ such that $\gamma_{k+1}[0,1] \subseteq R_k^\circ$, where $R_0 := T$.

We now define a curve $\gamma : [0,1] \to \mathbb{R}^2$ as

$$\gamma(t) := \begin{cases} \gamma_k(k(k+1)t - k^2 + 1), & \text{if} \quad t \in [(k-1)/k, k/(k+1)], \quad k = 1, 2, \ldots \\ x, & \text{if} \quad t = 1. \end{cases}$$

That is, the image of $[(k-1)/k, k/(k+1)]$ under $\gamma$ is $\gamma_k[0,1]$. It remains to show that $\gamma$ is continuous at 1. But this follows easily from the fact that as $k \to \infty$, $\operatorname{diam}(R_k) \to 0$ and consequently $\operatorname{diam}(\gamma_k[0,1]) \to 0$, and $\operatorname{dist}(\{x\}, \gamma_k[0,1]) \to 0$. The last assertion of the proposition follows from the construction of $\gamma$. □

Example 5.3 shows a tiles $T$ with a connected interior which is not arcwise connected. It is neither a reptile nor a self-affine tile.

**Proposition 2.3.** *Assume the same hypotheses of Theorem 2.2. Then*

(a) *the curve $\gamma$ in Theorem 2.2 can be chosen to be simple;*

(b) *for any $x, z \in T$, there exists a simple curve $\gamma : [0,1] \to T$ joining $x$ and $z$ with $\gamma(0,1) \subseteq T^\circ$.*

*Proof.* (a) Each of the arcs $\gamma_i$ in the proof of Theorem 2.2 can be chosen to be simple because $R_{i-1}^\circ$ is homeomorphic to an open disk. Notice that $\gamma_1[0,1]$ is compact, $\gamma_k[0,1] \subseteq R_{k-1}^\circ$, $\operatorname{diam}(R_k) \to 0$ and $x \in R_k$. Therefore, $\gamma_k[0,1] \cap \gamma_1[0,1] \neq \emptyset$ for only finitely many $k$. Let $k_1(> 1)$ be the largest of such $k$. So $\gamma_{k_1}(0) = y_{k_1-1}$ and $\gamma_{k_1}(1) = y_{k_1}$, with $y_0 := y$. Let

$$t_1 := \sup\bigl\{t \in [0,1] : \gamma_{k_1}(t) = \gamma_1(s) \text{ for some } s \in [0,1]\bigr\}.$$

Let $s_1$ be the number with $\gamma_1(s_1) = \gamma_{k_1}(t_1)$. Define a piece of a simple arc $\sigma_1 : [0, k_1/(k_1+1)] \to T^\circ$ with image $\gamma_1[0, s_1] \cup \gamma_{k_1}[t_1, 1]$ by

$$\sigma_1(t) := \begin{cases} \gamma_1(2s_1 t), & t \in [0, 1/2] \\ \gamma_{k_1}(at + b), & t \in [1/2, k_1/(k_1+1)]. \end{cases}$$

Here $a$ and $b$ are chosen so that $a/2 + b = t_1$ and $ak_1/(k_1 + 1) + b = 1$. Precisely, $a = 2(t_1 - 1)(k_1 + 1)/(1 - k)$ and $b = (1 + k - 2t_1k)/(1 - k)$.

Use a similar procedure to get $\sigma_2$ by altering and extending $\sigma_1$, but leaving it unchanged over $[0, 1/2]$. The details are as follows. As before, there are finitely many $k > k_1$ such that $\gamma_k[0,1] \cap \sigma_1[0, k_1/(k_1+1)] \neq \emptyset$. Let $k_2$ be the largest of such integers and let

$$t_2 := \sup\big\{t \in [0,1] : \gamma_{k_2}(t) = \sigma_1(s) \text{ for some } s \in [0,1]\big\}.$$

Let $s_2$ be such that $\sigma_1(s_2) = \gamma_{k_2}(t_2)$. Notice that $s_2 > 1/2$ since $\gamma_{k_2}$ has empty intersection with $\sigma_1[0, 1/2]$. Define a piece of a simple curve $\sigma_2 : [0, k_2/(k_2+1)] \to T^\circ$ with image $\sigma_1[0, s_2] \cup \gamma_{k_2}[t_2, 1]$ by

$$\sigma_2(t) := \begin{cases} \sigma_1(t), & t \in [0, 1/2] \\ \sigma_1(a_1t + b_1), & t \in [1/2, k_1/(k_1 + 1)] \\ \gamma_{k_2}(a_2t + b_2), & t \in [k_1/(k_1 + 1), k_2/(k_2 + 1)], \end{cases}$$

where $a_1$ and $b_1$ are chosen so that $a_1/2 + b_1 = 1/2$ and $a_1k_1/(k_1 + 1) + b_1 = s_2$, while $a_2$ and $b_2$ are chosen such that $a_2k_1/(k_1 + 1) + b_2 = s_2$ and $a_2k_2/(k_2 + 1) + b_2 = 1$.

Repeat the process to get simple curves $\sigma_i : [0, k_i/(k_i+1)] \to T^\circ$, with $k_i$ increasing to $\infty$. As $i$ increases, a larger and larger segment of the arcs with initial point $y$ becomes fixed.

Define $\gamma : [0, 1] \to T$, with $k_0 := 1$, as

$$\gamma(t) := \begin{cases} \sigma_1(t), & t \in [0, 1/2], \\ \sigma_i(t), & t \in [k_{i-1}/(k_{i-1} + 1), k_i/(k_i + 1)], \quad i = 1, 2, 3, \dots \\ x, & t = 1. \end{cases}$$

Then $\gamma$ is a simple arc connecting $y$ to $x$, and this proves (a).

(b) We only need to consider the case where $x, z$ are distinct points in $\partial T$. Let $y \in T^\circ$. Then by part (a) above, there exists a simple arc $\gamma_1 : [0, 1] \to T$ connecting $y$ to $x$ with $\gamma_1(0) = y$, $\gamma_1(1) = x$, and $\gamma_1[0, 1) \subseteq T^\circ$. Similarly, there exists a simple arc $\gamma_2$ connecting $y$ to $z$ with the analogous properties. Let

$$t_0 := \sup\big\{t \in [0, 1] : \gamma_1(t) = \gamma_2(s) \text{ for } s \in [0, 1]\big\} < 1.$$

Define $s_0$ to be the unique number such that $\gamma_2(s_0) = \gamma_1(t_0)$. Then, after a suitable reparametrization, $\gamma_1[t_0, 1] \cup \gamma_2[s_0, 1]$ is a required simple arc joining $x$ and $z$. □

Before we proceed, we state the Jordan Curve Theorem (see, for example, [HS], [M]) and a related result. They will be used frequently throughout the rest of this paper.

**Theorem 2.4.** *(Jordan Curve Theorem.) Let $\gamma$ be a simple closed curve in $\mathbb{R}^2$. Then*

(a) *$\mathbb{R}^2 \setminus \gamma$ consists of a bounded component $U$ and an unbounded component $V$;*

(b) *$\gamma = \overline{U} \setminus U = \overline{V} \setminus V$.*

In the proof of Proposition 2.6, we will use the version of the Jordan Curve Theorem on $\mathbb{S}^2$, the two-sphere: If $\gamma$ is a simple closed curve in $\mathbb{S}^2$, then $\mathbb{S}^2 \setminus \gamma$ consists of two regions.

Let $\gamma$ be a simple closed curve in $\mathbb{R}^2$ and let $R_\gamma$ denote the bounded component of $\mathbb{R}^2 \setminus \gamma$. We say that a set $S$ is *enclosed* by $\gamma$ if $S \subseteq R_\gamma$. For an arc $\alpha$ with end-points $x, y$, we call $\alpha' = \alpha \setminus \{x, y\}$ the *interior* of $\alpha$. Let $x, y$ be distinct points in $\mathbb{R}^2$, and suppose $\alpha, \beta$ are arcs joining $x$ and $y$, respectively. Then $\alpha, \beta$ are said to be *independent arcs* from $x$ to $y$ if $\alpha \cap \beta = \{x, y\}$.

The following result can be found in, e.g., [HS, Theorem 9.11, p.244], or [M, Theorem 7, p.20; Theorem 13, p.79].

**Lemma 2.5.** *Let $\alpha, \beta, \gamma$ be simple arcs in $\mathbb{R}^2$ joining two distinct points $x, y \in \mathbb{R}^2$. Suppose $\alpha, \beta, \gamma$ are mutually independent. Then*

(a) *Exactly one of the interior of the arcs is enclosed by the union of the other two arcs. Precisely, exactly one of the following is true: $\alpha' \subseteq R_{\beta\cup\gamma}$, $\beta' \subseteq R_{\alpha\cup\gamma}$ or $\gamma' \subseteq R_{\alpha\cup\beta}$.*

(b) *In the case $\gamma' \subseteq R_{\alpha\cup\beta}$, we have $R_{\alpha\cup\beta} = R_{\alpha\cup\gamma} \cup R_{\beta\cup\gamma} \cup \gamma'$. Similar results hold for the other two cases.*

We now show that if $T^\circ$ consists of only one component, then $T$ does not have holes.

**Proposition 2.6.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$ with $T^\circ$ being connected. Then $T^c$ is connected.*

*Proof.* Suppose the conclusion is false. Then $T^c$ is a disjoint union of open connected components $\cup_{i=0}^\infty V_i$, with $V_0$ being the only unbounded component and with $V_1 \neq \emptyset$.

Let $\{T_i\}_{i=0}^\infty$ be a tiling of $\mathbb{R}^2$ consisting of congruent copies of $T$, with $T_0 = T$. There is some $j \neq 0$ with $T_j \cap V_1 \neq \emptyset$, and therefore $T_j^\circ \cap V_1 \neq \emptyset$. Since $T_j^\circ \subseteq \cup_{i=0}^\infty V_i$ and $T_j^\circ$ is connected, it follows that $T_j^\circ \subseteq V_1$. In the following we will show that $\operatorname{diam}(T_j) = \operatorname{diam}(T) > \operatorname{diam}(V_1)$. This contradiction will complete the proof. That $T$ being a reptile or self-affine tile is needed in the reasoning. Example 5.2 shows a set $T$ enclosing a bounded component $V_1$ of $T^c$, with $\operatorname{diam}(T) = \operatorname{diam}(V_1)$.

Let $a, b \in \overline{V_1}$ such that

$$\|a - b\| = \operatorname{diam}(\overline{V_1}) = \operatorname{diam}(V_1) > 0.$$

Note that such $a, b$ exist because $\overline{V_1}$ is compact. Moreover, $a, b \in \partial V_1 \cap T$. Let $L[a, b]$ be the line segment joining $a$ and $b$, including the end-points. Let $L$ be the straight line containing $L[a, b]$. We claim that there must be a point $c \in L \setminus L[a, b]$ such that $c \in T$. This will finish the proof. Suppose this is false, i.e., $(L \setminus L[a, b]) \cap T = \emptyset$. We will derive a contradiction.

Let

$$S := \big(L \setminus L(a, b)\big) \bigcup \overline{V_1},$$

where $L(a,b) := [a,b] \setminus \{a,b\}$. For convenience, we assume that $L$ is the real axis in $\mathbb{R}^2$, $a$ lies on the positive part of the axis, and $b$ lies on the negative part.

*Step 1.* We will show that $\mathbb{R}^2 \setminus S$ consists of at least two components, with two of which being unbounded.

We first show that there is a simple arc $\gamma \subseteq \overline{V_1}$ joining $a$ and $b$ such that $\gamma' \subseteq V_1$. In the tiling $\{T_i\}$, suppose $T_1, \dots, T_m$ is the collection of all tiles with $T_i \cap V_1 \neq \emptyset$. Then $\overline{V_1} \subseteq \cup_{i=1}^m T_i$. On the other hand, it follows from the same argument as above that $T_i^\circ \subseteq V_1$ for $i = 1, \dots, m$. Consequently, $\overline{V_1} = \cup_{i=1}^m T_i$.

If $a$ and $b$ are in the same $T_i$, Proposition 2.3 gives the required $\gamma$. Suppose they belong to different pieces, say, $a \in T_1$ and $b \in T_2$. Pick $a' \in T_1^\circ$ and $b' \in T_2^\circ$. By Proposition 2.3, there are simple arcs $\gamma_{a,a'}$ and $\gamma_{b,b'}$, with respectively $a, a'$ and $b, b'$ as end-points, such that $\gamma'_{a,a'} \subseteq T_1^\circ$ and $\gamma'_{b,b'} \subseteq T_2^\circ$. Since $V_1$ is open and connected, there exists a simple arc $\gamma_{a',b'} \subseteq V_1$ joining $a'$ and $b'$. Then $\gamma := \gamma_{a,a'} \cup \gamma_{b,b'} \cup \gamma_{a',b'} \subseteq \overline{V_1}$ is a desired arc. In case it is not simple, use the technique in Proposition 2.3 to construct a simple arc from it. We still denote the resulting simple arc by $\gamma$. (In example 5.3, $(0, \pm 3)$ cannot be joint by an arc $\gamma \subseteq \overline{V_1}$ with $\gamma' \subseteq V_1$.)

As a consequence of the Jordan Curve Theorem on $\mathbb{S}^2$, applied to the simple closed curve $\sigma \cup \{\infty\}$, the simple curve $\sigma := (L \setminus L(a,b)) \cup \gamma$ divides $\mathbb{R}^2 \setminus \sigma$ into two unbounded regions $C_1$ and $C_2$, denoting respectively the northern and southern regions. Fix $R > 0$ such that $B_R(0,0) \supseteq \overline{V_1}$. Let $p_1 = (0, R+1)$, $p_2 = (0, -R-1)$. Then $p_i \in \mathbb{R}^2 \setminus S$, $i = 1,2$. Let $D_i := C_i \setminus S$, $i = 1,2$. Let $U_i$ be the component of $D_i$ containing $p_i$. Then $U_i \subseteq C_i$ are unbounded components of $\mathbb{R}^2 \setminus S$.

*Step 2.* $T^\circ$ has empty intersection with an unbounded component $U_1$ or $U_2$. In fact, under our assumption that $(L \setminus L[a,b]) \cap T = \emptyset$, we have $T \cap \sigma = \{a,b\}$. Hence $a, b \in \partial T$. If $T^\circ \cap U_i \neq \emptyset$ for $i = 1,2$, then $T^\circ$ is separated, contradictory to the hypothesis that $T^\circ$ is connected.

*Step 3.* Assume without loss of generality that $T^\circ \subseteq C_2$. In other words, we assume that $T^\circ \subseteq U_2$ or if $T^\circ$ is contained in a bounded component $U_3$ of $\mathbb{R}^2 \setminus S$, $U_3 \subseteq C_2$. We show that $V_1$ cannot be a component of $T^c$, by showing that it is not maximal connected.

Let $y_0 := \inf\{y : (0,y) \in U_1\}$. Then $p = (0, y_0) \in \partial U_1 \cap \partial V_1$. Since $\overline{U_1} \subseteq \overline{C_1}$, we have $p \in \overline{C_1}$. Also as $\gamma' \subseteq V_1$, $p$ doesn't belong to $\gamma'$ and hence cannot belong to $\sigma = \partial C_1$. We conclude that $p \in C_1$. Now, $T \subseteq \overline{C_2} = C_1^c$. Therefore there is an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq T^c$. Consequently $U_1 \cup V_1 \cup B_\epsilon(p)$ is a connected set in $T^c$, contradicting that $V_1$ is a component of $T^c$. □

**Remark 2.1.** *Using Proposition 2.6, we can explain why Theorem 1.1 is plausible. We show that a simple closed curve $\gamma : [0,1] \to T$ is null-homotopic in $T$.*

*Proof.* Assume $\gamma(0) = \gamma(1) = x$. Let $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$. Let $\overline{D}$ be the union of $\mathbb{S}^1$ and the disk it encloses. Define $\Gamma : \mathbb{S}^1 \to \mathbb{R}^2$ as

$$\Gamma\big(\cos(2\pi s), \sin(2\pi s)\big) = \gamma(s), \quad s \in [0, 1].$$

Then $\Gamma$ is a homeomorphism of $\mathbb{S}^1$ onto $\gamma[0, 1]$. By Schönflies Theorem [HS], $\Gamma$ can be extended to a homeomorphism $\overline{\Gamma} : \overline{D} \to R_\gamma \cup \gamma$, where $R_\gamma$ is the region bounded by $\gamma$. Define $G : [0, 1] \times [0, 1] \to \overline{D}$ as

$$G(s, 0) = \big(\cos(2\pi s), \sin(2\pi s)\big) \quad \text{and} \quad G(s, t) = (1 - t)G(s, 0) + (t, 0).$$

Then $H = \overline{\Gamma} \circ G$ is a homotopy between $\gamma$ and the constant curve $x$ in $\mathbb{R}^2$. As $T^c$ is connected, it has only one component, which is unbounded. It follows that $R_\gamma \subseteq T$. Therefore $H$ is a homotopy in $T$. □

We will not pursue a direct proof of Theorem 1.1. Instead, we will prove the stronger result Theorem 1.2, from which Theorem 1.1 follows.

## 3. Neighbors of $k$-level pieces

Let $R$ be a $k$-th level piece of $T$. $x \in \partial R$ is a *$k$-th level vertex point* (or simply *vertex point*) of $R$ if each of its neighborhoods has nonempty intersection with more than two $k$-th level pieces. Our goal in this section is to show that if $T^\circ$ is connected, then the number of vertex points of any $k$-th level piece is finite. We find this result to be of interest in its own right.

**Lemma 3.1.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$ with $T^\circ$ being connected. Let $a, b, c$ be three distinct points on $\partial T$. Then there exist $y \in T^\circ$ and simple arcs $\gamma_1, \gamma_2, \gamma_3$ connecting $y$ with $a, b,$ and $c$ respectively, such that the three arcs lie in $T^\circ$ except at the end-points, and they do not intersect each other except at $y$.*

*Proof.* Let $x \in T^\circ$. By Proposition 2.3, we can join $x$ and $a, b, c$ respectively by simple arcs $\alpha_1, \alpha_2, \alpha_3$ that lie in $T^\circ$ except at the three end-points $a$, $b$, and $c$.

Start from $x$ and move along $\alpha_1$ until it reaches its last intersection with either of the other two arcs. Denote the intersection by $x_1$ and note that $x_1 \in T^\circ$. We may assume without loss of generality that $\alpha_1$ last hits $\alpha_2$. Discard the portion of $\alpha_1$ before the intersection and denote the remaining arc by $\beta_1$. In other words, suppose $\alpha_1 : [0, 1] \to T$ is a parametrization of $\alpha_1$ with $\alpha_1(0) = x, \alpha_1(1) = a$. Let

$$s := \sup\big\{t \in [0, 1] : \alpha_1(t) \in \big(\alpha_2[0, 1] \cup \alpha_3[0, 1]\big)\big\} < 1,$$

$x_1 := \alpha_1(s)$ and $\beta_1 := \alpha_1 \setminus \alpha_1[0, s)$. Note that $\beta_1$ does not intersect $\alpha_2$ or $\alpha_3$ except at the end-point $x_1$.

Now, start from $x$ and move along $\alpha_3$ until it last intersects $\alpha_2$, at say $x_2 \in T^\circ$. Discard the portion of $\alpha_3$ before the intersection and denote the remaining arc by $\beta_2$.

Lastly, start from $x$, go along $\alpha_2$, and discard the portion of $\alpha_2$ before the two intersections $x_1$ and $x_2$ (which may coincide). Denote the part of $\alpha_2$ from $b$ to the second intersection by $\beta_3$ and the part between the intersections by $\beta_4$. The lemma now follows by letting $y$ be the second intersection, and defining

$$\gamma_1 := \beta_1, \quad \gamma_2 := \beta_3, \quad \gamma_3 := \beta_2 \cup \beta_4, \quad \text{or} \quad \gamma_1 := \beta_1 \cup \beta_4, \quad \gamma_2 := \beta_3, \quad \gamma_3 := \beta_2,$$

depending respectively on whether the second intersection is $x_1$ or $x_2$. □

**Theorem 3.2.** *Let $T$ be a reptile or a self-affine tile in $\mathbb{R}^2$ with $T^\circ$ being connected. Then the number of vertex points of any $k$-th level piece is finite.*

*Proof.* Suppose on the contrary that some $k$-th level piece $R$ has infinitely many distinct vertex points $\{x_n\}$. Let $x_n \to x$. Then $x$ must also be a vertex point of $R$. Moreover, by taking a subsequence if necessary, we can assume that there exists a $k$-th level neighbor $S$ of $R$ such that $x_n \in \partial S$ for all $n \geq 1$. Here $S$ could be a $k$-th level piece of a neighbor of $T$.

Choose $\delta > 0$ small enough so that the $\delta$-ball $B_\delta(x)$ centered at $x$ is contained in the neighborhood of $x$ consisting of $k$-th level pieces, and moreover, we assume $2\delta < \mathrm{diam}(R)$.

We now let $m > k$ be sufficiently large so that the $m$-th level pieces that form a neighborhood of $x$ are completely contained in $B_\delta(x)$. We first claim that there exist $m$-th level pieces $P \subseteq R$ and $Q \subseteq S$ such that $\partial P \cap \partial Q$ contains both $x$ and a subsequence of $\{x_n\}$ converging to $x$. In fact, since there are only finitely many $m$-th level pieces contained in $R$, one of them must contain a subsequence of the $\{x_n\}$. We call this piece $P$. By compactness, $P$ must contain $x$. Now, taking the subsequence of $\{x_n\}$ obtained and repeating the same argument for $S$ yields the existence of $Q$. By taking a subsequence, we may further assume that $\{x_n\} \subseteq \partial P \cap \partial Q$.

Consider the three distinct points $x, x_1, x_2 \in \partial P \cap \partial Q$. By Lemma 3.1, there exist $y \in P^\circ$ and three arcs $\gamma_{x,y}, \gamma_{x_1,y}, \gamma_{x_2,y}$, joining $y$ to $x, x_1, x_2$ respectively, such that all arcs lie in $P^\circ$ except at the end-points $x, x_1, x_2$ and they are nonintersecting except at $y$. Similarly, there exist $z \in Q^\circ$ and three arcs $\gamma_{x,z}, \gamma_{x_1,z}, \gamma_{x_2,z}$ with the analogous properties. Define three arcs joining $y$ and $z$ as

$$\alpha := \gamma_{x,y} \cup \gamma_{x,z}, \quad \beta := \gamma_{x_1,y} \cup \gamma_{x_1,z}, \quad \gamma := \gamma_{x_2,y} \cup \gamma_{x_2,z}.$$

Obviously, $\alpha, \beta, \gamma$ are simple and mutually independent. By Lemma 2.5, one of the $\alpha', \beta', \gamma'$ is enclosed by the union of the other two arcs. In other words, one of the simple closed curves $\alpha \cup \beta$, $\alpha \cup \gamma$ or $\beta \cup \gamma$ must enclose a vertex point in $\{x, x_1, x_2\}$. Therefore it must enclose points from more than two $k$-th level pieces, and hence must enclose an entire $k$-th level piece different from $R$ and $S$. On the other hand, since $2\delta < \mathrm{diam}(R)$, none of the closed curves $\alpha \cup \beta$, $\alpha \cup \gamma$, or $\beta \cup \gamma$ can enclose any $k$-th level piece. This contradiction completes the proof. □

Example 5.3 shows a tile $T$ in $\mathbb{R}^2$ with a connected interior together with a tiling. $T$ has infinitely many vertex points. $T$ is neither a reptile nor a self-affine tile.

## 4. Proof of Theorem 1.2

Let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve. We will call $\gamma' = \gamma(a,b)$ an *open arc*. Recall that a *region* is a nonempty open connect subset of $\mathbb{R}^2$. The Jordan Curve Theorem will be used frequently in this section, often without mention.

**Lemma 4.1.** *Let $\alpha$ and $\beta$ be independent simple arcs in $\mathbb{R}^2$ joining two distinct points $x, y \in \mathbb{R}^2$. Let $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$. Let $U = R_{\alpha\cup\beta}$ be the region bounded by $\alpha \cup \beta$. Let $0 < r < \|x - y\|$ and let $C$ be the circle of radius $r$ centered at $x$. Then $C \cap U \neq \emptyset$, and is the disjoint union of countably many maximal open circular arcs. One of these arcs has its two end-points lying on different arcs $\alpha$ and $\beta$.*

*Proof.* It suffices to prove the last statement.

We use the notation $\Gamma(\xi, \eta)$ to denote the open circular arc in $C \cap U$ with end-points $\xi$ and $\eta$. This is unambiguous because there is only one such open arc with these two end-points. In this notation, the ordering of $\xi$ and $\eta$ does not matter. Also, the open arcs in $C \cap U$ are of the form $\Gamma(\alpha(a), \alpha(b))$, $\Gamma(\beta(a), \beta(b))$, or $\Gamma(\alpha(a), \beta(b))$, with $a, b \in (0, 1)$. We are claiming that one of them is of the form $\Gamma(\alpha(a), \beta(b))$. We will suppose that this is not true and derive a contradiction.

Let $\mathcal{A}$ be the set of all maximal open circular arcs in $C \cap U$. Let $\mathcal{A}_\alpha \subseteq \mathcal{A}$ be the set of those arcs with end-points lying on $\alpha$. Define $\mathcal{A}_\beta$ similarly. Our supposition means that $\mathcal{A} = \mathcal{A}_\alpha \cup \mathcal{A}_\beta$. The derivation of a contradiction is divided into the following eight steps.

*Step 1.* Consider arcs $\Gamma(\alpha(a), \alpha(b)) \in \mathcal{A}_\alpha$. Assume that $a < b$. We claim that for different arcs $\Gamma(\alpha(a), \alpha(b)), \Gamma(\alpha(a'), \alpha(b')) \in \mathcal{A}$, either $(a, b) \cap (a', b') = \emptyset$, or they are nested: one is included in another. We postpone the proof of this claim to Lemma 4.2.

Notice that $(a, b)$ is not the domain of a parametrization of $\Gamma(\alpha(a), \alpha(b)) \in \mathcal{A}$. Instead, it is a parametrizing interval for $\alpha(a, b)$. Indeed, $\alpha(a, b) \cap \Gamma(\alpha(a), \alpha(b)) = \emptyset$. $(a, b), (a', b') \in \mathcal{A}_\alpha$ can nest, even though $\Gamma(\alpha(a), \alpha(b))$ and $\Gamma(\alpha(a'), \alpha(b'))$ cannot intersect by definition.

*Step 2.* There cannot exist an infinite increasing sequence of nested intervals $(a_j, b_j)$ with $\Gamma(\alpha(a_j), \alpha(b_j)) \in \mathcal{A}$. Suppose there is such a sequence. The length of the arcs $\Gamma(\alpha(a_j), \alpha(b_j))$ tends to 0 as $i \to \infty$. So $\alpha(\lim a_j) = \lim \alpha(a_j) = \lim \alpha(b_j) = \alpha(\lim b_j)$. The simplicity of $\alpha$ now implies that $\lim a_j = \lim b_j$, which is impossible for an increasing sequence of $(a_j, b_j)$. As a consequence, for every nested sequence of $\{(a_j, b_j)\}$, there is a biggest one that contains every interval in the sequence.

Let $\mathcal{I}_\alpha$ denote the set of maximal intervals $\{(a_i, b_i)\}_{i\in I}$ with $\Gamma(\alpha(a_i), \alpha(b_i)) \in \mathcal{A}_\alpha$. The index set $I = \{1, \dots, N\}$ for some positive integer $N$ or $I = \{1, 2, \dots\}$. Suppose the index $i$ order $\mathcal{I}_\alpha$ in decreasing length of its elements.

*Step 3.* Construct inductively, for each $i \in I$, a simple arc $\alpha_i$ connecting $x$ and $y$ as follows. $\alpha_0 := \alpha$. For $i \geq 0$, let $\alpha_{i+1}$ be the simple arc obtained from $\alpha_i$ by replacing $\alpha_i(a_{i+1}, b_{i+1}) = \alpha(a_{i+1}, b_{i+1})$ with the open circular arc $\Gamma(\alpha(a_{i+1}), \alpha(b_{i+1}))$. More precisely, using complex notation for convenience, suppose that $\alpha(a_{i+1}) = x + r\exp(\boldsymbol{i}\theta_0)$, $\alpha(b_{i+1}) = x + r\exp(\boldsymbol{i}\theta_1)$, where $\boldsymbol{i} := \sqrt{-1}$. Define $\alpha_{i+1}$ by

$$\alpha_{i+1}(t) := \begin{cases} \alpha_i(t), & t \in [0,1] \setminus (a_{i+1}, b_{i+1}) \\ x + r\exp\left[\boldsymbol{i}\Big(\frac{b_{i+1}-t}{b_{i+1}-a_{i+1}}\theta_0 + \frac{t-a_{i+1}}{b_{i+1}-a_{i+1}}\theta_1\Big)\right], & t \in (a_{i+1}, b_{i+1}). \end{cases}$$

If $I$ is finite with $N = \max(I)$, we simply define $\alpha_{i+1} := \alpha_N$, for all $i \geq N$.

*Step 4.* Since $\alpha, \beta$ do not cross the open circular arcs, the $\alpha_i$ are simple and $\alpha_i \cup \beta$ are simple closed curves. Let $U_0 := U$. For $i \in I$, let $U_i$ be the region bounded by $\alpha_i \cup \beta$. Let $W_i \subseteq U_{i-1}$ be the region bounded by $\alpha[a_i, b_i] \cup \Gamma(\alpha(a_i), \alpha(b_i))$. Then $U_{i+1} = U_i \setminus \overline{W}_{i+1}$.

Consider the arcs $\Gamma(\alpha(a), \alpha(b)) \in \mathcal{A}_\alpha$ with $(a, b) \subsetneqq (a_{i+1}, b_{i+1})$. Notice that $\alpha(a), \alpha(b) \in \alpha[a_{i+1}, b_{i+1}] \subseteq \partial W_{i+1}$ and $\Gamma(\alpha(a), \alpha(b)) \subseteq U_i$. Hence $\Gamma(\alpha(a), \alpha(b)) \subseteq \overline{W}_{i+1}$. Also, it follows from the definition of $W_{i+1}$ that $\Gamma(\alpha(a_{i+1}), \alpha(b_{i+1})) \subseteq \overline{W}_{i+1}$. Therefore we have

$$C \cap U_{i+1} = (C \cap U_i) \setminus \big\{\Gamma(\alpha(a), \alpha(b)) \in \mathcal{A}_\alpha : (a, b) \subseteq (a_{i+1}, b_{i+1})\big\}.$$

*Step 5.* Observe that $\alpha$ is uniformly continuous and either $I$ is finite or $b_i - a_i \to 0$ as $i \to \infty$. Hence, $\{\alpha_i\}$ is uniformly convergent. Let $\alpha_*$ be the limit. Then $\alpha_*$ is an arc joining $x$ and $y$. We claim that $\alpha_*$ is simple. To see this, suppose $\alpha_*(t_1) = \alpha_*(t_2)$. If $t_1$ belongs to some $(a_i, b_i) \in \mathcal{I}_\alpha$, then $\alpha_m(t_1) = \alpha_i(t_1)$ for all $m \geq i$. If $t_1$ does not belong to any $(a_i, b_i) \in \mathcal{I}_\alpha$, then $\alpha_*(t_1) = \alpha(t_1) = \alpha_m(t_1)$ for all $m \geq 0$. Similarly, there exists some $j$ such that $\alpha_*(t_2) = \alpha_m(t_2)$ for all $m \geq j$ (recall that $\alpha_0 := \alpha$). Let $k = \max\{i, j\}$. Then $\alpha_k(t_1) = \alpha_k(t_2)$. The simplicity of $\alpha_k$ implies that $t_1 = t_2$.

Similarly, $\alpha_* \cup \beta$ is a simple closed curve. That is, $\alpha'_* \cap \beta = \emptyset$. In fact, if $\alpha_*(t_0) = \beta(s_0)$ for some $t_0 \in (0, 1)$ and $s_0 \in [0, 1]$, there would exist some $k$ such that $\alpha_k(t_0) = \alpha_*(t_0) = \beta(s_0)$, contradicting the simplicity of $\alpha_k \cup \beta$.

*Step 6.* Let $U_*$ be the bounded component of $\mathbb{R}^2 \setminus (\alpha_* \cup \beta)$. Then

$$U_* \subseteq \bigcap_{i=0}^{\infty} U_i.$$

Otherwise, we can derive a contradiction as follows.

Suppose $U_* \setminus \bigcap_{i=0}^{\infty} U_i \neq \emptyset$, then $U_* \setminus U_i \neq \emptyset$ for some $i$. We claim that $U_* \setminus \overline{U_i} \neq \emptyset$. To see this, pick $q \in U_* \setminus U_i$. If $q \in U_* \setminus \overline{U_i}$, we are done. Suppose $q \in \partial U_i$. Then $q \in \partial \overline{U_i}^c$, as the Jordan Curve Theorem says that $\partial U_i = \partial \overline{U_i}^c$. Let $B_\epsilon(q) \subseteq U_*$. Then $B_\epsilon(q)$ contains a point in $U_* \cap \overline{U_i}^c$, and the claim is established.

Now pick $p_1 \in U_* \cap \overline{U_i}^c$ and $p_2 \in \overline{U_*}^c \cap \overline{U_i}^c$. An arc joining them in $\overline{U_i}^c$ must contain a point $p \in \alpha_* \cap \overline{U_i}^c$. On the other hand, notice that for any $i$, $\alpha_j \subseteq \overline{U_i}$ for $j \geq i$. Hence $\alpha_* \subseteq \overline{U_i}$. This contradiction establishes the claim. It follows that

$$C \cap U_* \subseteq (C \cap U) \setminus \bigcup \{\varphi : \varphi \in \mathcal{A}_\alpha\} = \bigcup \{\varphi : \varphi \in \mathcal{A}_\beta\}.$$

*Step 7.* We repeat the above arguments for $\beta$ and $U_*$. Let $\beta_0 := \beta$, $V_0 := U_*$. Let $\mathcal{I}_\beta = \{(c_i, d_i)\}_{i \in J}$ be the collection of maximal intervals, a subset of the collection of intervals that correspond to maximal open circular arcs $\Gamma(\beta(c_i), \beta(d_i)) \in \mathcal{A}_\beta = C \cap U_*$. Order the elements of $\mathcal{I}_\beta$, say, in decreasing length of the intervals. Construct $\beta_i$ as we have constructed the $\alpha_i$. Let $V_i$ be the region bounded by $\alpha_* \cup \beta_i$. As before we have

$$C \cap V_{i+1} = (C \cap V_i) \setminus \big\{\Gamma(\beta(c), \beta(d)) \in \mathcal{A}_\beta : (c, d) \subseteq (c_{i+1}, d_{i+1})\big\}.$$

Pass to the limit to get $\beta_*$. Let $V_*$ be the region bounded by $\alpha_* \cup \beta_*$. Then we have $C \cap V_* = \emptyset$.

*Step 8.* We can get a contradiction now. Observe that $a = \inf\{a_i\} > 0$ and $c = \inf\{c_i\} > 0$. Hence $\alpha_*[0, a] = \alpha[0, a]$ and $\beta_*[0, c] = \beta[0, c]$. That is, the beginning portions of $\alpha$ and $\beta$ are unchanged. So $x \in \overline{V_*}$. Similarly, the tail portions remain unchanged and $y \in \overline{V_*}$. Since $V_*$ is connected, $C \cap V_* \neq \emptyset$, contradicting the conclusion in the last paragraph. □

To complete the proof of Lemma 4.1, it remains to establish the following.

**Lemma 4.2.** *Assume the same hypotheses of Lemma 4.1 and use the same notation as in the proof of it. Then the intervals $(a, b)$ with $\Gamma(\alpha(a), \alpha(b)) \in \mathcal{A}_\alpha$ are either disjoint or nested.*

*Proof.* Suppose the conclusion of the lemma does not hold. Let $\Gamma(\alpha(a_1), \alpha(b_1))$ and $\Gamma(\alpha(a_2), \alpha(b_2))$ be circular arcs in $\mathcal{A}_\alpha$ with $a_1 < a_2 < b_1 < b_2$. The ordering of $\alpha(a_1), \alpha(b_1), \alpha(a_2), \alpha(b_2)$ along $C$ does not affect the following reasoning. We use the additional notation $\Gamma[\xi, \eta] := \Gamma(\xi, \eta) \cup \{\xi, \eta\}$. Define simple arcs

$$\gamma_1 := \alpha[a_2, b_1], \quad \gamma_2 := \Gamma[\alpha(a_1), \alpha(b_1)] \cup \alpha[a_1, a_2], \quad \gamma_3 := \Gamma[\alpha(a_2), \alpha(b_2)] \cup \alpha[b_1, b_2].$$

It follows from definition and the simplicity of $\alpha$ that $\gamma_1, \gamma_2, \gamma_3$ are mutually independent arcs sharing the same end-points $\alpha(a_2)$ and $\alpha(b_1)$.

By Lemma 2.5, one of the open arcs $\gamma_i'$ is enclosed in the union of the other two arcs. We will derive a contradiction in each of the three cases below.

*Case 1.* $\gamma_1'$ is enclosed in $\gamma_2 \cup \gamma_3$.

We first claim that $R_{\gamma_1 \cup \gamma_2} \subseteq U = R_{\alpha \cup \beta}$. To see this, recall that $\Gamma(\alpha(a_1), \alpha(b_1)) \subseteq U$. Hence for any point $z \in \Gamma(\alpha(a_1), \alpha(b_1)) \subseteq \gamma_2$, there is an $\epsilon > 0$ such that $B_\epsilon(z) \subseteq U$. Since $B_\epsilon(z) \cap R_{\gamma_1 \cup \gamma_2} \neq \emptyset$, we have $U \cap R_{\gamma_1 \cup \gamma_2} \neq \emptyset$. Let $p_1 \in U \cap R_{\gamma_1 \cup \gamma_2}$. Now, suppose $R_{\gamma_1 \cup \gamma_2}$ contains some point $p_2 \in U^c$. Then an arc joining $p_1$ and $p_2$ in $R_{\gamma_1 \cup \gamma_2}$ must contain a point $p \in \partial U = \alpha \cup \beta$. So $p \in \tilde{\beta} := \alpha[0, a_1) \cup \alpha(b_1, 1] \cup \beta$ because $p \notin \gamma_1 \cup \gamma_2$. Since $\tilde{\beta}$ is connected and does not intersect $\gamma_1 \cup \gamma_2$, we have $\tilde{\beta} \subseteq R_{\gamma_1 \cup \gamma_2}$. In particular, $\alpha(b_2) \in \tilde{\beta} \subseteq R_{\gamma_1 \cup \gamma_2}$. This is a contradiction because $\alpha(b_2) \in \gamma_3'$ is not enclosed by $\gamma_1 \cup \gamma_2$. Thus, $R_{\gamma_1 \cup \gamma_2} \subseteq U$.

Similarly, $R_{\gamma_1 \cup \gamma_3} \subseteq U$.

Let $z \in \gamma_1' = \alpha(a_2, b_1)$. Then there is an $\epsilon > 0$ such that $B_\epsilon(z) \subseteq R_{\gamma_2 \cup \gamma_3}$. By Lemma 2.5, $B_\epsilon(z) \subseteq R_{\gamma_1 \cup \gamma_2} \cup R_{\gamma_1 \cup \gamma_3} \cup \gamma_1' \subseteq \overline{U}$. This contradicts that $z \in \alpha[a_2, b_1] \subseteq \partial U$.

*Case 2.* $\gamma_2'$ is enclosed by $\gamma_1 \cup \gamma_3$.

Again, $R_{\gamma_1 \cup \gamma_2} \subseteq U$. The reasoning is the same as above.

Focus on $R_{\gamma_2 \cup \gamma_3}$. We claim that it contains points from both $U$ and $\overline{U}^c$. It contains points of $U$, since $\Gamma(\alpha(a_1), \alpha(b_1)) \subseteq U \cap \partial R_{\gamma_2 \cup \gamma_3}$. $R_{\gamma_2 \cup \gamma_3}$ must also contain points from $\overline{U}^c$. Suppose otherwise, i.e., $R_{\gamma_2 \cup \gamma_3} \subseteq \overline{U}$. Then $R_{\gamma_2 \cup \gamma_3} \subseteq U$ as $R_{\gamma_2 \cup \gamma_3}$ is open. Let $z \in \alpha(a_1, a_2) \subseteq \gamma_2' \subseteq R_{\gamma_1 \cup \gamma_3}$. Let $\epsilon > 0$ be sufficiently small so that $B_\epsilon(z) \subseteq R_{\gamma_1 \cup \gamma_3}$. Then by Lemma 2.5, $B_\epsilon(z) \subseteq R_{\gamma_1 \cup \gamma_2} \cup R_{\gamma_2 \cup \gamma_3} \cup \gamma_2' \subseteq \overline{U}$, contradicting $z \in \alpha(a_1, a_2) \subseteq \partial U$.

As a result, $R_{\gamma_2 \cup \gamma_3} \cap \partial U \neq \emptyset$. In fact

$$R_{\gamma_2 \cup \gamma_3} \cap \partial U = \beta \cup \alpha[0, a_1) \cup \alpha(b_2, 1] := \gamma_4'. \tag{4.1}$$

The left is included in the right because $\alpha(a_2, b_1) = \gamma_1'$ is outside of $R_{\gamma_2 \cup \gamma_3}$, and $\alpha[a_1, a_2], \alpha[b_1, b_2]$ are parts of $\gamma_2$ and $\gamma_3$. For the reverse inclusion, notice that $\gamma_4'$ is connected, and does not cross $\gamma_2 \cup \gamma_3$. Hence the entire $\gamma_4'$ must be contained in $R_{\gamma_2 \cup \gamma_3}$. $\gamma_4$ is a simple arc joining $\alpha(a_1)$ and $\alpha(b_2)$. Now focus on the two subregions of $R_{\gamma_2 \cup \gamma_3}$ separated by $\gamma_4$, namely,

$$R_1 := R_{\gamma_4 \cup \alpha[b_1, b_2] \cup \Gamma(\alpha(a_1), \alpha(b_1))} \quad \text{and} \quad R_2 := R_{\gamma_4 \cup \alpha[a_1, a_2] \cup \Gamma(\alpha(a_2), \alpha(b_2))}.$$

They both contain points in $U$, since each of them has either $\Gamma(\alpha(a_1), \alpha(b_1)) \subseteq U$ or $\Gamma(\alpha(a_2), \alpha(b_2)) \subseteq U$ as part of its boundary. Since $\partial U \setminus \cup_{i=1}^4 \gamma_i = \emptyset$, $R_1, R_2$ must be subsets of $U$. But now look at a point $z \in \gamma_4'$. Let $\epsilon > 0$ be sufficiently small such that $B_\epsilon(z) \subseteq R_{\gamma_2 \cup \gamma_3} = R_1 \cup R_2 \cup \gamma_4' \subseteq \overline{U}$. This contradicts $z \in \gamma_4' \subseteq \partial U$.

*Case 3.* $\gamma_3'$ is enclosed by $\gamma_1 \cup \gamma_2$.

Derive a contradiction as in Case 2. ☐

**Lemma 4.3.** *Let $T$ be a reptile or self-affine tile in $\mathbb{R}^2$ with $T^\circ$ being connected and let $x \in \partial T$. Let $R, S \subseteq T$ be distinct $k$-th level pieces of $T$ containing $x$. Then there exist simple arcs $\gamma_r$, $\gamma_s$ and $\gamma_t$, which are mutually nonintersecting except at end-points and have the following properties.*

(a) *$\gamma_r \subseteq R$, has $x$ and $p_r \neq x$, $p_r \in \partial R$ as end-points, and $\gamma_r' \subseteq R^\circ$.*

(b) *$\gamma_s \subseteq S$, has $x$ and $p_s \neq x$, $p_s \in \partial S$ as end-points, and $\gamma_s' \subseteq S^\circ$.*

(c) *$\gamma_t \subseteq T^\circ \setminus (R^\circ \cup S^\circ)$, has $p_r$ and $p_s$ as end-points, and $\gamma_t' \subseteq T^\circ \setminus (R \cup S)$.*

*Hence $\gamma_r \cup \gamma_s \cup \gamma_t$ is a simple closed curve in $T$. In the case $p_r = p_s$, $\gamma_t$ is a point.*

*Proof.* Pick $y \in R^\circ$ and $z \in S^\circ$ arbitrarily. Let $D$ be the open unit disk in $\mathbb{R}^2$. By Theorem 2.1, there exists a homeomorphism $h : T^\circ \to D$. Let $\ell : [0,1] \to D$ parametrizes the line segment joining $h(y)$ and $h(z)$, i.e.,

$$\ell(t) = (1-t)h(y) + th(z), \quad t \in [0,1].$$

Define

$$\begin{aligned} t_0 &:= \sup\big\{t \in [0,1] : \ell(t) \in h(R^\circ)\big\}, & t_1 &:= \inf\big\{t \in [t_0,1] : \ell(t) \in h(S^\circ)\big\}, \\ q_r &:= \ell(t_0) \in \partial\big(h(R^\circ)\big), & q_s &:= \ell(t_1) \in \partial\big(h(S^\circ)\big), \\ p_r &:= h^{-1}(q_r) \in \partial R \cap T^\circ, & p_s &:= h^{-1}(q_s) \in \partial S \cap T^\circ. \end{aligned}$$

Notice that $t_0 \le t_1$. Let $\ell_1$ be the line segment joining $q_r$ and $q_s$. Then $\ell_1' \subseteq h(T^\circ) \setminus (h(R \cap T^\circ) \cup h(S \cap T^\circ))$. Let $\gamma_t := h^{-1}(\ell_1)$. Then $\gamma_t$ is a simple arc joining $p_r$ and $p_s$, with $\gamma_t' = h^{-1}(\ell_1') \subseteq T^\circ \setminus (R \cup S)$.

By Proposition 2.3, there exists a simple arc $\gamma_r$ in $R$ joining $x$ and $p_r$ such that $\gamma_r(0,1) \subseteq R^\circ$. Similarly, there is a simple arc $\gamma_s$ in $S$ joining $x$ and $p_s$ satisfying $\gamma_s(0,1) \subseteq S^\circ$.

It is straightforward to check that $\gamma_r$, $\gamma_s$ and $\gamma_t$ satisfy the required conditions. □

**Lemma 4.4.** *Assume the same hypotheses of Lemma 4.3. Let $U_k$ be the neighborhood of $x$ consisting of the $k$-th level pieces of $T$ and those of its neighbors. Then there is a simple curve $\sigma$ with end-points $y \in R^\circ$, $z \in S^\circ$ and $\sigma \subseteq U_k \cap T^\circ$.*

*Proof.* Let $\gamma = \gamma_r \cup \gamma_s \cup \gamma_t$ be the simple closed curve constructed in Lemma 4.3. Let $y$ and $z$ respectively be any point in $\gamma_r' \subseteq R^\circ$ and $\gamma_s' \subseteq S^\circ$. Notice that with the only exceptional point $x$, the portion of $\gamma$ starting from $y$, going through $x$ and ending at $z$ lies in $U_k \cap T^\circ$. We will modify a portion of $\gamma$ near $x$ to get a required simple curve.

Observe that $\operatorname{dist}(\{x\}, \gamma_t) > 0$. Hence there exists $\delta > 0$ such that

$$\overline{B_\delta(x)} \subseteq U_k, \quad \gamma_t \cap \overline{B_\delta(x)} = \emptyset, \quad \text{and} \quad \delta < \min\{\|x-y\|, \|x-z\|\}.$$

So $y, z$ are not in $\overline{B_\delta(x)}$. Let $C := \partial B_\delta(x)$.

Let $\mathbb{R}^2 \setminus \gamma = U \cup V$, where $U$ and $V$ are respectively the bounded and unbounded components. Since $T^c$ is connected (Proposition 2.6), $T^c \cap \gamma = \emptyset$, and $T^c \cap V \ne \emptyset$, it follows that $T^c \subseteq V$. So $U \subseteq T$ and hence $U \subseteq T^\circ$.

By Lemma 4.1, $C \cap U \subseteq T^\circ$ is a union of countably many disjoint maximal open circular arcs, with one of them having end-points lying on different arcs $\gamma_r'$ and $\gamma_s'$. Suppose it is $\Gamma(y', z')$, with $y' \in \gamma_r \subseteq R^\circ$, $z' \in \gamma_s' \subseteq S^\circ$. Let $\gamma_{y,y'} \subseteq \gamma_r'$ be the arc between $y$ and $y'$, and $\gamma_{z,z'} \subseteq \gamma_s'$ be the arc between $z$ and $z'$. Then

$$\sigma := \gamma_{y,y'} \cup \Gamma(y', z') \cup \gamma_{z,z'} \subseteq U_k \cap T^\circ$$

is a simple curve we want. □

**Remark 4.1.** *By Proposition 2.3 and Lemma 4.4, any pair of points in $R \setminus \partial T$ and $S \setminus \partial T$ can be joined by an arc in $U_k \cap T^\circ$.*

A boundary point $x$ of a simply connected region $\Omega \subseteq \mathbb{R}^2$ is called a *simple boundary point* if $x$ has the following property (see [R]). For every sequence $\{x_n\} \subseteq \Omega$, $x_n \to x$, there is a curve $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma[0,1) \subseteq \Omega$, and a sequence $\{t_n\} \subseteq (0,1)$, $t_n \to 1$, such that $\gamma(t_n) = x_n$.

We will use the following result (see [R]). If every boundary point of a bounded simply connected region $\Omega \subseteq \mathbb{R}^2$ is simple, then $\overline{\Omega}$ is homeomorphic to the closed unit disk.

*Proof of Theorem 1.2.* Since $T^\circ$ is an open disk (Theorem 2.1) it suffices to show that each point on $\partial T$ is simple. We use an argument similar to that in Bandt and Wang ([BW]).

Let $x \in \partial T$ and let $\{x_n\}$ be a sequence of points in $T^\circ$ converging to $x$. Let $U_k$ be the neighborhood of $x$ consisting of the $k$-th level pieces of $T$ and those of the neighbors of $T$ containing $x$. $x$ belongs to a finite number of pieces from $T$ and a finite number of pieces from the neighbors of $T$. Since $x_n \to x$, there is an $n_k$ such that $x_n \in U_k$ for all $n \geq n_k$. We define

$$\begin{aligned} n_k &: = \min\{n \in \mathbb{N} : x_n \in U_k \text{ for all } n \geq n_k\}, \\ X_k &: = \{x_n\} \cap (U_k \setminus U_{k+1}). \end{aligned}$$

Then $\{x_n\} = \cup_{k=0}^{\infty} X_k$. For each $X_k \neq \emptyset$ and for each $x_n \in X_k$, Remark 4.1 allows us to construct an arc $\sigma_{x_n,x_{n+1}} \subseteq U_k \cap T^\circ$ connecting $x_n$ and $x_{n+1}$. (Note that if $X_k = \{x_{n_k}, \ldots, x_{n_k+\ell}\}$, then $x_{n_k+\ell+1} \in X_{k'}$, where $k' = \inf\{m > k : X_m \neq \emptyset\}$.) Define a curve $\gamma : [0,1] \to T$, with

$$\gamma\left(\frac{n-1}{n}\right) := x_n, \quad \gamma\left[\frac{n-1}{n}, \frac{n}{n+1}\right] := \sigma_{x_n,x_{n+1}}, \quad \gamma(1) := x.$$

Since $\cup_{n \geq n_k} \sigma_{x_n,x_{n+1}} \subseteq U_k$ and $\lim_{k\to\infty} \operatorname{diam}(U_k) \to 0$, $\gamma$ is really continuous at 1, and hence is a curve. Therefore $x$ is simple. This completes the proof of Theorem 1.2. □

## 5. Examples and comments

The following is an example of an open topological disk $E$ whose closure $\overline{E}$ is simply connected but is not a topological disk.

**Example 5.1.** *Let $\delta > 0$ and let $E$ be the open topological disk defined by an infinite union of rectangles as*

$$E := \Big((-\delta, 1) \times (-\delta, 0)\Big) \bigcup \Big((-\delta, 0) \times [0,1)\Big) \bigcup \left(\bigcup_{k=0}^{\infty} \left(\frac{3}{2^{k+2}}, \frac{1}{2^k}\right) \times [0,1)\right).$$

*(See Figure 1.) Then $\overline{E}$ is simply connected but not a topological disk.*

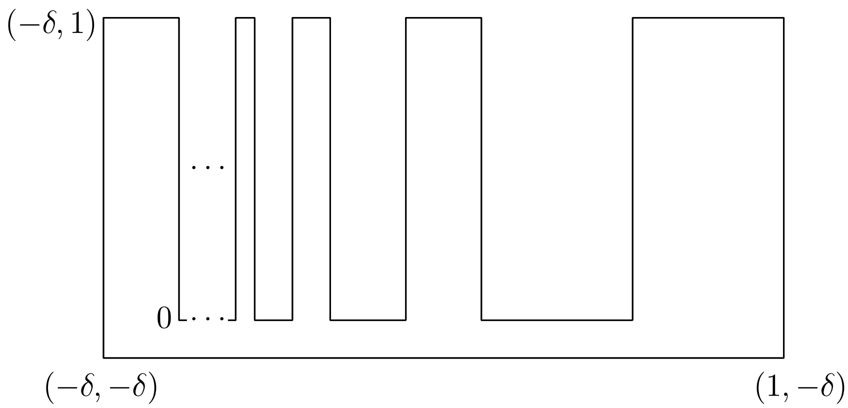


FIGURE 1. The "comb" like set $\overline{E}$ in Example 5.1.

*Proof.* It is easy to see that $\overline{E}$ is contractible and hence it is simply connected. If $\overline{E}$ is homeomorphic to the closed unit disk, $\partial E$ would be homeomorphic to $\mathbb{S}^1$. However, $\partial E$ cannot be the image of a curve. To see this suppose the contrary holds. Let $\alpha : [0,1] \to \mathbb{R}^2$ be a parametrization of $\partial E$, with $\alpha(t_0) = (0,0)$ for some $t_0 \in (0,1)$. Now for whatever $\delta > 0$, $\alpha(t_0 - \delta, t_0 + \delta)$ is not contained in $B_{1/2}(0,0)$. Therefore $\alpha$ cannot be continuous at $t_0$, contradicting that $\alpha$ is a parametrization of $\partial E$. □

In the following example, $T$ is a set with connected interior and $T$ "encloses" a region with the same diameter. This is shown in the proof of Proposition 2.6 to be impossible if $T$ is a reptile or a self-affine tile with connected interior.

**Example 5.2.** *Let $P$ be the closed and filled triangle with vertices at $(0, \pm 3)$ and $(1, 0)$. Let $P^\epsilon := \{x \in P, d(x, \partial P) > \epsilon\}$. Let $Q$ be a portion of a fattened, trumpet-shaped topological sine curve:*

$$Q := \left\{(x,y) : x \in (0, 1/4), y \in \left(2e^{-2x}\Big(\sin\frac{1}{x} - \frac{1}{2}\Big), 2e^{-2x}\Big(\sin\frac{1}{x} + \frac{1}{2}\Big)\right)\right\}.$$

*(See Figure 2.) Let $\epsilon < 1/4$ be sufficiently small. Let $T := P \setminus (P^\epsilon \cup Q)$. Then $T^c$ consists of two components $V_0 := \mathbb{R}^2 \setminus P$, $V_1 := P^\epsilon \cup Q$. We have diam$(V_1) =$ diam$(T) = 6$, and $\mu(V_1) > \mu(T)$, where $\mu$ is the two-dimensional Lebesgue measure.*

*Proof.* Notice that $Q$ is really included in $P$. The reason is as follows. The fattened topologist's sine curve $Q$ is sandwiched between $3e^{-2x}$ and $-3e^{-2x}$, which intersect the $y$-axis at $\pm 3$. Moreover, for $x \in (0, 1/4)$, $d(3e^{-2x})/dx = -6e^{-2x} < -6e^{-1/2} < -3$, the slope of the upper side of the triangle. As a consequence, $V_1 = P^\epsilon \cup Q \subseteq P$.

$V_0$ and $V_1$ are separated, since the line segment with $(0, \pm 3)$ as end-points is in $T$. It is straightforward to check that diam$(V_1) =$ diam$(T) = 6$. Lastly, it is clear that for all $\epsilon > 0$ sufficiently small, $\mu(V_1) > \mu(P^\epsilon) > \mu(T)$. □

FIGURE 2. The sets in Example 5.2. The outer triangle is $P$ and the inner triangle is $P^\epsilon$. The fattened topologist's sine curve $Q$ lies inside $P$.

Another variant of the fattened topologist's sine curve provides an example of a tile in $\mathbb{R}^2$ with connected interior. It is not a reptile and exhibits properties very different from those described in this paper.

(a) The tile $T$.

(b) $T$ with two vertical neighbors.

FIGURE 3. Figure (a) shows the tile $T$ drawn with $\ell = 1$. Figure (b) shows $T$ together with $T + (0, 1)$ and $T + (0, -1)$, indicating how $T$ tiles. Figure (b) also show that each neighborhood of any point on $T_1$ intersects at least three pieces on the right and at least one piece on the left.

**Example 5.3.** *Let $n = 2\ell$ be a positive even integer. Let $T = T_1 \cup T_2$, where*

$$T_1 = \big\{(0, y) \in \mathbb{R}^2 : y \in [-1 - 1/n, 1 + 1/n]\big\},$$
$$T_2 = \big\{(x, y) \in \mathbb{R}^2 : x \in (0, 1], y \in [\sin(1/x) - 1/n, \sin(1/x) + 1/n]\big\}.$$

*(See Figure 3.) Let $\mathcal{L}$ be the lattice generated by the vectors $(1, 0)$ and $(0, 2/n)$. Then $T + \mathcal{L}$ is a tiling of $\mathbb{R}^2$. $T^\circ$ is connected, but $T$ is not arcwise connected. $T$ has infinitely many vertex points.*

*Proof.* It is obvious that $T^\circ$ is connected and that $T + \mathcal{L}$ is a tiling of $\mathbb{R}^2$. As the usual topologist's sine curve, $T$ is not arcwise connected.

As in Section 3, a *vertex point* is a point with every one of its neighborhoods having nonempty intersection with the interior of more than two neighbors. The only difference now is that we do not have vertex points of different levels. For a point $x \in T_1$, every one of its neighborhoods has nonempty intersection with the interiors of at least $n+1$ neighbors on the right, and at least one neighbor on the left. So points in $T_1$ are vertex points and $T$ contains infinitely many vertex points. The vertex points actually form a line segment. Figure 2 is drawn with $\ell = 1$. It shows the set $T$ together with two translates of $T$. According to Theorem 3.2, $T$ cannot be a reptile. □

The following example shows that for reptiles, Theorem 1.2 cannot be extended to $\mathbb{R}^3$.

**Example 5.4.** *Start with the closed rectangular solid* $[0,1] \times [0,4] \times [0,3]$*, which is composed of* 12 *unit cubes. Remove the cube* $[0,1] \times (2,3) \times (1,2)$ *from the solid, and adhere to it the cube* $[0,1] \times [1,2] \times [1,2]$*. Call the resulting set* $T$ *(see Figure 4). Then* $T^\circ$ *is connected,* $T$ *is a* 1728*-reptile, but* $T$ *is not simply connected.*

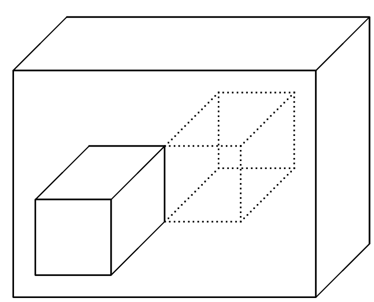

FIGURE 4. Figure showing the set $T$ in Example 5.4. The dotted cube is removed, and a unit cube is adhered to the front.

*Proof.* Obviously, $T^\circ$ is connected but $T$ is not simply connected. $T$ is a torus. Two copies of $T$ facing each other can be put together to form a rectangular solid, say, $R := [0,2] \times [0,4] \times [0,3]$. Therefore $T$ is a tile.

To see that $T$ is a reptile, we notice that the lengths of the sides of $R$ are 2, 3, and 4. Therefore 72 copies of $R$, or 144 copies of $T$, can be put together to form a cube $C$ of side length 12. Shrink $C$ down to a unit cube $C_1$ by a linear factor of $1/12$. Since $T$ can be tiled by 12 copies of $C_1$, it can be tiled by 1728 congruent pieces, each similar to $T$ itself, each shrunk by a linear factor of $1/12$. □

In view of Theorem 1.2, it is interesting to investigate the case in which $T^\circ$ is not connected. In particular, one can ask under what conditions is the closure of each component of $T^\circ$ simply connected. Note that the example by Grünbaum in [CFG, Figure C17] is a tile with disconnected interior such that the closure of one of the components is not simply connected. For reptiles with infinitely many components, it is known that the closure of each component of the Heighway dragon is a topological disk (see [NN]). The structure of the Lévy dragon is explored in Bailey *et al.* [BKS].

**Acknowledgements.** Part of this work was carried out while the first author was visiting the Department of Mathematics of the Chinese University of Hong Kong in 2001. He would like to thank Professor Ka-Sing Lau for his invitation and constant support, and he is grateful to members of the Department for their hospitality and help during his visits. Both authors would like to thank Hing-Sun Luk, Bruce McLean and Mary Russell for some helpful conversations.

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