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Unified Characterization of Exponential Objects in TOP, PRTOP and PARATOP

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UNIFIED CHARACTERIZATION OF EXPONENTIAL OBJECTS IN TOP, PRTOP AND PARATOP

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Abstract. A unified internal characterization of exponential objects in the categories of topological, pretopological and paratopological spaces (with continuous maps) is presented as an application of a theorem on product of $\mathbb{D}$-compactoid filters.

Résumé: Une caractérisation unifiée des objets exponentiels dans les catégories des espaces topologiques, prétopologiques et paratopologiques (munies des applications continues) est présentée comme application d'un théorème concernant les produits de filtres $\mathbb{D}$-compactoides.

1. Introduction and terminology

It is well known that the category **TOP** of topological spaces (and continuous maps) fails to be cartesian-closed, or in other words, fails to have "good" function spaces. Namely, there is in general no topology τ to put on sets $C(X,Z)$ of continuous functions from X to Z to ensure that the exponential law

$$C(X \times Y, Z) = C(Y, C_\tau(X,Z)) \tag{1.1}$$

is satisfied ([1]) for every triplet of topological spaces (X,Y,Z) (see [11], [1]). To remedy this situation, one can allow for more structures on $C(X,Z)$ than only topologies, i.e., embed **TOP** in a larger category that is cartesian-closed, or one can restrict the objects to those satisfying (1.1). More specifically, a topological space X is called *exponential* in **TOP** if for every topological space Z there exists a topology τ on $C(X,Z)$ such that (1.1) is satisfied for every topological space Y. Not surprisingly, a category used for the former approach would be instrumental in getting internal characterizations of exponential objects, as observed by F. Schwarz. More specifically, it is known from [17] that an object X of an epireflective subcategory **L** of a topological cartesian-closed category **C** is exponential in **L** if and only if the reflector $L : \mathbf{C} \to \mathbf{L}$ commutes with the product in the following way:

$$L(X \times Y) \leq X \times LY, \tag{1.2}$$

for every **C**-object Y. F. Schwarz used this approach (with $\mathbf{C} = \mathbf{Conv}$ the category of convergence spaces and continuous maps) to characterize exponential objects in **TOP**, while the author used it in [14], [15] to characterize among other things exponential objects in the categories **PRTOP** of pretopological spaces (which were first characterized in [12]) and **PARATOP** of paratopological spaces (and continuous maps). The later category was introduced by S. Dolecki [5] and is instrumental

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[1]The equality in (1.1) stands for a bijection via the transposition map ${}^t : C(X \times Y, Z) \to C(Y, C(X,Z))$ defined by ${}^t f(y)(x) = f(x,y)$.

to characterize countably biquotient maps, strongly Fréchet (also called countably bisequential) spaces and many other notions. But despite some similarity in both proofs and results, all the known internal characterizations of exponential objects in **PRTOP** and **TOP** needed separate proofs so far. It is the aim of this paper to present a long sought completely unified treatment of exponential objects in **TOP** and in **PRTOP**. The case of **PARATOP** is also obtained as a by-product. The key is to interpret convergence of a filter in various reflections (in **Conv**) of the underlying convergence structure in terms of compactoidness in the underlying convergence, for various classes of filters and relatively to various families.

Recall that by a *convergence space* (X, ξ) I mean a set endowed with a relation ξ between points of X and filters on X, denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \underset{\xi}{\to} x$, whenever x and $\mathcal{F}$ are in relation, and satisfying $\lim \mathcal{F} \subset \lim \mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$; $\{x\}^\uparrow \to x$ (2) for every $x \in X$ and $\lim (\mathcal{F} \wedge \mathcal{G}) = \lim \mathcal{F} \cap \lim \mathcal{G}$ for every filters $\mathcal{F}$ and $\mathcal{G}$. A map $f : (X, \xi) \to (Y, \tau)$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$. If ξ and τ are two convergences on X, we say that ξ *is finer than* τ,in symbols $\xi \geq \tau$, if $Id_X : (X, \xi) \to (X, \tau)$ is continuous. The category **Conv** of convergence spaces and continuous maps is topological (3) and cartesian-closed [4, Theorem 5] (4). A convergence is called *atomic* if it has at most one non-isolated point.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh,* in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A subset A of X is ξ-*closed* if $\lim_\xi \mathcal{F} \subset A$ whenever $A \in \mathcal{F}^{\#}$. The family of ξ-closed sets defines a topology $T\xi$ on X called *topological modification* of ξ. The neighborhood filter of $x \in X$ for this topology is denoted $\mathcal{N}_\xi(x)$ and the closure operator for this topology is denoted cl_ξ . A convergence is a topology if $x \in \lim_\xi \mathcal{N}_\xi(x)$. By definition, the adherence of a filter (in a convergence space) is:

$$\mathrm{adh}_\xi \mathcal{F} = \bigcup_{\mathcal{G}\#\mathcal{F}} \lim_\xi \mathcal{G}. \tag{1.3}$$

In particular, the adherence of a subset A of X is the adherence of its *principal filter* $\{A\}^\uparrow$. The *vicinity filter* $\mathcal{V}_\xi(x)$ of x for ξ is the infimum of the filters converging to x for ξ. A convergence ξ is a *pretopology* if $x \in \lim_\xi \mathcal{V}_\xi(x)$. Notice that a convergence ξ is respectively a topology, a pretopology, a *paratopology,* a *pseudotopology* if $x \in \lim_\xi \mathcal{F}$ whenever $x \in \bigcap_{\mathbb{D} \ni \mathcal{D}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{D}$, where $\mathbb{D}$ is respectively, the class $\mathrm{cl}_\xi^\natural(\mathbb{F}_1)$ of principal filters of ξ-closed sets, the class $\mathbb{F}_1$ of principal filters, the class $\mathbb{F}_\omega$ of countably based filters, the class $\mathbb{F}$ of all filters. In other words, the map $\mathrm{Adh}_\mathbb{D}$

(2) If $\mathcal{A} \subset 2^X$, $\mathcal{A}^\uparrow = \{B \subset X : \exists A \in \mathcal{A}, A \subset B\}$.

(3) In other words, for every sink $(f_i : (X_i, \xi_i) \to X)_{i\in I}$, there exists a *final convergence* structure on X : the finest convergence on X making each f_i continuous. Equivalently, for every source $(f_i : X \to (Y_i, \tau_i))_{i\in I}$ there exists an *initial convergence*: the coarsest convergence on X making each f_i continuous.

(4) In other words, for any pair (X, ξ), (Y, τ) of convergence spaces, there exists the coarsest convergence $[\xi, \tau]$ -called *continuous convergence*- on the set $C(\xi, \tau)$ of continuous functions from X to Y making the evaluation map

$$ev : (X, \xi) \times (C(\xi, \tau), [\xi, \tau]) \to (Y, \tau)$$

(jointly) continuous.

defined by

$$\lim_{\mathrm{Adh}_{\mathbb{D}}\xi}\mathcal{F}=\bigcap_{\mathbb{D}\ni\mathcal{D}\#\mathcal{F}}\mathrm{adh}_{\xi}\,\mathcal{D} \tag{1.4}$$

defines the reflector from **Conv** onto the (sub)category of respectively topological, pretopological, paratopological and pseudotopological spaces when $\mathbb{D}$ is respectively, the class $\mathrm{cl}_{\xi}^{\natural}(\mathbb{F}_1)$ of principal filters of ξ-closed sets, the class $\mathbb{F}_1$ of principal filters, the class $\mathbb{F}_{\omega}$ of countably based filters, the class $\mathbb{F}$ of all filters. A class of filters $\mathbb{D}$ (under mild conditions on $\mathbb{D}$ [5]) defines a reflective subcategory of **Conv** (and the associated reflector) via (1.4). Dually, it also defines (under mild conditions on $\mathbb{D}$) the coreflective subcategory of **Conv** of $\mathbb{D}$-*based convergence spaces*, and the associated coreflector $B_{\mathbb{D}}$ is

$$\lim_{B_{\mathbb{D}}\xi}\mathcal{F}=\bigcup_{\mathbb{D}\ni\mathcal{D}\leq\mathcal{F}}\lim_{\xi}\mathcal{D}. \tag{1.5}$$

If $o:2^X\to 2^X$ and $\mathcal{A}\subset 2^X$, then $o^{\natural}(\mathcal{A})=\{o(A):A\in\mathcal{A}\}$. If $\mathbb{D}$ is a class of filters, then $o^{\natural}(\mathbb{D})=\{\mathcal{D}\in\mathbb{D}:\mathcal{D}=o^{\natural}(\mathcal{D})\}$. If ξ and σ are two convergences on X, we say that ξ is σ-*regular* if $\lim_{\xi}\mathcal{F}\subset\lim_{\xi}\mathrm{adh}_{\sigma}^{\natural}(\mathcal{F})$ for every filter $\mathcal{F}$. To a convergence ξ, we can associate two (Alexandroff) topologies $\xi^{\bullet}$ and ξ^{*} defined by (see [7], [6] for details).

$$\mathrm{cl}_{\xi^{\bullet}}A=\bigcup_{x\in A}\mathrm{cl}_{\xi}\{x\}\text{ and }\mathrm{cl}_{\xi^{*}}A=\{y:\mathrm{cl}_{\xi}\{y\}\cap A\neq\varnothing\}.$$

Notice that

$$\mathcal{A}\#\,\mathrm{cl}_{\xi^{\bullet}}^{\natural}\mathcal{B}\iff\mathrm{cl}_{\xi^{*}}^{\natural}\mathcal{A}\#\mathcal{B}.$$

A convergence ξ that is ξ^{*}-regular is called $*$-*regular* [2] ([5]).

Let $\mathbb{D}$ be a class of filters on a convergence space (X,ξ) and let $\mathcal{A}$ be a family of subsets of X. A filter $\mathcal{F}$ is $\mathbb{D}$-*compactoid at* $\mathcal{A}$ (for ξ) if

$$\mathcal{D}\in\mathbb{D},\mathcal{D}\#\mathcal{F}\Longrightarrow\mathrm{adh}_{\xi}\,\mathcal{D}\#\mathcal{A}. \tag{1.6}$$

Notice that a subset K of a (topological or more generally convergence) space X is respectively compact, countably compact, Lindelöf if $\{K\}^{\uparrow}$ is $\mathbb{D}$-compactoid at $\{K\}$ if $\mathbb{D}$ is respectively, the class of all, of countably based, of countably deep ([6]) filters. Compactoidness not only generalizes compactness, but also convergence. In particular:

Theorem 1. *Let $\mathbb{D}$ be a class of filters.*

1. *$x\in\lim_{\mathrm{Adh}_{\mathbb{D}}\xi}\mathcal{F}$ if and only if $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\{x\}$ for ξ. In particular, $x\in\lim_{P\xi}\mathcal{F}$ if and only if $\mathcal{F}$ is $\mathbb{F}_1$-compactoid at $\{x\}$ for ξ.*
2. *$x\in\lim_{T\xi}\mathcal{F}$ if and only if $\mathcal{F}$ is $\mathbb{F}_1$-compactoid at $\mathcal{N}_{\xi}(x)$ for ξ.*

To be precise, answering a problem of F. Schwarz [17], [6], [14], [15] give characterizations of *quasi-exponential objects* in **L** (where **L** ranges over **TOP**, **PRTOP** and **PARATOP**) that is, of objects X satisfying (1.2) for every convergence space Y among objects of the cartesian-closed hull of **L** rather than just among **L**-object. Of course, exponential objects in **L** are the quasi-exponential objects that are also

[5] $\mathrm{cl}_{\xi^{\bullet}}A$ is often denoted $\downarrow A=\{y:\exists x\in A,\,y\sqsubseteq x\}$ where $\sqsubseteq$ denotes the specialization order (e.g., [9]) of the topology $T\xi$. Analogously $\mathrm{cl}_{\xi^{*}}A$ is $\uparrow A=\{y:\exists x\in A,\,x\sqsubseteq y\}$. Therefore, a convergence is $*$-regular in the sense of [3] if and only if it is *up-nice* in the sense of R. Heckmann [10].

[6] A filter $\mathcal{F}$ is *countably deep* if $\bigcap\mathcal{A}\in\mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

L-objects. In particular, calling a convergence space (X,ξ) *finitely generated if* $\xi = B_{\mathbb{F}_1}\xi$ and *bisequential* if $\xi \geq SB_{\mathbb{F}_\omega}\xi$ ([7]), we have:

Theorem 2. [14]

(1) *A pseudotopological space is quasi-exponential in **PRTOP** if and only if it is finitely generated.*
(2) *A pseudotopological space is quasi-exponential in **PARATOP** if and only if it is bisequential.*

A convergence ξ is called *core compact* if for every filter $\mathcal{F}$ with $x \in \lim_\xi \mathcal{F}$ and every $F \in \mathcal{F}$ there exists $K_F \in \mathcal{F}$ that is compactoid at F and T*-core compact* if for every filter $\mathcal{F}$ with $x \in \lim_\xi \mathcal{F}$ and every $V \in \mathcal{N}_\xi(x)$ there exists $F_V \in \mathcal{F}$ that is compactoid at V.

Theorem 3. [6]

(1) *A core compact convergence space is quasi-exponential in **TOP**;*
(2) *Every convergence space that is quasi-exponential in **TOP** is T-core compact.*

However, it was not known wether the two conditions are really different. In the present paper, I show that $*$-regular quasi-exponential objects in **TOP** are exactly the T-core compact ones, settling the question left in [6].

2. Product of $\mathbb{D}$-compactoid filters

[8, Theorem 2] was applied successfully in [16] to a large variety of product problems, including stability under (finite) product of global properties like countable and pseudo compactness and Lindelöfness, local properties like Fréchetness or strong Fréchetness, and properties of maps like perfectness and its variants and quotientness and its variants. It is the common principle behind a surprisingly large number of classical theorems. With the following variant of [8, Theorem 2] (for $\mathbb{M} = \mathbb{J} = \mathbb{F}$), I will be able to show that internal descriptions of exponential objects in **TOP,** in **PRTOP** and in the category **PARATOP** of paratopological spaces, are also consequences of this same principle.

A filter $\mathcal{F}$ on X *is compactoidly* $\mathbb{D}$*-meshable at* $\mathcal{A}$ if for every $A \in \mathcal{A}$ and every ultrafilter finer than $\mathcal{F}$ there exists a filter $\mathcal{D}$ in $\mathbb{D}$ that is compactoid at A. This is a particular case (for $\mathbb{M} = \mathbb{J} = \mathbb{F}$) of a general notion introduced in [8] of a $\mathbb{M}$-*compactoidly* $\mathbb{J}$ to $\mathbb{D}$ *meshable filter* that depends on three classes of filters.

A class $\mathbb{D}$ of filters is *composable* if for any X and Y, the (possibly degenerate) filter $\mathcal{H}\mathcal{D}$ generated by $\{HD : H \in \mathcal{H}, D \in \mathcal{D}\}$[8] belongs to $\mathbb{D}(Y)$ whenever $\mathcal{D} \in \mathbb{D}(X)$ and $\mathcal{H} \in \mathbb{D}(X \times Y)$, with the convention that every class of filters contains the degenerate filter. Notice that

$$\mathcal{H}\#(\mathcal{F}\times\mathcal{G}) \Longleftrightarrow \mathcal{H}\mathcal{F}\#\mathcal{G} \Longleftrightarrow \mathcal{H}^-\mathcal{G}\#\mathcal{F}, \tag{2.1}$$

where $\mathcal{H}^-\mathcal{G} = \{H^-G = \{x \in X : (x,y) \in H \text{ and } y \in G\} : H \in \mathcal{H}, G \in \mathcal{G}\}^\uparrow$.

Theorem 4. *Let* $\mathbb{D}$ *be a composable class of filters that includes* $\mathbb{F}_1$ *and let* (X,ξ) *be a convergence space. The following are equivalent:*

[7] It is easy to verify that this definition coincide for topological spaces with the usual notion [13].
[8] $HD = \{y \in Y : (x,y) \in H \text{ and } x \in D\}$.

(1) *$\mathcal{F}$ is compactoidly $\mathbb{D}$-meshable at $\mathcal{A} \subset 2^X$ in (X,ξ);*
(2) *for every convergence space Y and $\mathcal{B} \subset 2^Y$, and for every filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at $\mathcal{B}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compactoid at $\mathcal{A} \times \mathcal{B}$;*
(3) *for every convergence space Y, every filter $\mathcal{G}$ on Y which is $\mathbb{D}$-compactoid at $\{y_0\}$ and every $\mathcal{H} \subset 2^{X\times Y}$ such that $\mathcal{H}^-\{y_0\} \subset \mathcal{A}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compactoid at $\mathcal{H}$;*
(4) *for every atomic convergence space Y, and for every filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at $\{y_0\}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\mathcal{A} \times \{y_0\}$.*

Proof. $(1 \Longrightarrow 2)$

Let $\mathcal{D}$ be a $\mathbb{D}$-filter such that $\mathcal{D}\#\mathcal{F} \times \mathcal{G}$. The filter $\mathcal{D}^-(\mathcal{G})\#\mathcal{F}$ and $\mathcal{F}$ is compactoidly $\mathbb{D}$-meshable at $\mathcal{A}$, so that for every $A \in \mathcal{A}$, there exists a compactoid $\mathbb{D}$-filter $\mathcal{C}_A\#\mathcal{D}^-(\mathcal{G})$ at A. Now $\mathcal{D}(\mathcal{C}_A)\#\mathcal{G}$ and $\mathcal{D}(\mathcal{C}_A)$ is a $\mathbb{D}$-filter, so that for each $B \in \mathcal{B}$, there exists a filter $\mathcal{M}_B\#\mathcal{D}(\mathcal{C}_A)$ which converges to a point $y_B \in B$. Moreover $\mathcal{D}^-\mathcal{M}_B\#\ \mathcal{C}_A$ so that there exists $\mathcal{U}_{A,B}\#\mathcal{D}^-\mathcal{M}_B$ that converges to some point of A. Therefore $\operatorname{adh}\mathcal{D} \cap (A \times B) \neq \varnothing$. Hence, $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$-compactoid at $\mathcal{A} \times \mathcal{B}$.

$(1 \Longrightarrow 3)$. Let $\mathcal{D}$ be a $\mathbb{D}$-filter such that $\mathcal{D}\#(\mathcal{F} \times \mathcal{G})$. Since $\mathcal{D}^-\mathcal{G}\#\mathcal{F}$, for every $H \in \mathcal{H}$, there exists a $\mathbb{D}$-filter $\mathcal{L}_H\#\mathcal{D}^-\mathcal{G}$ which is compactoid at $H^-y_0 \in \mathcal{A}$. The $\mathbb{D}$-filter $\mathcal{D}\mathcal{L}_H$ meshes with $\mathcal{G}$ so that there exists $\mathcal{W}_H\#\mathcal{D}\mathcal{L}_H$ so that $y_0 \in \lim_Y \mathcal{W}_H$. Moreover, $\mathcal{D}^-\mathcal{W}_H\#\mathcal{L}_H$. Thus there exists $\mathcal{U}_H\#\mathcal{D}^-\mathcal{W}_H$ and $x_H \in \lim_X \mathcal{U}_H \cap H^-y_0$. Hence $(x_H, y_0) \in \operatorname{adh}_{X\times Y} \mathcal{D} \cap H$.

$(2 \Longrightarrow 4)$ and $(3 \Longrightarrow 4)$ are obvious.

$(4 \Longrightarrow 1)$.

If $\mathcal{F}$ is not compactoidly $\mathbb{D}$-meshable at $\mathcal{A}$ (on X), then there exists $A_0 \in \mathcal{A}$ and an ultrafilter $\mathcal{U}$ of $\mathcal{F}$ such that for every $\mathbb{D}$-filter $\mathcal{D} \leq \mathcal{U}$, there exists an ultrafilter $\mathcal{W}_\mathcal{D}$ of $\mathcal{D}$ such that $\lim_X \mathcal{W}_\mathcal{D} \cap A_0 = \emptyset$.

Consider the convergence space (Y,τ) whose underlying set is $X \cup \{y_0\}$ in which every point of X is isolated and $\mathcal{H}$ converges to y_0 if and only if there exists a $\mathbb{D}$-filter $\mathcal{D} \leq \mathcal{U}$ such that $\mathcal{H} \geq \mathcal{W}_\mathcal{D} \wedge \{y_0\}^\uparrow$. By construction $\mathcal{U}$ is $\mathbb{D}$-compactoid at $\{y_0\}$ in Y. However, $\mathcal{F} \times \mathcal{U}$ is not $\mathbb{F}_1$-compactoid at $A_0 \times \{y_0\}$: In the space $X \times Y$, the set $\Delta = \{(x,x) : x \in X\}$ meshes with $\mathcal{F} \times \mathcal{U}$ because $\mathcal{F}\#\mathcal{U}$ in X, but $\operatorname{adh}_{X\times Y} \Delta \cap (A_0 \times \{y_0\}) = \varnothing$. Indeed, if $\mathcal{H}$ is a filter on Δ, then there exists a filter $\mathcal{H}_0$ on X such that $\mathcal{H}$ is generated by $\{\{(x,x) : x \in H\} : H \in \mathcal{H}_0\}$. If $(x, y_0) \in \lim_{X\times Y} \mathcal{H}$ then $y_0 \in \lim_Y \mathcal{H}_0$ (and $x \in \lim_X \mathcal{H}_0$). Hence there exists a $\mathbb{D}$-filter $\mathcal{D} \leq \mathcal{U}$ such that $\mathcal{H}_0 = \mathcal{W}_\mathcal{D}$ (because $\mathcal{H}_0$ cannot be $\{y_0\}^\uparrow$), so that $\lim_X \mathcal{H}_0 \cap A_0 = \emptyset$. Thus $x \notin A_0$. □

In the case where $\mathcal{A}=\mathcal{N}_\xi(x_0)$ and ξ is $*$-regular, we can give an alternative converse to $(3 \Longrightarrow 1)$.

Proposition 1. *Let (X,ξ) be a $*$-regular convergence space. If for every atomic convergence space (Y,τ), and for every filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at $\{y_0\}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\mathcal{N}_{\xi\times\tau}(x_0, y_0)$, then $\mathcal{F}$ is compactoidly $\mathbb{D}$-meshable at $\mathcal{N}_\xi(x_0)$.*

Proof. If $\mathcal{A} = \mathcal{N}_\xi(x_0)$, then in the construction carried on in the $(4 \Longrightarrow 1)$ part of the proof of Theorem 4, A_0 can be chosen ξ-open. Moreover $\mathcal{F} \times \mathcal{U}$ is not $\mathbb{F}_1$-compactoid at $\mathcal{N}_{\xi\times\tau}(x_0, y_0)$ because $(x_0, y_0) \notin \operatorname{cl}_{\xi\times\tau}(\operatorname{adh}_{\xi\times\tau} \Delta)$. Indeed, by the same argument as in $(4 \Longrightarrow 1)$, $\operatorname{adh}_{\xi\times\tau} \Delta \subset A_0^c \times \{y_0\} \cup \bigcup_{x\in X} (\operatorname{cl}_\xi x \times \{x\})$ and the set $A_0^c \times \{y_0\} \cup \bigcup_{x\in X} (\operatorname{cl}_\xi x \times \{x\})$ can be shown to be $(\xi \times \tau)$-closed : First notice that

$A_0^c \times \{y_0\}$ is $(\xi \times \tau)$-closed. Assume $\mathcal{H} \times \mathcal{M}$ is a filter on $\bigcup_{x \in X} (\mathrm{cl}_\xi\, x \times \{x\})$ converging to (x, y) for $\xi \times \tau$. If $y \neq y_0$ then $\mathcal{M} = \{y\}^\uparrow$ and $\mathcal{H}$ is a filter on $\mathrm{cl}_\xi\, y$, which is ξ-closed, so that $x \in \mathrm{cl}_\xi\, y$. If $y = y_0$ then $\mathcal{M} = \mathcal{W}_\mathcal{D}$ for some $\mathcal{D}$ and $\mathcal{H} \geq \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{M}$ so that $\mathrm{cl}^\natural_{\xi^*} \mathcal{H} \# \mathcal{M}$. Moreover $x \in \lim_\xi \mathcal{H} \subset \lim_\xi \mathrm{cl}^\natural_{\xi^*} \mathcal{H}$ by $*$-regularity of ξ. But $\lim_\xi \mathrm{cl}^\natural_{\xi^*} \mathcal{H} \subset \mathrm{adh}_\xi \mathcal{M} \subset A_0^c$ so that $(x, y) \in A_0^c \times \{y_0\}$. □

In case $\mathbb{D} = \mathbb{F}_1$ and $\mathcal{A} = \{x_0\}$, Theorem 4 applies to the effect that

Corollary 1. *Let (X, ξ) be a pseudotopology. The following are equivalent:*

1. *(X, ξ) is finitely generated;*
2. *$X \times PY \geq P(X \times Y)$ for every convergence space Y;*
3. *X is quasi-exponential in **PRTOP**.*

Proof. If (X, ξ) is a finitely generated pseudotopology, then $\mathcal{F}$ is compactoidly $\mathbb{F}_1$-meshable at $\{x_0\}$ whenever $x_0 \in \lim_X \mathcal{F}$. Therefore, for any convergence space Y and every $\mathcal{G}$ such that $y_0 \in \lim_{PY} \mathcal{G}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\{(x_0, y_0)\}$, that is $(x_0, y_0) \in \lim_{X \times PY} (\mathcal{F} \times \mathcal{G})$ because $\mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\{y_0\}$. Hence $X \times PY \geq P(X \times Y)$ for every convergence space Y. Conversely, if $X \times PY \geq P(X \times Y)$ for every convergence space Y then whenever $x_0 \in \lim_X \mathcal{F}$, we have that $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\{(x_0, y_0)\}$ for every filter $\mathcal{G}$ that is $\mathbb{F}_1$-compactoid at $\{y_0\}$ in a convergence space Y. Hence $\mathcal{F}$ is compactoidly $\mathbb{F}_1$-meshable at $\{x_0\}$. In particular every ultrafilter of $\mathcal{F}$ contains a set converging to x_0. Therefore, a finite union of such converging sets –also converging to x_0– belongs to $\mathcal{F}$. Hence X is finitely generated. □

In case $\mathbb{D} = \mathbb{F}_1$, $\mathcal{A} = \mathcal{N}_\xi(x_0)$ and $\mathcal{H} = \mathcal{N}_{\xi \times \tau}(x_0, y_0)$, Theorem 4 ($1 \Longrightarrow 3$) particularizes to ($1 \Longrightarrow 2$) in the result below. Assuming that (X, ξ) is a $*$-regular convergence space, Proposition 1 leads to ($3 \Longrightarrow 1$) in the result below.

Corollary 2. *Let (X, ξ) be a $*$-regular convergence space. The following are equivalent:*

1. *(X, ξ) is a T-core-compact;*
2. *$X \times TY \geq T(X \times Y)$ for every convergence space Y;*
3. *$X \times PY \geq T(X \times Y)$ for every convergence space Y;*
4. *X is quasi-exponential in **TOP**.*

Proof. The proof of ($1 \Longrightarrow 2$) is similar to that of Corollary 1 except that we need to observe two things. The first one is that $(\mathcal{N}_{\xi \times \tau}(x_0, y_0))^- \{y_0\} \subset \mathcal{N}_\xi(x_0)$. The second is that $\mathcal{F}$ is compactoidly $\mathbb{F}_1$-meshable at $\mathcal{N}_\xi(x_0)$ if and only if for every $V \in \mathcal{N}_\xi(x_0)$ there exists $K_V \in \mathcal{F}$ which is compactoid at V. If $\mathcal{F}$ is compactoidly $\mathbb{F}_1$-meshable at $\mathcal{N}_\xi(x_0)$ then for every $V \in \mathcal{N}_\xi(x_0)$ and every ultrafilter $\mathcal{U}$ of $\mathcal{F}$, there exists $U_{V,\mathcal{U}} \in \mathcal{U}$ which is compactoid at V. Therefore, there exists finitely many ultrafilters $\mathcal{U}_1, .., \mathcal{U}_n$ of $\mathcal{F}$ such that $\bigcup_{i=1}^n U_{V,\mathcal{U}_i} \in \mathcal{F}$. Evidently $\bigcup_{i=1}^n U_{V,\mathcal{U}_i}$ is compactoid at V. The other implication is trivial.
($2 \Longrightarrow 3$) is clear and ($3 \Longrightarrow 1$) follows from Proposition 1. □

Notice that Corollary 2 characterizes quasi-exponential objects in **TOP** among $*$-regular convergences rather than topologies, improving upon the results of [6], [7]. It also shows that the upper Kuratowski convergence (which can be identified with the continuous convergence $[\xi, \$]$, where \$ denotes the two points Sierpiński space) on the set of closed subsets of (X,ξ) (equivalently the Scott convergence on the lattice of open subsets; see [7] for details) is topological whenever it is pretopological as long as ξ is a $*$-regular convergence. This fact was as well only known for topological spaces (X,ξ).

In case $\mathbb{D} = \mathbb{F}_\omega$ and $\mathcal{A} = \{x_0\}$, Theorem 4 applies to the effect that

Corollary 3. *Let (X,ξ) be a pseudotopology. The following are equivalent:*

(1) *(X,ξ) is bisequential;*
(2) *$X \times P_\omega Y \geq P_\omega(X \times Y)$ for every convergence space Y;*
(3) *$X \times P_\omega Y \geq P(X \times Y)$ for every convergence space Y;*
(4) *X is quasi-exponential in **PARATOP**.*

Proof. It suffice to notice that the condition that $\mathcal{F}$ is compactoidly $\mathbb{F}_\omega$-meshable at $\{x_0\}$ whenever $x_0 \in \lim_\xi \mathcal{F}$ rephrases as $\xi \geq SB_{\mathbb{F}_\omega} S\xi$, which coincides with bisequentiality for pseudotopologies. □

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