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by

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OPTIMAL BIVARIATE RANKED SET SAMPLE DESIGN FOR THE MATCHED PAIRS SIGN TEST

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Abstract

Optimal alternative bivariate ranked set sample designs for the matched pairs sign test are obtained. Our investigation reviled that the optimal bivariate ranked set sample designs for matched pairs sign test are those with quantifying order statistics with labels $\left\{\left(\frac{r+1}{2},\frac{r+1}{2}\right)\right\}$, when the set size r is odd and $\left\{\left(\frac{r}{2},\ \frac{r}{2}\right),\left(\frac{r}{2}+1,\ \frac{r}{2}+1\right)\right\}$ when the set size r is even. The exact null distributions, asymptotic distributions and Pitman efficiencies of those designs are derived. Numerical analysis of the power of the proposed optimal designs is included. Illustration using real data with Bootstrap algorithm for P-value estimation is used.

Key Words**:** Bivariate Ranked Set Sample, Bootstrap method; power of the test, P-value of the test; Pitman's relative efficiency, matched pairs sign test.

AMS: 62G10

1. Introduction

Hennekens and Buring (1987) argued that matching as a technique for the control of confounding has great intuitive appeal and has been widely used over the years in many epidemiological studies. Unlike randomization and restriction, which used to control for confounding in the design stage of a study, matching is a strategy that must include elements of both design and analysis. Examples for matched pairs studies are found in identical twins, before-and-after and other studies, where subjects are matched based on some confounding factors.

These types of studies produce data consisting of observations in a bivariate random sample $\{(X_i,\ Y_i),\ i=1,\ 2,\ ...,\ n\}$, where there are n pairs of observations. Within each pair $(X_i,\ Y_i)$ a comparison is made, and the pair is classified as "+" if $X_i < Y_i$, "$-$" if $X_i > Y_i$ or "0" if $X_i = Y_i$. Here the measurement scale needs only to be ordinal. Other needed assumptions are (1) The bivariate variables $(X_i,\ Y_i)$, $i=1,\ 2,\ \ldots,\ n$, are mutually independent. (2) The pairs $(X_i,\ Y_i)$ are internally consistent in that if $P(+)>P(-)$ for one pair $(X_i,\ Y_i)$, then $P(+)>P(-)$ for all pairs. The same is true for P(+)<P($-$) and P(+)=P($-$), see Conover (1980) and Samawi et al. (2006 b).

The types of null hypotheses that can be tested using the matched pairs sign test are:

(1) $H_o:\ P(+) = P(-) = \frac{1}{2}$.

(2) $H_o:\ E(X_i) = E(Y_i)$,

for all i, which is interpreted as X_i and Y_i have the same location parameter.

(3) $H_o:$ The median of X_i equals the median of Y_i for all i,

(see Conover 1980.)

The matched pairs sign test statistic which denoted by T_{BVSRS} , for testing the above hypotheses equals the number of "+" pairs, that is

$$T_{BVSRS} = \sum_{i=1}^{n} I(X_i < Y_i) \qquad (1.1)$$

where

$$I(X_i < Y_i) = \begin{cases} 1 & \text{if } X_i < Y_i \\ 0 & \text{otherwise} . \end{cases}$$

Conover (1980) suggested discarding all tied pairs and let n equal the number of the remaining pairs. Depending on whether the alternative hypothesis is one-tailed or two-tailed, and if $n \leq 20$, then one can use the binomial distribution with the values n and p=1/2 for finding the critical region of approximately size α . For n larger than 20 and when the null hypothesis is true then $T_{BVSRS} \sim N(\frac{n}{2}, \frac{n}{4})$. Therefore the critical region can be defined based on the normal distribution. It has been argued that T_{BVSRS} is an unbiased and a consistent test statistic when testing

$H_o: P(+) = P(-)$.

However, for testing

$H_o: E(X_i) = E(Y_i)$, for all i

and

$H_o:$ The median of X_i = the median of Y_i for all i ,

T_{BVSRS} is neither unbiased nor consistent (see Conover 1980.)

In most statistical applications the data used is assumed to consist of a simple random sample (SRS). However, it becomes obvious in some situations that quantification of sampling units with respect to the variable of interest is costly as compared with the physical acquisition of the unit. Cost savings of quantifying sampling units can be achieved by using ranked set sampling (RSS) method which was introduced first by McIntyre (1952) without any mathematical proof, to estimate the population mean and later called RSS by Halls and Dell (1966).

The RSS procedure can be described as follows: Randomly sample a group of sampling units from the target population. Then randomly partition the group into disjoint subsets each having a pre-assigned size *r*. In most practical situations, the size *r* will be 2, 3 or 4. Rank the elements in each subset by a suitable method of ranking such as prior information, visual inspection or by the subject-matter experimenter himself, … etc. Then the *i-th* order statistic from the *i-th* subset, $X_{i(i)}$, $i = 1, \ldots, r$, will be quantified (actual measurement). Therefore, $X_{1(1)}, X_{2(2)}, \ldots, X_{r(r)}$ constitutes the RSS. This represents one cycle. The whole procedure can be repeated *m*-times as needed, to get a RSS of size $n=mr$. For the theoretical aspects of RSS, see Takahasi and Wakimoto (1968) or Dell and Clutter(1972). For more about univariate RSS and its variations, see Kaur et al. (1995) and Patil et al. (1999).

Recently, ranked set samples were used for quantiles and distribution estimation by Stokes and Sager (1988), Chen (2000), Samawi (2001), and Samawi and Al-Saleh (2004). Optimality of ranked set sample scheme for inference on population quantiles was suggested by Chen (2001). Other authors have used the RSS sampling method to improve parametric and non-parametric statistical inference. For non-parametric methods, RSS was considered by Bohn and Wolfe (1992, 1994), Kvam and Samaniego (1994) and Hettmansperger (1995). Koti and Babu (1996) showed that the RSS sign test provides a more powerful test than the SRS sign test. Barabesi (1998) provided a simpler and faster method for computing the exact distribution of the RSS sign test.

The optimality of the RSS sign test has been established by several researchers in the literature via Pitman asymptotic efficacy. It was shown that the median ranked set sample (MRSS) is the best among all possible sampling schemes in the ranked set sampling environment for the sign test procedure; for example see Öztürk (1999), Öztürk and Wolfe (2000) and Samawi and Abu- Dayyeh (2002).

Another RSS procedure for estimation of bivariate characteristics using bivariate ranked set sampling (BVRSS) was introduced by Al-Saleh and Zheng (2002). Their procedure can be described as follows:

Suppose (X, Y) is a bivariate random vector with the joint probability density function (p.d.f.) $f(x, y)$.

1. A random sample of size r^4 is identified from the population and randomly allocated into r^2 pools each of size r^2, where each pool is a square matrix with r rows and r columns.
2. In the first pool, identify the minimum value by judgment w.r.t. the first characteristic X, for each of the r rows.
3. For the r minima obtained in Step 2, choose the pair that corresponds to the minimum value of the second characteristic Y, identified by judgment, for actual quantification. This pair, which resembles the label (1, 1), is the first element of the BVRSS sample.
4. Repeat Steps 2 and 3 for the second pool, but in step 3, the pair that corresponds to the second minimum value *w.r.t.* the second characteristic, Y, is chosen for actual quantification. This pair resembles the label (1, 2).
5. The process continues until the label (r, r) is resembled from the r^2-*th* (last) pool.

This will produce a BVRSS of size r^2. The procedure can be repeated m times to obtain a sample of size $n=mr^2$.

Let $[(X^{z}_{ijk}, Y^{z}_{ijk}),\ i=1,\ 2,\ ...,\ r\ ,\ j=1,\ 2,\ ...,\ r,\ k=1,\ 2,\ ...,\ m\ \&\ z=1,...,r^{2}]$ be mr^{4} *i.i.d* ordered pairs from a bivariate probability density function, say *f(x, y); (x, y)* $\in \mathrm{R}^{2}$. Following the Al-Saleh and Zheng (2002) definition of BVRSS let $\left[\ (X_{[i](j)\,k},\ Y_{(i)[j]k}),\ i=1,\ 2,\ ...,\ r;\ \ j=1,\ 2,\ ...,\ r\ \&\ k=1,\ 2,\ ...,\ m\right]$ denotes such a sample from f(x, y). Therefore, the pdf of $(X_{[i](j)k},\ Y_{(i)[j]k})$, k=1, 2, …, m is given by Al-Saleh and Zheng (2002) as

$$f_{X_{[i](j)},\,Y_{(i)[j]}}(x,\ y)(x,\ y)=c_{1}(F_{Y_{[j]}}(y))^{i-1}(1-F_{Y_{[j]}}(y))^{r-i}(F_{X}(x))^{j-1}(1-F_{X}(x))^{r-j}\,f(x,y) \quad (1.2)$$

where

$$c_{1}=\frac{r\,!}{(i-1)!\ (r\text{-}i)!}\frac{r\,!}{(j-1)!\ (r\text{-}j)!}$$

and

$$f_{Y_{[\mathrm{j}]}}(y)=\int_{-\infty}^{\infty} f_{X_{(j)}}(x)\,f_{Y|X}(y\,|\,x)dx.$$

Note that from Al-Saleh and Zheng (2002) we have the following results:

$$\frac{1}{r^{2}}\sum_{j=1}^{r}\sum_{i=1}^{r}f_{X_{[i](j)},\,\mathrm{Y}_{(i)[j]}}(x,\ y)\ =f\ (x,\ y),$$

$$\frac{1}{r^{2}}\sum_{j=1}^{r}\sum_{i=1}^{r}f_{X_{[i](j)}}(x)\ =f_{X}\ (x)$$

and

$$\frac{1}{r^{2}}\sum_{j=1}^{r}\sum_{i=1}^{r}f_{Y_{(i)[j]}}(y)\ =f_{Y}\ (y).$$

Samawi et al. (2006 a), used the idea of BVRSS to improve the efficiency of bivariate sign test for one-sample bivariate location model.

This paper introduces the optimal BVRSS designs (OBVRSS) for matched pairs sign test. Numerical comparisons between the performance of the OBVRSS matched pairs sign test and the performance of the BVSRS and BVRSS sign tests via Pitman's asymptotic efficiency and asymptotic power are investigated. The exact null distribution and the asymptotic null distribution and power of the OBVRSS sign test are derived. It will be shown that OBVRSS substantially improves the efficiency and the power of the matched pairs sign test. We also introduce a bootstrap method for finding the P-value of the matched pairs test for small sample sizes and demonstrate the procedure using real data from the Iowa 65+ Rural Health Study (RHS).

2. Alternative BVRSS Designs for Matched Pairs Sign Test

An alternative bivariate ranked set sampling (ABVRSS) is a sampling protocol that quantifies the same order statistics in each pool using similar BVRSS protocol. Samawi at el. (2006 c) described ABVRSS as follows: Define $\ell(A)$ to be the cardinality of a set *A*, then $\ell(A)$ = the number of elements in a set *A*. Let J_{ABVRSS} = {set of all possible alternative BVRSS designs} = {J_1, J_2 , ..., $J_{\ell(J_{ABVRSS})}$ }, for example, when *r=2*, then J_{ABVRSS} = {{(1, 1)}, {(1, 2)}, {(2, 1)}, {(2, 2)}, {(1, 1), (1, 2)}, {(1, 1), (2, 1)}, {(1, 1), (2, 2)}, {(1, 2), (2, 1)}, {(1, 2),(2, 2)}, {(2, 1),(2, 2)}, {(1, 1), (1, 2), (2, 1)}, {(1, 1), (1, 2), (2, 2)}, {(1, 1), (2, 1), (2, 2)}, {(1, 2), (2, 1), (2, 2)}, {(1, 1), (1, 2), (2, 1), (2, 2)}}.

Then for *r=2*, $\ell(J_{ABVRSS}) = \sum_{i=1}^{4}\binom{4}{i} = 15$. In general, for a set of size *r*,

$\ell(J_{ABVRSS}) = \sum_{i=1}^{r^2}\binom{r^2}{i} = 2^{r^2} - 1$. Now, for an integer s, $s \in \left\{1,\ 2,\ ....,\ 2^{r^2}-1\right\}$, let $J_s \in J_{ABVRSS}$ be the set of judgment ranks of ordered pairs labels for the observations to be quantified. Our sampling protocol involves selecting $m\ell(J_s)r^2$ units from an infinite population. These units are partitioned into $m\ell(J_s)$ pools each having r^2 units.

From each pool, by using the same procedure discussed for a BVRSS protocol by Al-Saleh and Zheng (2002), we quantify only one of the ordered pair's labels in J_s; therefore, they are mutually independent.

Let $J_s = \{(c_1, d_1), ..., (c_{\ell(J_s)}, d_{\ell(J_s)})\}$, then

$\left\{\left(X_{[c_1](d_1)k}, Y_{(c_1)[d_1]k}\right), ..., \left(X_{[c_{\ell(J_s)}](d_{\ell(J_s)})k}, Y_{(c_{\ell(J_s)})[d_{\ell(J_s)}]k}\right)\right\}$, where $k = 1, 2, ..., m$, will be ABVRSS from $F(x, y)$ with $n = m\ell(J_s)$. The ABVRSS sign test statistic can be defined as $T_{ABVRSS} =$ the number of "+"; or; $= \#(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k})$ for all u and k, that is

$$T_{ABVRSS} = \sum_{u=1}^{l(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}) = \sum_{u=1}^{l(J_s)} T_u \tag{2.1}$$

where

$$T_u = \sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}).$$

Clearly, T_u, $u = 1, 2, ..., l(J_s)$ are stochastically independent and each T_u has a binomial distribution with parameters m and $p_u = P(X_{[c_u](d_u)} < Y_{(c_u)[d_u]})$. The mean and the variance of T_{ABVRSS} are respectively

$$E(T_{ABVRSS}) = E[\sum_{u=1}^{l(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k})] = m\sum_{u=1}^{l(J_s)} P_u \tag{2.2}$$

and

$$V(T_{ABVRSS}) = V[\sum_{u=1}^{l(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k})] = m \sum_{u=1}^{l(J_s)} P_u(1-P_u) \quad (2.3)$$

Also, the exact distribution of T_{ABVRSS} is given by

$$P(T_{ABVRSS} = t) = \sum_{L_{xy}} \prod_{u=1}^{l(J_s)} \binom{m}{v_u} p_u^{v_u} (1-p_u)^{m-v_u} \quad (2.4)$$

for $t = 0,1,2,...,ml(J_s)$; where $L_{xy} = \{(v_u : u = 1, 2, ..., l(J_s)) : \sum_{u=1}^{l(J_s)} v_u = t;\ 0 \le v_u \le m.\}$.

According to our setting, ordinary BVRSS is a special case when $J_s = J_{\ell(J_{ABVRSS})}$.

Unfortunately, most of the exact distributions in (2.4) and the results in (2.2) and (2.3) from the ABVRSS designs, $J_s,\ s = 1,2,...,l(J_{ABVRSS})$, depend on the given underlying bivariate distribution function even under the null hypothesis. Thus, finding the exact critical value and the P-value of the test requires knowledge of the underlying distribution function. However, under the null hypothesis and with a diagonally symmetric underlying bivariate density function, for some of the alternative designs, T_{ABVRSS} has binomial distribution. Similar to univariate case described by Q̈ztürk (2001), ABVRSS design is called symmetric if

$c_u + c_{r-u+1} = r+1$ and $d_u + d_{r-u+1} = r+1$ or $c_u = d_u = \frac{r+1}{2}$ (when r is odd). Based on the above assumptions, the following results are proved.

Theorem 2.1: Assume that *f(x, y)=f(-x, -y)* is diagonally symmetric about 0. If ABVRSS design is symmetric then the following are true:

1-When *r* is odd, $f_{X_{[\frac{r+1}{2}](\frac{r+1}{2})}, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(x, y) = f_{X_{[\frac{r+1}{2}](\frac{r+1}{2})}, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(-x, -y)$; diagonally symmetric.

2- When r is odd or even, $f_{X_{[c_u](d_u)},\, Y_{(c_u)[d_u]}}(-x,-y) = f_{X_{[r-c_u+1](r-d_u+1)},\, Y_{(r-c_u+1)[r-d_u+1]}}(x,\, y)$.

Proof. 1. Using equ. (1.2) and let $c = \dfrac{r!}{(\frac{r-1}{2})!(\frac{r-1}{2})!}$,

$$f_{X_{[\frac{r+1}{2}](\frac{r+1}{2})},\, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(-x,-y) = c_1(F_{Y_{[\frac{r+1}{2}]}}(-y))^{\frac{r-1}{2}}(1-F_{Y_{[\frac{r+1}{2}]}}(-y))^{\frac{r-1}{2}}(F_X(-x))^{\frac{r-1}{2}}(1-F_X(-x))^{\frac{r-1}{2}} f(-x,-y)$$

$F_{Y_{[\frac{r+1}{2}]}}(-y) = 1 - F_{Y_{[\frac{r+1}{2}]}}(y)$ because

$$f_{Y_{[\frac{r+1}{2}]}}(-y) = \int_{-\infty}^{\infty} c(F_X(-x))^{\frac{r-1}{2}}(1-F_X(-x))^{\frac{r-1}{2}} f(-x,-y)dx = \int_{-\infty}^{\infty} c(1-F_X(x))^{\frac{r-1}{2}}(F_X(x))^{\frac{r-1}{2}} f(x,y)dx$$
$$= f_{Y_{[\frac{r+1}{2}]}}(y),$$

is symmetric about 0. Therefore, by the diagonally symmetric assumption of *f(x,y),*

$$f_{X_{[\frac{r+1}{2}](\frac{r+1}{2})},\, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(-x,-y) = c_1(1-F_{Y_{[\frac{r+1}{2}]}}(y))^{\frac{r-1}{2}}(F_{Y_{[\frac{r+1}{2}]}}(y))^{\frac{r-1}{2}}(1-F_X(x))^{\frac{r-1}{2}}(F_X(x))^{\frac{r-1}{2}} f(x,y)$$
$$= f_{X_{[\frac{r+1}{2}](\frac{r+1}{2})},\, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(x,y).$$

Proof 2. Again by (1.2)

$$f_{X_{[c_u](d_u)},\, Y_{(c_u)[d_u]}}(-x,-y) = c_1(F_{Y_{[d_u]}}(-y))^{c_u-1}(1-F_{Y_{[d_u]}}(-y))^{r-c_u}(F_X(-x))^{d_u-1}(1-F_X(-x))^{r-d_u} f(-x,-y)$$

And similar argument as in proof 1, we have

$$f_{X_{[c_u](d_u)},\, Y_{(c_u)[d_u]}}(-x,-y) = c_1(1-F_{Y_{[r-d_u+1]}}(y))^{c_u-1}(F_{Y_{[r-d_u+1]}}(y))^{r-c_u}(1-F_X(x))^{d_u-1}(F_X(x))^{r-d_u} f(x,y)$$
$$= f_{X_{[r-c_u+1](r-d_u+1)},\, Y_{(r-c_u+1)[r-d_u+1]}}(x,y)$$

because also,

$$f_{Y_{[d_u]}}(-y) = \int_{-\infty}^{\infty} c(F_X(-x))^{d_u-1}(1-F_X(-x))^{r-d_u} f(-x,-y)dx$$

$$= \int_{-\infty}^{\infty} c(1-F_X(x))^{d_u-1}(F_X(x))^{r-d_u} f(x,y)dx = f_{Y_{[r-d_u+1]}}(y).$$

Theorem 2.2: Assuming no tied pairs $(X_{[c_u](d_u)k} = Y_{(c_u)[d_u]k})$ for all u and k and *f(x, y)=f(-x, -y)*. Under the null hypothesis $H_o: P(+) = P(-) = \frac{1}{2}$ and the assumption of symmetric ABVRSS design, for fixed r we have the following:

1- $E(T_{ABVRSS}) = \frac{n}{2} = \frac{ml(J_s)}{2}$.

2- $V_o = V(T_{ABVRSS}) = \frac{ml(J_s)}{2}[1 - \frac{2}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2]$.

A special case is $V_o = V(T_{ABVRSS}) = \frac{ml(J_s)}{4}$ {if $J_s = (\frac{r+1}{2}, \frac{r+1}{2})$, when r is odd}

3- For large *m*, T_{ABVRSS} has approximately $N(\frac{n}{2},\ V_o)$, where $n = m\,l(J_s)$,

Proof: Prove of 1 and 2 follow directly from Theorem 2.1.

Proof 3, Discard all tied pairs and let *n* equal the number of pairs that are not ties. From parts 1 and 2 in the Theorem, we have $E(T_{ABVRSS}) = \frac{ml(J_s)}{2} = \frac{n}{2}$ and

$V_0 = V(T_{ABVRSS}) = \frac{ml(J_s)}{2}[1 - \frac{2}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2] = \frac{n}{4}[2 - \frac{4}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2]$. Note that $[2 - \frac{4}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2]$ is finite fixed number. Therefore, using similar argument as in Hettmansperger (1995) the proof follows.

Clearly, when *r* is odd, T_{ABVRSS} has a binomial distribution with *n*=number of trials=$ml(J_s)$ and probability of success *p*=0.5. Again, similar to Conover (1980), discard all tied pairs and let *n* equal the number of the remaining pairs. Depending on whether the alternative hypothesis is one-tailed or two-tailed, and if $n \le 20$, then use the binomial distribution with the values *n* and *p*=0.5 for finding the critical region of approximately test size α. For *n* larger than 20 use normal approximation in Theorem 2.2 part 3. For *r* even, a consistent estimator for V_o is given by

$$\hat{V}_o = \frac{ml(J_s)}{4}(2 - \frac{4}{l(J_s)}\sum_{u=1}^{l(J_s)} \hat{P}_u^2), \text{where } \hat{P}_u = \frac{1}{m}\sum_{k=1}^{m} I(X_{[c_u](d_u)k} < Y_{(c_u)[d_u]k}).$$

Depending on whether the alternative hypothesis it is one-tailed or two-tailed and if $n \ge 20$, the asymptotic test procedure is to reject the null hypothesis $H_o: P(+) = P(-) = \frac{1}{2}$ in favor of the alternative {e.g. $H_a: P(+) > P(-)$} if $Z_o = \frac{T_{ABVRSS} - \frac{n}{2}}{\sqrt{\hat{V}_o}} > z_\alpha$, where z_α is the 100(1-α)% quantile of the standard normal distribution.

3. The Asymptotic Relative Efficiency

The performance of the matched pairs sign test using ABVRSS will be compared with the matched pairs sign test using BVSRS based on the criterion of Pitman's asymptotic relative efficiency (ARE). The Pitman's regularity conditions are satisfied for both T_{ABVRSS} and T_{BVSRS} because all moments of the tests are in terms of probabilities, and hence are bounded above by 1. The Pitman's ARE of T_{BVRSS} versus T_{BVSRS} is defined as

$$ARE(T_{BVRSS}, T_{BVSRS}) = \frac{e^2(T_{BVRSS})}{e^2(T_{BVSRS})}, \tag{3.1}$$

where the efficiency of a test statistics *T* is given by *e (T)* and $e(T) = \lim_{n \to \infty} \left. \frac{\frac{\partial E(T)}{\partial \theta}}{\sqrt{n \ \text{var}(T)}} \right|_{H_o}$.

Using the above definition and noting that $F_D(0) = P(X < Y) = P(X - Y < 0) = P(D < 0)$, the efficiencies of T_{ABVRSS} and T_{BVSRS} are obtained as

$$e(T_{BVRSS}) = \left. \frac{2\sum_{u=1}^{l(J_s)} \frac{f_{Du}(0)}{l(J_s)}}{\sqrt{[2 - \frac{4}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2]}} \right|_{H_o} \tag{3.2}$$

and

$$e(T_{BVSRS}) = 2 f_D(0) \tag{3.3}$$

respectively. Note that θ could be the central parameter of D or the shifted parameter such that $P(X < Y + \theta) = 0.5$ under the null hypothesis.

Therefore, by (3.1), (3.2) and (3.3)

$$ARE(T_{BVRSS}, T_{BVSRS}) = \left. \frac{(\sum_{u=1}^{l(J_s)} \frac{f_{Du}(0)}{l(J_s)})^2}{f_{Du}^2(0)[2 - \frac{4}{l(J_s)} \sum_{u=1}^{l(J_s)} P_u^2]} \right|_{H_o},$$

where

$F_{Du}(0) = P(X_{[c_u](d_u)} < Y_{(c_u)[d_u]}) = P(X_{[c_u](d_u)} - Y_{(c_u)[d_u]} < 0) = P(Du < 0)$.

3.1 Numerical comparisons

Assume that the bivariate random variable (*X*, *Y*) has a bivariate normal distribution. We computed $ARE(T_{ABVRSS}, T_{BVSRS})$ for {*r*=2, 3, and 4, and correlation coefficient ($\rho = \pm 0.5$, and ± 0.9)}.

Table 3.1 shows Pitman's asymptotic relative efficiency $ARE(T_{ABVRSS}, T_{BVSRS})$ for only the most efficient ABVRSS symmetrical designs. Table 3.2 and 3.3 show the asymptotic power of T_{ABVSRS} optimal designs and T_{BVRSS} respectively.

From Table 3.1 the optimal bivariate ranked set sample designs for matched pairs sign test are those with quantifying order statistics with labels $\left\{\left(\frac{r+1}{2}, \frac{r+1}{2}\right)\right\}$, when the set size r is odd and $\left\{\left(\frac{r}{2}, \ \frac{r}{2}\right), \left(\frac{r}{2}+1, \ \frac{r}{2}+1\right)\right\}$ when the set size r is even. The matched pairs sign tests using the optimal designs will be denoted by T_{OBVRSS}. Clearly, via Pitman's asymptotic relative efficiency, the performance of T_{OBVRSS} is superior to T_{BVSRS}, the ordinary T_{BVRSS} and all alternative symmetrical designs (T_{ABVRSS}). Note that all proposed alternative designs including ordinary BVRSS performed better for negative correlation. The asymptotic relative efficiency $ARE(T_{ABVRSS}, T_{BVSRS})$ increases as the set size r increases in all cases. Also, it is clear that the $ARE(T_{ABVRSS}, T_{BVSRS})$ increases as the negative ρ decreases away from zero. When the correlation coefficient ρ is positive, $ARE(T_{ABVRSS}, T_{BVSRS})$, decreases as ρ increases. However, ABVRSS is still more efficient than BVSRS.

Table 3.1: Pitman's asymptotic relative efficiency $ARE(T_{BVRSS}, T_{BVSRS})$. The results for negative correlation coefficients are in bold and parenthesis.

ABVRSS design	r	$\rho = \pm 0.5$	$\rho = \pm 0.9$
Ordinary BVRSS	2	1.15(**1.43**)	1.04(**1.59**)
{(1,2), (2,1)}	2	1.08(**1.12)**	1.02(**1.06)**
{(1,1), (2,2)}	2	1.23(**1.80)**	1.05(**2.14**)
Ordinary BVRSS	3	1.29(**1.77**)	1.09(**2.38**)
{(2,1), (2,3)}	3	1.42(**1.67**)	1.10(**1.57**)
{(2,2)}	3	1.67(**3.02**)	1.16(**4.28**)
Ordinary BVRSS	4	1.43(**2.14**)	1.12(**2.56**)
{(2,2), (3,3)}	4	1.96(**3.94**)	1.24(**5.97**)

4. Matched Pairs Sign Tests Using Optimal Designs

In this section we introduce the optimal bivariate ranked set sampling protocols (OBVRSS) for the matched pairs sign test. Some theoretical results of the test using those (OBVRSS) designs are derived. Also, we investigate the power of the test for those designs.

Case 1: Set size r is odd.

Let $\left(X_{[\frac{r+1}{2}](\frac{r+1}{2})k}, Y_{(\frac{r+1}{2})[\frac{r+1}{2}]k}\right)$, $k = 1, 2, ..., n$ be $OBVRSS_O$ from $F(x, y)$**.** Then the sign test statistic based on $OBVRSS_O$ is

$$T_{OBVRSS_O} = \sum_{k=1}^{n} I(X_{[\frac{r+1}{2}](\frac{r+1}{2})k} < Y_{(\frac{r+1}{2})[\frac{r+1}{2}]k}). \tag{4.1}$$

By Theorem 2.2 and under H_o T_{OBVRSS_O} has a binomial distribution with parameters n and $p=\frac{1}{2}$.

Also, under the alternative hypothesis, $H_a: P(+)>P(-)$, T_{OBVRSS_O} has a binomial distribution with parameters n and $p_o = P(X_{[\frac{r+1}{2}](\frac{r+1}{2})} < Y_{(\frac{r+1}{2})[\frac{r+1}{2}]})$.

Moreover, for n>20, by Theorem 2.2, the asymptotic power of testing the hypothesis $H_o: P(+)=P(-)$ versus the alternative {without loss of generality consider $H_a: P(+)>P(-)$} for T_{OBVRSS_O} and T_{BVSRS} are defined by

$$\beta_{OBVRSS_O} = 1-\Phi[(z_\alpha\sqrt{\frac{n}{4}} + n/2 - np_o)/\sqrt{V_{ao}}],$$

where $V_{ao} = np_o(1-p_o)$ is the variance of T_{OBVRSS_O} under the alternative hypotheses and

$$\beta_{BVSRS} = 1-\Phi[(z_\alpha\sqrt{n/4} + n/2 - nP(X < Y))/\sqrt{nP(X < Y)(1-P(X < Y))}].$$

Therefore, under the null hypothesis $\beta_{BVRSS} = 1-\Phi(z_\alpha)=\alpha$ and $\beta_{BVSRS} = 1-\Phi(z_\alpha)=\alpha$.

Case 2: Set size r is even.

Let $\left\{\left(X_{[\frac{r}{2}](\frac{r}{2})k}, Y_{(\frac{r}{2})[\frac{r}{2}]k}\right),\left(X_{[\frac{r}{2}+1](\frac{r}{2}+1)k}, Y_{(\frac{r}{2}+1)[\frac{r}{2}+1]k}\right)\right\}$, $k=1, 2, ..., m$, be OBVRSS$_E$ from $F(x, y)$ and $n=2m$**.** Then the sign test statistic based on OBVRSS$_E$ is

$$T_{OBVRSS_E} = \sum_{k=1}^{m}[I(X_{[\frac{r}{2}](\frac{r}{2})k} < Y_{(\frac{r}{2})[\frac{r}{2}]k}) + I(X_{[\frac{r}{2}+1](\frac{r}{2}+1)k} < Y_{(\frac{r}{2}+1)[\frac{r}{2}+1]k})] \tag{4.2}$$

Also, the exact distribution of T_{OBVRSS_E} is given by

$$P(T_{OBVRSS_E}=t)=\begin{cases}\sum_{j=0}^{t}\binom{m}{j}p_{1o}^{j}(1-p_{1o})^{m-j}\binom{m}{t-j}p_{2o}^{t-j}(1-p_{2o})^{m-t+j} & \text{if } t\le m\\ \sum_{j=t-m}^{m}\binom{m}{j}p_{1o}^{j}(1-p_{1o})^{m-j}\binom{m}{t-j}p_{2o}^{t-j}(1-p_{2o})^{m-t+j} & \text{if } m<t\le n,\end{cases} \tag{4.3}$$

where

$p_{1o}=P(X_{[\frac{r}{2}](\frac{r}{2})}<Y_{(\frac{r}{2})[\frac{r}{2}]})$ and $p_{2o}=P(X_{[\frac{r}{2}+1](\frac{r}{2}+1)}<Y_{(\frac{r}{2}+1)[\frac{r}{2}+1]})$.

Unfortunately, the exact null distribution as well as the alternative distribution depends on the underlying bivariate distribution. Therefore, for large *m* use the asymptotic z-test similar to that introduced in Section 2. For small *m<20*, a bootstrap algorithm will be introduced.

Moreover, by Theorem 2.2, the asymptotic power of testing the hypothesis $H_o: P(+)=P(-)$ versus the alternative {without loss of generality consider $H_a: P(+)>P(-)$ } for T_{OBVRSS_E} is defined by $\beta_{OBVRSS_E}=1-\Phi[(z_\alpha\sqrt{\frac{n}{4}}+n/2-m(p_1+p_2))/\sqrt{V_a}]$, where $V_a=m[p_1(1-p_1)+p_2(1-p_2)]$ is the variance of T_{OBVRSS_E} under the alternative hypothesis, $p_1=P(X_{[\frac{r}{2}](\frac{r}{2})}<Y_{(\frac{r}{2})[\frac{r}{2}]})$ and $p_2=P(X_{[\frac{r}{2}+1](\frac{r}{2}+1)}<Y_{(\frac{r}{2}+1)[\frac{r}{2}+1]})$. Also, V_o (as in Theorem 2.2).

Tables 4.1 and 4.2 show the asymptotic power for {(*r*=2, *m=10*), (*r*=2, *m*= 12), (*r*=3, *m*=27) and (*r*=4, *m*=18)}, shifted parameter of center of the two marginal distributions ($\theta=0, 0.1,\ 0.5,$ and 1), level of significance { $\alpha=0.05$ } and correlation coefficient ($\rho=\pm 0.5$ and ± 0.9)}.

Table 4.1 gives evidence towards T_{BVSRS} being unbiased and consistent in this case, although such evidence is not a conclusive proof. The power of T_{BVSRS} increases as the sample size increases and the shift parameter on the variable Y increases away from 0.

Table 4.1: Asymptotic power for T_{OBVRSS} and T_{BVSRS}, when $\alpha = 0.05$. The results for negative correlation coefficients are in bold and are in parenthesis.

		$\rho = \pm 0.5$		$\rho = \pm 0.9$	
$n=r^2 m$	θ	OBVRSS	BVSRS	OBVRSS	BVSRS
n=24 (*r*=2, *m*=12)	0	0.0500 (**0.0500**)	0.0500 (**0.0500**)	0.0500 (**0.0500**)	0.0500 (**0.0500**)
	0.1	0.3083 (**0.2904**)	0.3033 (**0.2807**)	0.3741 (**0.2867**)	0.3708 (**0.2768**)
	0.5	0.5822 (**0.4677**)	0.5431 (**0.4101**)	0.9688 (**0.4605**)	0.9596 (**0.3901**)
	1	0.9564 (**0.7414**)	0.9023 (**0.5977**)	1.0000 (**0.7262**)	1.0000 (**0.5528**)
n=27 (*r*=3, *m*=27)	0	0.0500 (**0.0500**)	0.0500 (**0.0500**)	0.0500 (**0.0500**)	0.0500 (**0.0500**)
	0.1	0.3323 (**0.3170**)	0.3170 (**0.2938**)	0.3964 (**0.3197**)	0.3859 (**0.2898**)
	0.5	0.6668 (**0.5628**)	0.5609 (**0.4259**)	0.9879 (**0.5828**)	0.9664 (**0.4056**)
	1	0.9968 (**0.9141**)	0.9141 (**0.6160**)	1.0000 (**0.9456**)	1.0000 (**0.5707**)

Also, Table 4.1 shows that T_{OBVRSS} is more powerful than T_{BVSRS} for all studied sample sizes, shifted parameter and ρ values. There is a draw back of power when ρ is negative, T_{OBVRSS} gained more efficiency when ρ is negative than when ρ is positive (see, also Table 3.1). Also, the superiority of T_{OBVRSS} over T_{BVSRS} increases as the set size *r* increases. There is evidence towards T_{OBVRSS} being unbiased and consistent in this case; such evidence is again not a conclusive proof. From Theorem 2.2 T_{OBVRSS} has a similar asymptotic distribution as T_{BVSRS} but with smaller asymptotic variance. Therefore, it is safe to say that T_{OBVRSS} has similar asymptotic properties as T_{BVSRS} for testing $H_o : P(+) = P(-)$, i.e. T_{OBVRSS} is also unbiased and a consistent test procedure. However, T_{OBVRSS} is more efficient and more powerful than T_{BVSRS} and T_{BVRSS}.

Remark: This paper provides both cases when *r* is odd and when *r* is even. However, in practice, since the case when *r* is odd in OBVRSS is similar to BVSRS without any extra complications in computation, and using OBVRSS for matched pairs sign test is more efficient and more powerful than using BVSRS, it is recommended to choose *r* to be odd always.

4.1 Bootstrap Algorithm for Estimating the P-value of the Test

The null distribution of our nonparametric tests, T_{OBVRSS}, in Section 4 depend on the underlying bivariate distribution function, especially when r is even. Thus, the exact P-value calculation for sample size *n*<20 is not feasible without knowing the underlying distribution. In this section we introduce a simple bootstrap method for calculating the P-value of the sign test for any given bivariate data. For general description of the bootstrap method of estimation see Efron and Tibshirani (1993).

Suppose that a bivariate random sample of size n<20 is drawn from a population using the BVSRS sampling method. This implies that $\{(X_i, Y_i), i = 1, 2, ..., n\}$ is a random sample. The bootstrap algorithm for approximating the bootstrap P-value of the test for testing the hypothesis $H_o: P(+) = P(-)$ versus the alternative {e.g. $H_a: P(+) > P(-)$} is :

1)Calculate the sample test statistic (say $T = \sum_{i=1}^{n} I(X_i < Y_i)$) from the original sample.

2) Estimate θ from the data; say $\hat{\theta}$. Shift Y_i to $Y_i - \hat{\theta}$, *i*=1, 2, …, *n*, where θ =median (*Y*-*X*)

3) Define $\hat{F}(x, y)$ by placing a mass probability $p_i = \frac{1}{n}$ on $(X_i, Y_i), i = 1, 2, ..., n$.

4)Generate a resample $(X_i^*, Y_i^*), i = 1, 2, ..., n$ from $\hat{F}(x, y)$.

5) Find $T_b^* = \sum_{i=1}^{n} I(X_i^* < Y_i^*)$.

6)Repeat steps 3 and 4 B times.

Then the bootstrap P-value, $P^* = P(T^* \geq T \mid \hat{F}(x,y))$, can be approximated by

$P^* = \frac{1}{B}\sum_{b=1}^{B} I(T_b^* \geq T)$. However, since OBVRSS$_E$, has two different ordered pair labels, slight modification of the above algorithm is needed as follows:

1- Divide the sample into 2 mutually exclusive strata each containing m i.i.d ordered pair labels.

2- Independently from each stratum generate a resample with replacement of size m by placing a mass probability ($\frac{1}{m}$) on each original observation in that stratum.

3- Combine both resamples and do similar steps as in (5) and (6) above.

5. Illustration Using Real Data from the Iowa Rural Health Study (RHS)

The RHS is a prospective longitudinal cohort study of 8 years from 1981 to 1989 of 3,673 individuals (1,420 men and 2,253 women) aged 65 or older living in Washington and Iowa counties of the state of Iowa. This study is one of four supported by the National Institute on Aging and collectively referred to as EPESE (Established Populations for Epidemiological Studies of the Elderly), see Rubenstein and Lemke (1993) and Brock et al. (1986).

The life histories of 2,717 non-institutional individuals who could walk across a small room without any help were obtained from RHS and divided into two cohorts; one containing 1134 who exercised daily by walking and the other containing 1583 who did not exercise daily by walking. The purpose of this illustration is to test the hypothesis that those elderly people (age 65+) who exercise by outdoor daily walking tend to be younger than those who do not exercise by outdoor daily walking in the Iowa 65+ rural health study (RHS.)

We created matched pairs of the cohort of daily walking with the cohort of non-daily walking based on their gender and some health conditions. Thus the total number of pairs included in our target population was 1134. Let the random variable *X* represent the age at baseline of the elderly who exercised by outdoor daily walking and *Y* represent the age at baseline of the elderly who did not exercise by outdoor daily walking. Due to the availability of the age at baseline for all matched pairs in this illustration, ranking was done on both variables *X* and *Y* using the actual ages. However, in real life situation, selecting BVRSS should be done as follows: For example, when *r*=3, nine pre-matched pairs of the elderly should be randomly selected. From the first three pairs select the pair with the second youngest age with respect to *X* (age at baseline of the elderly who exercised by outdoor daily walking). From the second three pairs, again the pair with the second youngest age with respect to *X* and so on. Quantify the values of *X* and *Y* for the pair with the second youngest age with respect to *Y* (the age at baseline of the elderly who did not exercise by outdoor daily walking). This resemble the label (2,2). Repeat this 16 times. Two samples, OBVRSS and BVSRS, of size *n*=16, were drawn from the population of matched pairs (Table 5.1).

The observed test statistics from the observed samples are T_{OBVRSS} =12 and T_{BVSRS} =12. The exact P-values of T_{BVSRS} and T_{OBVRSS} can be obtained by using the binomial distribution for *n*=16 and *p*=0.5. The P-value for both tests is found to be 0.0384. For illustration purposes, we use the bootstrap method with 5000 bootstrap replications to obtain the P-value for both T_{OBVRSS} and T_{BVSRS} . The results of our simulations are as follows: The approximate bootstrap P-value of T_{OBVRSS} is 0.007 with bootstrap MSE (based on 1000 iterations) of 0.0000013. However, the approximate bootstrap P-value of T_{BVSRS} is 0.025 with bootstrap MSE based on 1000 iterations of 0.0000044. Thus the P-value of the RSS is less than the P-value of the SRS and therefore one is more likely to reject H_0 with RSS and that may be due the fact that T_{OBVRSS} is more powerful than T_{BVSRS} .

Table 5.1. The drawn samples of size 16

	BVRSS Sample of (*r*=2 and *m*=4)					BVSRS of size *n*=16			
	Daily walking		Non-daily walking			Daily walking		Non-daily walking	
No.	Age *X*	Gender	Age Y	Gender		Age *X*	Gender	Age Y	Gender
1	73 (2,2)	Male	72	Male		68	Male	68	Male
2	72 (2,2)	Male	74	Male		72	Male	74	Male
3	74 (2,2)	Male	77	Male		74	Male	77	Male
4	68 (2, 2)	Male	74	Male		76	Male	83	Female
5	66 (2,2)	Male	67	Male		66	Male	67	Female
6	67 (2, 2)	Male	71	Male		79	Male	66	Female
7	75 (2, 2)	Female	74	Male		66	Female	68	Female
8	77(2, 2)	Female	79	Female		83	Female	78	Female
9	77 (2,2)	Female	69	Female		79	Female	78	Female
10	75 (2,2)	Female	79	Female		72	Female	74	Female
11	70(2,2)	Female	72	Female		66	Female	73	Female
12	71 (2, 2)	Female	71	Female		76	Female	81	Female
13	67 (2,2)	Female	69	Female		68	Female	75	Female
14	72 (2, 2)	Female	80	Female		67	Female	76	Female
15	71(2, 2)	Female	78	Female		79	Female	86	Female
16	67 (2, 2)	Female	73	Female		67	Female	73	Female

* The parenthesis is the observation label according to the OBVRSS procedure.

In conclusion, whenever OBVRSS, especially when *r* is odd, can be obtained, it is recommended to be used instead of BVSRS for the bivariate matched pairs sign test.

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