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Multifractal Structure of Noncompactly Supported Infinite Measures

by

Sze-man Ngai
Department of Mathmatical Sciences
Georgia Southern University, Statesboro, GA 30460-8093

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SZE-MAN NGAI

Abstract. Many important definitions in the theory of multifractal measures on $\mathbb{R}^d$, such as the L^q-spectrum, L^∞-dimensions, and the Hausdorff dimension of a measure, cannot be applied to noncompactly supported or infinite measures. We propose definitions that extend the original definitions to positive Borel measures on $\mathbb{R}^d$ which are finite on bounded sets, and recover many important results that hold for compactly supported finite measures. In particular, we prove that if the L^q-spectrum is differentiable at $q = 1$, then the derivative is equal to the Hausdorff dimension of the measure.

1. Introduction

Given a finite positive Borel measure μ on $\mathbb{R}^d$ with compact support, denoted by $\operatorname{supp}(\mu)$, a central problem in multifractal theory is to compute its *dimension spectrum* $f(\alpha)$, which is the Hausdorff dimension of the component of $\operatorname{supp}(\mu)$ consisting of points $x \in \operatorname{supp}(\mu)$ such that the *local dimension* of μ at x is α, i.e.,

$$f(\alpha) = \dim_{\mathrm{H}}\left\{x \in \operatorname{supp}(\mu) : \lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha\right\}.$$

The idea and physical significance of multifractal measures have their origins in the work of Mandelbrot [M] in the 1970's. In general it is very difficult to compute $f(\alpha)$ directly. In the mid 1980's, physicists Frisch and Parisi [FP], Halsey *et al.* [HJKPS] introduced an indirect method which is similar to the one described below. For $q \in \mathbb{R}$, define the *L^q-spectrum* of μ as

$$\tau(q) = \varliminf_{\delta\to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $\{B_\delta(x_i)\}_i$ is a family of disjoint closed δ-balls with center $x_i \in \operatorname{supp}(\mu)$ and the supremum is taken over all such families. They showed heuristically that $f(\alpha)$ is equal to the *Legendre transform* τ^* of τ. ($\tau^*(\alpha) := \inf\{q\alpha - \tau(q) :\ q \in \mathbb{R}\}$.) This relationship is known as the *multifractal formalism* (or the *thermodynamic formalism*).

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Since the multifractal formalism provides a very powerful technique for computing the dimension spectrum, a lot of efforts have been made to justify its validity for different classes of measures, especially those generated by *iterated function systems (IFSs)*. These include the results of Cawley and Mauldin [CM], Edgar and Mauldin [EM], Olsen ([O1], [O2]), Falconer [F3], Riedi [R], Patzschke [P], and Hanus *et al.* [HMU]. The main assumption in their works is the *open set condition (OSC)* (see [Hut], [F2]). The multifractal formalism for iterated function systems that do not satisfy the OSC have also been studied by many authors, including Daubechies and Lagarias [DL], Lau and Ngai [LN2], Feng ([Fe1], [Fe2]), Ye [Y], and Shmerkin [S].

In all the above investigations, the measure is assumed to be finite and compactly supported. There are interesting measures that are not compactly supported. Barnsley and Elton [BE] showed that measures with noncompact support arise naturally when noncontractive (or strictly expansive) maps are introduced into an IFS. They showed that the introduction of such maps allows for (1) the inclusion of maps that describe symmetries of the attractor, (2) the inclusion of random irregularities, and (3) the description of unbounded images. We refer the reader to [BE, Section 1] for some two-dimensional examples that illustrate the benefits of allowing noncontractive maps in an IFS.

Noncontractive IFSs are also of theoretical interest in their own right. Barnsley and Elton introduced the family of measures defined by the following IFS:

$$S_1(x) = \frac{1}{2}x, \qquad S_2(x) = x + 1, \qquad x \in \mathbb{R}, \tag{1.1}$$

with positive probability weights p_1, p_2. It is shown in [BE] that any such an invariant measure has support $[0, \infty)$ and is singular with respect to Lebesgue measure. However, little is known about the multifractal structure of such measures.

It is the purpose of this paper to lay the foundation for studying the multifractal structure of noncompactly supported measures. We also allow the measure to be infinite, provided all bounded subsets have finite measure.

Let μ be a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets, i.e., $\mu(E) < \infty$ if E is a bounded subset of $\mathbb{R}^d$. Such a measure is necessarily σ-finite. The d-dimensional Lebesgue measure is such an example. For $\delta > 0$ and $x \in \mathbb{R}^d$, let $B_\delta(x) := \{y \in \mathbb{R}^d : |y - x| \leq \delta\}$ denote the closed δ-ball centered at x, and for each subset $E \subseteq \mathbb{R}^d$ let $\mathcal{P}_\delta(E)$ denote the collection of δ-packings of E by closed balls, i.e.,

$$\mathcal{P}_\delta(E) := \big\{\{B_\delta(x_i)\}_i : x_i \in E, B_\delta(x_i) \text{ are disjoint}\big\}.$$

We say that an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of bounded subsets of $\mathbb{R}^d$ is *admissible* if $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d$ and any bounded subset of $\mathbb{R}^d$ is contained in some E_n.

To define the L^q-spectrum we fix an admissible sequence $\{E_n\}$. For $q \in \mathbb{R}$, $\delta > 0$, and $n \in \mathbb{N}$ sufficiently large so that $E_n^* := E_n \cap \operatorname{supp}(\mu) \neq \emptyset$, we define

$$S_\delta^n(q) := \sup_{\mathcal{B} \in \mathcal{P}_\delta(E_n^*)} \sum_{B \in \mathcal{B}} \mu(B)^q, \tag{1.2}$$

Define

$$\tau_n(q) := \underline{\lim}_{\delta \to 0^+} \frac{\ln S_\delta^n(q)}{\ln \delta}.$$

We observe that for any fixed $q \in \mathbb{R}$, $\{\tau_n(q)\}_n$ is a decreasing sequence. Using this observation, we can define the L^q-spectrum of μ.

Definition 1.1. *Let μ be a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets. The* (lower) L^q-spectrum *of μ is defined as*

$$\tau(q) = \tau_\mu(q) := \lim_{n \to \infty} \tau_n(q), \qquad q \in \mathbb{R}.$$

It is clear from the definition that if the support of μ is compact, then there exists some $n_0 \in \mathbb{N}$ such that $\tau_n(q) = \tau_{n_0}(q)$ for all $n \geq n_0$, and thus the above definition of $\tau(q)$ coincides with the standard one in the literature.

Remark 1.1. *The definition of $\tau(q)$ is independent of the choice of the admissible sequence. The proof is straightforward. Thus we will frequently use the admissible sequence of concentric balls $\{B_n(0)\}$ and let $B_n^* := B_n(0) \cap \operatorname{supp}(\mu)$.*

Remark 1.2. *In the definition of $\tau(q)$, suppose we define μ_n to be the restriction of μ to E_n, i.e.,*

$$\mu_n(E) := \mu(E \cap E_n), \quad \textit{for all Borel sets } E.$$

Let $\tau^{(n)}$ denote the L^q-spectrum of μ_n. Then for $q \geq 0$,

$$\tau(q) = \lim_{n \to \infty} \tau^{(n)}(q).$$

To see this, we use the admissible sequence of balls $\{B_n(0)\}$ and notice that if $q \geq 0$, then for $n \in \mathbb{N}$ sufficiently large so that $B_n^* \neq \emptyset$, and $\delta > 0$ sufficiently small,

$$\sup_{\mathcal{B} \in \mathcal{P}_\delta(B_{n+1}^*)} \sum_{B \in \mathcal{B}} \mu_{n+1}(B)^q \geq \sup_{\mathcal{B} \in \mathcal{P}_\delta(B_n^*)} \sum_{B \in \mathcal{B}} \mu(B)^q \geq \sup_{\mathcal{B} \in \mathcal{P}_\delta(B_n^*)} \sum_{B \in \mathcal{B}} \mu_n(B)^q.$$

This leads to

$$\tau^{(n+1)}(q) \leq \tau_n(q) \leq \tau^{(n)}(q),$$

and the result follows by letting $n \to \infty$.

Remark 1.3. *In the definition of $\tau(q)$, we assume that μ is finite on any bounded subset of $\mathbb{R}^d$. In fact, there exists a continuous σ-finite measure on $\mathbb{R}^2$ and a set C of positive Hausdorff dimension such that for all $x \in C$ and all $\delta > 0$, $\mu(B_\delta(x)) = \infty$. We will construct such a measure in Example 2.1. (Atomic measures with such properties can be constructed more easily; for example, let $\mu := \sum_{r\in\mathbb{Q}} \delta_r$, where $\mathbb{Q}$ is the set of all rational numbers and δ_r is the Dirac measure at r.) For such measures, $\tau(q)$ cannot be defined as above. Thus, we will not consider general σ-finite measures in this paper.*

We will show that $\tau(q)$ is an increasing concave function (see Proposition 2.2). Recall that the *effective domain* of a concave function $f : \mathbb{R} \to [-\infty, \infty)$ is defined as

$$\operatorname{Dom} f = \{x : -\infty < f(x) < \infty\}.$$

The *Legendre transform* (or *concave conjugate*) of a concave function f is the function $f^* : \mathbb{R} \to [-\infty, \infty)$ defined by

$$f^*(\alpha) = \inf\{\alpha x - f(x) : x \in \mathbb{R}\}.$$

For $x \in \operatorname{Dom} f$, we let $\partial f(x) \subseteq \mathbb{R}$ be the *subdifferential* of f at x, i.e.,

$$\partial f(x) = \{\alpha : f(y) \le f(x) + \alpha(y - x) \text{ for all } y \in \mathbb{R}\}.$$

Then $f^*(\alpha) + f(\alpha) = \alpha x$ for all $\alpha \in \partial f(x)$ (see [Ro]). f is *strictly concave* at x if there exists $\alpha \in \mathbb{R}$ such that

$$f(y) < f(x) + \alpha(y - x) \quad \text{for all } y \ne x. \tag{1.3}$$

Suppose $\tau(q)$ is the L^q-spectrum of a measure μ. We denote the special subdifferentials $\partial\tau(0)$ and $\partial\tau(1)$ by $[\alpha_0^-, \alpha_0^+]$ and $[\alpha_1^-, \alpha_1^+]$, respectively. Also, let $\tau'_+(q)$ and $\tau'_-(q)$ denote the right-hand and left-hand derivatives of τ at q, respectively. It is known (see, e.g., [LN2, Proposition 2.3]) that $\operatorname{Dom}\tau^*$ is an interval and $(\operatorname{Dom}\tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$, where

$$\alpha_{\min} := \inf\{\alpha \in \partial\tau(q) : q \in \operatorname{Dom}\tau\}, \quad \alpha_{\max} := \sup\{\alpha \in \partial\tau(q) : q \in \operatorname{Dom}\tau\}.$$

For each $\alpha > 0$, define

$$\underline{K}(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$
$$\overline{K}(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$
$$\underline{L}(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \ge \alpha \right\},$$
$$\overline{U}(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \alpha \right\}.$$

The following result shows that $\tau^*(\alpha)$ is an upper bound for the Hausdorff dimension of the above multifractal components of $\operatorname{supp}(\mu)$.

Theorem 1.1. *Let μ be a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets.*

(a) *If $\alpha_{\min} < \alpha < \alpha_0^+$, then $\dim_{\mathrm{H}} \overline{K}(\alpha) \le \dim_{\mathrm{H}} \overline{U}(\alpha) \le \tau^*(\alpha)$.*
(b) *If $\alpha_0^+ \le \alpha < \alpha_{\max}$, then $\dim_{\mathrm{H}} \underline{K}(\alpha) \le \dim_{\mathrm{H}} \underline{L}(\alpha) \le \tau^*(\alpha)$.*

Let $\alpha, \alpha_1, \alpha_2 \in (\operatorname{Dom} \tau^*)^\circ$ with $\alpha_1 < \alpha_2$. For any integer $n \in \mathbb{N}$ sufficiently large so that $B_n^* \ne \emptyset$, we define

$$N_\delta^n(\alpha) := \sup_{\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)} \#\{B \in \mathcal{B}_\delta^n : \mu(B) \ge \delta^\alpha\},$$
$$\widetilde{N}_\delta^n(\alpha) := \sup_{\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)} \#\{B \in \mathcal{B}_\delta^n : \mu(B) < \delta^\alpha\},$$
$$N_\delta^n(\alpha_1, \alpha_2) := \sup_{\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)} \#\{B \in \mathcal{B}_\delta^n : \delta^{\alpha_2} \le \mu(B) < \delta^{\alpha_1}\},$$

where $\#A$ denotes the cardinality of the set A.

By assuming the strict concavity of τ^*, we recover the following result that holds for compactly supported measures. Loosely speaking, the right side of (1.4) captures the notion of certain box counting dimension of the set of points with local dimension α.

Theorem 1.2. *Let μ be defined as in Theorem 1.1. Let $\alpha \in (\operatorname{Dom} \tau^*)^\circ$ and suppose that τ^* is strictly concave at α. Then*

$$\tau^*(\alpha) = \lim_{\epsilon\to 0^+} \lim_{n\to\infty} \varlimsup_{\delta\to 0^+} \frac{\ln N_\delta^n(\alpha-\epsilon, \alpha+\epsilon)}{-\ln \delta}. \tag{1.4}$$

Remark 1.4. *$N_\delta^n(\alpha)$, $\widetilde{N}_\delta^n(\alpha)$, and $N_\delta^n(\alpha_1, \alpha_2)$ can be defined by using a general admissible sequence instead of the balls. Since the right side of (1.4) does not depend on the choice of the admissible sequence, it is sufficient to use the balls.*

The definition of the Hausdorff dimension of an infinite measure will be given in Section 5 (see Definition 5.1). The following result recovers a theorem proved independently by Heurteaux [H] and the author [N].

Theorem 1.3. *Let μ be defined as in Theorem 1.1.*

(a) *For μ a.e. $x \in \operatorname{supp}(\mu)$,*

$$\tau'_+(1) \le \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \tau'_-(1). \tag{1.5}$$

(b) *If the L^q-spectrum $\tau(q)$ of μ is differentiable at $q = 1$, then $\dim_{\mathrm{H}}(\mu) = \tau'(1)$.*

We remark that we do not assume self-similarity, self-affinity, or self-conformality in this paper. It is of interest to get sharper results by making such assumptions, together with appropriate separation conditions.

This paper is organized as follows. In Section 2, we study the basic properties of the L^q-spectrum $\tau(q)$ and the L^q-dimensions. In Section 3 we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. Finally in Section 5 we study the Hausdorff dimension of a measure and prove Theorem 1.3.

2. L^q-spectrum and L^q-dimension

We first construct, by making use of the construction of the standard Cantor set, a continuous σ-finite measure on $\mathbb{R}^2$ and a set C of positive Hausdorff dimension such that for all $x \in C$ and all $\delta > 0$, $\mu(B_\delta(x)) = \infty$ (see Remark 1.3).

Let

$$S_1(x) = \frac{1}{3}x, \qquad S_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Define

$$C_0 = [0,1] \quad \text{and} \quad C_k := \bigcup_{i_1,\dots,i_k=1}^{2} S_{i_1} \circ \cdots \circ S_{i_k}(C_0), \quad k \ge 1.$$

Thus, $C := \bigcap_{k=0}^{\infty} C_k$ is the standard Cantor set.

Let $P_0 = \{0, 1\}$, the end-points of C_0, and for all $k \ge 1$, let P_k be the end-points of the subintervals in C_k that are not in $\bigcup_{i=0}^{k-1} P_i$. For example, $P_1 = \{1/3, 2/3\}$ and $P_2 = \{1/9, 2/9, 7/9, 8/9\}$. For each $k \ge 0$ and each $p \in P_k$, draw an open semi-disk $D_{1/3^{k+1}}(p)$ with center p and radius $1/3^{k+1}$ so that $D_{1/3^{k+1}}(p) \cap C_k = \{p\}$ and so that the diameter of $D_{1/3^{k+1}}(p)$ is vertical. Note that the semi-disks $\{D_{1/3^{k+1}}(p) : p \in P_k,\ k \ge 0\}$ are disjoint.

Now for each $k \geq 0$ and $p \in P_k$ we construct a σ-finite measure μ_p supported on $D_{1/3^{k+1}}(p)$ such that

$$\mu_p(B_\delta(p)) = \infty \quad \text{for all } \delta > 0.$$

This can be done easily by first dividing $D_{1/3^{k+1}}(p)$ into countably many disjoint sectors $\{S_n(p)\}_{n=1}^\infty$ and then assigning a weight of $1/n$ uniformly to $S_n(p)$.

Example 2.1. *For each $k \geq 0$, let P_k and μ_p, $p \in P_k$, be defined as above and let C be the standard Cantor set. Define $P := \bigcup_{k=0}^\infty P_k$ and $\mu := \sum_{p \in P} \mu_p$. Then μ is continuous and σ-finite. Moreover, for all $x \in C$ and all $\delta > 0$, $\mu(B_\delta(x)) = \infty$.*

Proof. Clearly, μ is continuous and σ-finite. Since $\overline{P} = C$, the Cantor set, its Hausdorff dimension equals $\ln 2/\ln 3 > 0$. Moreover, for each $x \in C$, there exists a sequence in P that converges to x. Hence for any $\delta > 0$, $\mu(B_\delta(x)) = \infty$. This completes the proof. □

Example 2.1 shows that for general σ-finite measures, (1.2) need not be defined and a different treatment is needed. For simplicity we will not work in such a general setting.

In the rest of this section, we study some basic properties of the L^q-spectrum defined in Section 1. We will define the box dimension for unbounded sets, and also extend the definition of L^∞-dimensions. Unless otherwise stated, we assume throughout the rest of this section that μ is a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets, and we fix the admissible sequence $\{B_n(0)\}$ (see Remark 1.1).

Proposition 2.2. *Let $\tau(q)$ be the L^q-spectrum of a positive Borel measure μ on $\mathbb{R}^d$ that is finite on bounded sets.*

(a) *τ is an increasing function.*
(b) $\tau(1) = 0$.
(c) *τ is concave on* $\operatorname{Dom}\tau$.

Proof. We first observe that if $c > 0$ is a constant and $\tilde{\mu} := c\mu$, then the values of $\tau_n(q)$ for $\tilde{\mu}$ and μ are the same. Hence for each fixed, sufficiently large n, we can assume, without loss of generality, that $\mu(B_{n+1}(0)) = 1$ so that $\mu(B_\delta(x)) \leq 1$ for all $x \in B_n^*$ and all $\delta > 0$ sufficiently small.

(a) Observe that for $q_1 < q_2$ and $n \in \mathbb{N}$ sufficiently large,

$$\sum_{x_i \in B_n^*} \mu(B_\delta(x_i))^{q_1} \geq \sum_{x_i \in B_n^*} \mu(B_\delta(x_i))^{q_2}.$$

The assertion now follows from definitions.

(b) For each $n \in \mathbb{N}$ sufficiently large,

$$\tau_n(1) = \underline{\lim}_{\delta \to 0^+} \frac{\ln \left(\sup \sum_{x_i \in B_n^*} \mu(B_\delta(x_i)) \right)}{\ln \delta} \geq \underline{\lim}_{\delta \to 0^+} \frac{\ln \mu(B_{n+1}^*)}{\ln \delta} = 0.$$

Thus, $\tau(1) \geq 0$.

On the other hand, for each $\delta > 0$, we can find a finite sequence of disjoint balls $\{B_\delta(x_i)\}_i$, with $x_i \in \operatorname{supp}(\mu)$, such that $\sum_i \mu(B_\delta(x_i)) \geq c_1 \mu(B_n(0))$, where c_1 is a constant independent of n. (This can be done, say, as follows. First, partition $B_n(0)$ by using a $\delta/\sqrt{d}$-mesh of disjoint cubes. Let Q_1 be a cube with maximum μ measure. Let $x_i \in \mathbb{Q}^d \cap \operatorname{supp}(\mu)$, where $\mathbb{Q}$ is the set of rational numbers. Then $Q_1 \subseteq B_\delta(x_1)$. Among all δ-mesh cubes whose distance from Q_1 is greater than δ, choose one that has maximum μ measure and call it Q_2. Let $x_2 \in Q_2 \cap \operatorname{supp}(\mu)$. Then $Q_2 \subseteq B_\delta(x_2)$ and $B_\delta(x_1) \cap B_\delta(x_2) = \emptyset$. Continue.) Now,

$$\sup \sum_{x_i \in B_n^*} \mu(B_\delta(x_i)) \geq c_1 \mu(B_n(0)),$$

which leads to $\tau_n(1) \leq 0$. Hence $\tau(1) \leq 0$. This completes the proof of (b).

(c) For n sufficiently large so that $B_n^* \neq \emptyset$, the concavity of each τ_n on $\operatorname{Dom} \tau_n$ follows directly by an application of Hölder's inequality; we omit the details. Now let $q_1, q_2 \in \operatorname{Dom} \tau$. Then $q_1, q_2 \in \operatorname{Dom} \tau_n$ for all n sufficiently large. Moreover, for $0 < \lambda < 1$,

$$\begin{aligned} \tau(\lambda q_1 + (1-\lambda) q_2) &= \lim_{n \to \infty} \tau_n(\lambda q_1 + (1-\lambda) q_2) \\ &\geq \lim_{n \to \infty} (\lambda \tau_n(q_1) + (1-\lambda) \tau_n(q_2)) \\ &= \lambda \tau(q_1) + (1-\lambda) \tau(q_2). \end{aligned}$$

Hence τ is concave on $\operatorname{Dom} \tau$. □

As for measures with compact support, we define the *(lower) L^q-dimension* of μ for each $q > 1$ as

$$\underline{\dim}_q(\mu) := \frac{\tau(q)}{q-1}.$$

However, the L^∞-dimensions cannot be defined as for compactly supported measures. For example, if μ is not compactly supported, the quantity

$$\frac{\ln \left(\inf_{x \in \operatorname{supp}(\mu)} \mu(B_\delta(x)) \right)}{\ln \delta}$$

always tends to ∞ as $\delta \to 0^+$. Therefore this limit cannot be used to define $\underline{\dim}_{-\infty}(\mu)$, which should represent the asymptotic slope of $\tau(q)$ as $q \to -\infty$. Let's examine the following example for some inspiration.

Example 2.3. *Let μ be the discrete measure defined on all positive integers by*

$$\mu(\{k\}) = \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}.$$

Then $\tau(q) = 0$ for all $q \in \mathbb{R}$.

Proof. Let n be any positive integer and let $0 < \delta < 1/2$. Then for any $q \in \mathbb{R}$, it follows from definitions that

$$\tau_n(q) = \varliminf_{\delta \to 0^+} \frac{\ln\left(\sup \sum_{k=1}^{n} 1/k^{2q}\right)}{\ln \delta} = 0.$$

Thus $\tau(q) = 0$ for all $q \in \mathbb{R}$. □

In the above example, it is desirable to define $\underline{\dim}_{-\infty}(\mu) = 0$. Note that if we define

$$\underline{\dim}^{(n)}_{-\infty}(\mu) := \varliminf_{\delta \to 0^+} \frac{\ln\left(\inf_{x \in B_n^*} \mu(B_\delta(x))\right)}{\ln \delta},$$

then

$$\underline{\dim}^{(n)}_{-\infty}(\mu) = \varliminf_{\delta \to 0^+} \frac{\ln(1/n^2)}{\ln \delta} = 0.$$

Hence $\underline{\dim}_{-\infty}(\mu)$ is 0 if it is defined as $\lim_{n\to\infty} \underline{\dim}^{(n)}_{-\infty}(\mu)$.

Motivate by the above example, we define the L^∞-dimensions as follows. Let $\{E_n\}_{n=1}^{\infty}$ be an admissible sequence and let $E_n^* := E_n \cap \operatorname{supp}(\mu)$. Define

$$\underline{\dim}^{(n)}_{\infty}(\mu) := \varliminf_{\delta \to 0^+} \frac{\ln(\sup_{x \in E_n^*} \mu(B_\delta(x)))}{\ln \delta}, \qquad \underline{\dim}^{(n)}_{-\infty}(\mu) := \varliminf_{\delta \to 0^+} \frac{\ln(\inf_{x \in E_n^*} \mu(B_\delta(x)))}{\ln \delta},$$

$$\underline{\dim}_{\infty}(\mu) := \lim_{n\to\infty} \underline{\dim}^{(n)}_{\infty}(\mu), \qquad \underline{\dim}_{-\infty}(\mu) := \lim_{n\to\infty} \underline{\dim}^{(n)}_{-\infty}(\mu).$$

The corresponding upper dimensions can be defined analogously by replacing $\varliminf_{\delta \to 0^+}$ with $\varlimsup_{\delta \to 0^+}$.

It is straightforward to verify that $\{\underline{\dim}^{(n)}_{\infty}(\mu)\}_n$ is a decreasing sequence, bounded below by 0. Hence $\underline{\dim}_{\infty}(\mu)$ is well defined and nonnegative. On the other hand, $\{\underline{\dim}^{(n)}_{-\infty}(\mu)\}_n$ is an increasing sequence, not necessarily bounded above, and hence $\underline{\dim}_{-\infty}(\mu)$ can be infinite. It is also easy to see that these definitions coincide with the standard ones if the support of μ is compact.

Remark 2.1. *It can be shown directly that the above definitions are independent of the choice of the admissible sequence. So again, we will frequently use the sequence of balls $B_n(0)$ and $B_n^* := B_n(0) \cap \operatorname{supp}(\mu)$.*

Proposition 2.4. *$\alpha_{\min} = \lim_{q\to\infty} \underline{\dim}_q(\mu) = \underline{\dim}_{\infty}(\mu) \le \overline{\dim}_{\infty}(\mu) \le d$.*

Proof. We first show $\overline{\dim}_\infty(\mu) \le d$. Fix some $n \in \mathbb{N}$ sufficiently large so that $B_n^* \neq \emptyset$. Note that B_n^* can be covered by $C\delta^{-d}$ δ-balls with centers in B_n^*, where C is some constant independent of δ. One of these balls must have μ measure at least $\mu(B_n^*)/C\delta^{-d}$. Hence,

$$\sup_{x\in B_n^*} \mu(B_\delta(x)) \ge \frac{\mu(B_n^*)}{C\delta^{-d}},$$

which leads to $\overline{\dim}_\infty^{(n)}(\mu) \le d$. Thus, $\overline{\dim}_\infty(\mu) \le d$.

To show that $\lim_{q\to\infty} \underline{\dim}_q(\mu) = \underline{\dim}_\infty(\mu)$ we let

$$a := \underline{\dim}_\infty(\mu) \qquad \text{and} \qquad \phi(q) := aq - \tau(q).$$

We can assume $q > 1$. Then for all n sufficiently large, we have

$$\sup_{x\in B_n^*} \mu(B_\delta(x))^q \le S_\delta^n(q),$$

which yields $q\underline{\dim}_\infty^{(n)}(\mu) \ge \tau_n(q)$. By letting $n \to \infty$, we get $aq \ge \tau(q)$. Hence $\phi(q) \ge 0$.

We claim that $\phi(q)$ is decreasing. Let $\epsilon > 0$ be arbitrary and let

$$a_n := \varliminf_{\delta\to 0^+} \frac{\ln\left(\sup_{x\in B_n^*} \mu(B_\delta(x))\right)}{\ln\delta}.$$

Then for $1 < q_1 < q_2$, a similar argument as that in [LN2, Proposition 3.4] yields

$$\tau_n(q_1) - q_1(a_n - \epsilon) \le \tau_n(q_2) - q_2(a_n - \epsilon).$$

Letting $\epsilon \to 0$, followed by $n \to \infty$, yields $\phi(q_2) \le \phi(q_1)$ and the claim follows.

We conclude that $\phi(q)$ is bounded and therefore $\lim_{q\to\infty} \phi(q)/q = 0$. It follows that $\lim_{q\to\infty} \tau(q)/q = a$ and $\lim_{q\to\infty} \underline{\dim}_q(\mu) = a = \underline{\dim}_\infty(\mu)$.

Lastly, we show $\alpha_{\min} = a$. By the proof above, the function $\phi(q) = aq - \tau(q)$ decreases to some nonnegative constant, say ℓ, as $q \to \infty$. Thus, $h(q) = aq - \ell$ is the asymptote to $\tau(q)$ as $q \to \infty$. Hence $\alpha_{\min} = a$. □

Recall that for any concave function τ, $(\operatorname{Dom}\tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$ (see [LN2, Proposition 2.3]). The proof of the following proposition is similar to that of [LN2, Proposition 3.5].

Proposition 2.5. *Let τ^* be the Legendre transform of τ.*

(a) $(\operatorname{Dom}\tau^*)^\circ = (\alpha_{\min}, \alpha_{\max}) \subseteq (0, \infty)$ *and* $\tau^* \ge 0$ *on* $\operatorname{Dom}\tau^*$.
(b) *Let $\alpha_0 \in \partial\tau(0)$. Then τ^* has a maximum at α_0 with $\tau^*(0) = -\tau(0)$. Consequently, τ^* is increasing on $[\alpha_{\min}, \alpha_0]$ and decreasing on $[\alpha_0, \alpha_{\max})$.*

For measures with compact support, $-\tau(0)$ is equal to the upper box dimension of the support of μ. Clearly, the usual definition of box dimension cannot be applied to unbounded sets. In order to recover this result, we will first extend the definition of box dimension to unbounded sets.

Definition 2.1. *Let E be a subset of $\mathbb{R}^d$, not necessarily bounded, and let $\{E_n\}$ be an admissible sequence of $\mathbb{R}^d$. We define the* lower *and* upper box dimensions *of E, respectively, as*

$$\underline{\dim}_{\mathrm{B}}(E) = \lim_{n\to\infty} \underline{\dim}_{\mathrm{B}}(E\cap E_n) \quad \textit{and} \quad \overline{\dim}_{\mathrm{B}}(E) \;=\lim_{n\to\infty} \overline{\dim}_{\mathrm{B}}(E\cap E_n).$$

If $\underline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(E)$ we call the common value the box dimension *of E and denote it by $\dim_{\mathrm{B}}(E)$.*

Clearly, $\{\underline{\dim}_{\mathrm{B}}(E\cap B_n(0))\}_n$ and $\{\overline{\dim}_{\mathrm{B}}(E\cap B_n(0))\}_n$ are increasing sequences in n and bounded above by d. Thus, $\underline{\dim}_{\mathrm{B}}(E)$ and $\overline{\dim}_{\mathrm{B}}(E)$ are well defined and

$$\underline{\dim}_{\mathrm{B}}(E) \le \overline{\dim}_{\mathrm{B}}(E) \le d.$$

We also note that if E is bounded, then the definitions of $\underline{\dim}_{\mathrm{B}}(E)$, $\overline{\dim}_{\mathrm{B}}(E)$, and $\dim_{\mathrm{B}}(E)$ coincide with the usual ones.

Remark 2.2. *The definitions of the box-dimensions are independent of the choice of the admissible sequence. The proof is straightforward.*

Proposition 2.6. $\tau(0) = -\overline{\dim}_{\mathrm{B}}(\mathrm{supp}(\mu))$.

Proof. For any $n \in \mathbb{N}$ sufficiently large so that $B_n^* \neq \emptyset$,

$$\tau_n(0) = \underline{\lim}_{\delta\to 0^+} \frac{\ln \sup \sum_{x_i\in B_n^*} \mu(B_\delta(x_i))^0}{\ln \delta} = \underline{\lim}_{\delta\to 0^+} \frac{\ln N_\delta}{\ln \delta} = -\overline{\lim}_{\delta\to 0^+} \frac{\ln N_\delta}{-\ln \delta},$$

where N_δ is the maximum number of disjoint δ-balls with centers in B_n^*. By the usual definition of the upper box dimension, $\tau_n(0) = -\overline{\dim}_{\mathrm{B}}(B_n^*)$. The assertion now follows by letting $n \to \infty$. □

The following propositions can be proved by first establishing analogous ones for τ_n, using arguments for compactly supported measures (see [LN2, Propositions 3.3, 3.4]), and then letting $n \to \infty$. We omit the details.

Proposition 2.7. $\mathrm{Dom}\,\tau = \mathbb{R}$ *if and only if* $\overline{\dim}_{-\infty}(\mu) < \infty$. $\mathrm{Dom}\,\tau = [0,\infty)$ *if and only if* $\underline{\dim}_{-\infty}(\mu) = \infty$.

Example 2.9 below illustrates the second possibility of the dichotomy in Proposition 2.7.

Proposition 2.8. $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu)$.

The following example illustrates some of the definitions and results in this section.

Example 2.9. *Consider the family of IFSs* $\{S_{1,n}, S_{2,n}\}$, $n \in \mathbb{N}$, *defined as:*

$$S_{1,n}(x) = \frac{x}{3} + (n-1), \qquad S_{2,n}(x) = \frac{x}{3} + \Big(n - \frac{1}{3}\Big), \qquad n \in \mathbb{N},$$

Then the attractor of $\{S_{1,n}, S_{2,n}\}$ *is the standard Cantor set on* $[n-1, n]$. *Let* μ_n *be the weighted Cantor measure defined by* $\{S_{1,n}, S_{2,n}\}$ *with probability weights* $p_{1,n} = 1/2^n$ *and* $p_{2,n} = (2^n - 1)/2^n$. *Let* $\mu := \sum_{n=1}^{\infty} \mu_n$. *Then*

(a) $\alpha_{\min} = \underline{\dim}_{\infty}(\mu) = 0$; $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu) = \infty$ *and thus* $\operatorname{Dom}\tau = [0, \infty)$.

(b) $\tau(q) = \begin{cases} -\infty, & -\infty < q \le 0 \\ (q-1)\ln 2/\ln 3, & 0 \le q \le 1 \\ 0, & q > 1. \end{cases}$

(c) $\tau^*(\alpha) = \begin{cases} \alpha, & 0 \le \alpha \le \ln 2/\ln 3 \\ \ln 2/\ln 3, & \alpha > \ln 2/\ln 3. \end{cases}$

(d) $\operatorname{Dom}\tau = [0, \infty)$ *and* $\overline{\dim}_{\mathrm{B}}(\operatorname{supp}(\mu)) = \ln 2/\ln 3$.

Proof. $\alpha_{\min}$ and $\alpha_{\max}$ can be computed directly, by using the construction of the weighted Cantor measure, as follows:

$$0 \le \frac{\ln\big(\sup_{x \in B_n^*} \mu(B_{1/3^k}(x))\big)}{\ln(1/3^k)} \le \frac{\ln[(2^k-1)/2^k]}{\ln(1/3^k)} \to 0.$$

Thus $\underline{\dim}_{\infty}^{(n)}(\mu) = 0$ and hence, by Proposition 2.4, $\alpha_{\min} = \underline{\dim}_{\infty}(\mu) = 0$. On the other hand, for all $n \in \mathbb{N}$,

$$\frac{\ln\big(\sup_{x \in B_n^*} \mu(B_{1/3^k}(x))\big)}{\ln(1/3^k)} \ge \frac{\ln(1/2^k)}{\ln(1/3^k)} \to \infty.$$

Thus, $\overline{\dim}_{-\infty}^{(n)}(\mu) = 0$ and hence, by Proposition 2.8, $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu) = \infty$. That $\operatorname{Dom}\tau = [0, \infty)$ now follows from Proposition 2.7.

(b) In view of (a), it suffices to derive $\tau(q)$ for $0 \le q \le 1$. We claim that if $\tau_{\nu_1}(q) \le \tau_{\nu_2}(q)$ on $[0, 1]$, then $\tau_{\nu_1+\nu_2}(q) = \tau_{\nu_1}(q)$ on $[0, 1]$. This is in fact a consequence of the following inequalities, which hold for all k sufficiently large:

$$\sum_{B \in \mathcal{B}} \nu_1(B)^q \le \sum_{B \in \mathcal{B}} [\nu_1(B) + \nu_2(B)]^q \le \sum_{B \in \mathcal{B}} 2^q [\nu_1(B)^q + \nu_2(B)^q], \quad \mathcal{B} \in \mathcal{P}_\delta(E_k^*).$$

Now, for each n, it is well known (see [CM]) that

$$\tau_{\mu_n}(q) = \frac{\ln\big[1/2^{nq} + (2^n-1)^q/2^{nq}\big]}{-\ln 3}.$$

The claim now implies that

$$\tau(q) = \tau_{\mu_1}(q) = \frac{\ln 2}{\ln 3}(q-1).$$

(c) follows directly from (a), while (d) follows directly from (a), (b), and Proposition 2.6. ☐

3. Multifractal components

Let μ be a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets. Fix the admissible sequence $\{B_n(0)\}$, let $\mathcal{B}_\delta \in \mathcal{P}_\delta(\operatorname{supp}(\mu))$, and let $\alpha, \alpha_1, \alpha_2 \in (\operatorname{Dom}\tau^*)^\circ$ with $\alpha_1 < \alpha_2$. For any integer $n \in \mathbb{N}$ sufficiently large so that $B_n^* \neq \emptyset$, we let $N_\delta^n(\alpha)$, $\widetilde{N}_\delta^n(\alpha)$, and $N_\delta^n(\alpha_1, \alpha_2)$ be defined as in Section 1.

Lemma 3.1. *Let τ be the L^q-spectrum of a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) such that each bounded subset of $\mathbb{R}^d$ has finite measure. Suppose $\alpha_{\min} < \alpha < \alpha_0^+$, $q \in \partial\tau^*(\alpha)$, and $\xi > 0$. Fix a positive integer n sufficiently large so that $B_n^* \neq \emptyset$. Then for any $\epsilon > 0$, there exists $\delta_\epsilon(n) > 0$ (depending on n) such that for all $0 < \delta < \delta_\epsilon(n)$,*

$$N_\delta^n(\alpha \pm \epsilon) \leq \delta^{-\tau^*(\alpha) - (\xi \pm q)\epsilon}.$$

If $\alpha_0^+ < \infty$, then the same result holds for all α satisfying $\alpha_{\min} < \alpha \leq \alpha_0^+$.

Proof. Note that if $\alpha_{\min} < \alpha < \alpha_0^+$ or $\alpha_{\min} < \alpha = \alpha_0^+ < \infty$, and if $q \in \partial\tau^*(\alpha)$, then $q \geq 0$.

Let $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$ such that $\mu(B) \geq \delta^{\alpha+\epsilon}$ for all $B \in \mathcal{B}_\delta^n$. Then

$$S_\delta^n(q) \geq \sum_{B \in \mathcal{B}_\delta^n} \mu(B)^q \geq \delta^{q(\alpha+\epsilon)} \# \mathcal{B}_\delta^n.$$

Taking supremum over all such families of balls, we have

$$S_\delta^n(q) \geq \delta^{q(\alpha+\epsilon)} N_\delta^n(\alpha+\epsilon). \tag{3.1}$$

Since

$$\tau(q) \leq \tau_n(q) = \varliminf_{\delta \to 0^+} \frac{\ln S_\delta^n(q)}{\ln \delta}, \tag{3.2}$$

there exists $\delta_\epsilon(n) > 0$ (depending on n) such that for all $0 < \delta < \delta_\epsilon(n)$,

$$\tau(q) - \xi\epsilon \leq \tau_n(q) - \xi\epsilon \leq \frac{\ln S_\delta^n(q)}{\ln \delta},$$

and thus

$$S_\delta^n(q) \leq \delta^{\tau_n(q) - \xi\epsilon} \leq \delta^{\tau(q) - \xi\epsilon}. \tag{3.3}$$

Combining (3.1) and(3.3), we have, for all $0<\delta<\delta_\epsilon(n)$,

$$N_\delta^n(\alpha+\epsilon)\le \delta^{-q(\alpha+\epsilon)}\delta^{\tau(q)-\xi\epsilon}=\delta^{-\tau^*(\alpha)-(\xi+q)\epsilon}.$$

The proof for $N_\delta^n(\alpha-\epsilon)$ is similar. □

For $\alpha_0^+\le\alpha<\alpha_{\max}$, we have the following analog of Lemma 3.1:

Lemma 3.2. *Let τ be as in Lemma 3.1. Let $\alpha_0^+\le\alpha<\alpha_{\max}$, $q\in\partial\tau^*(\alpha)$ and $\xi>0$. Fix a positive integer n sufficiently large so that $B_n^*\ne\emptyset$. Then for any $\epsilon>0$, there exists $\delta_\epsilon(n)>0$ (depending on n) such that for all $0<\delta<\delta_\epsilon(n)$,*

$$\widetilde{N}_\delta^n(\alpha\pm\epsilon)\le\delta^{-\tau^*(\alpha)-(\xi\pm q)\epsilon}.$$

Proof. Note that $q\le 0$. Let $\mathcal{B}_\delta^n\in\mathcal{P}_\delta(B_n^*)$ such that $\mu(B)<\delta^{\alpha+\epsilon}$ for all $B\in\mathcal{B}_\delta^n$. Then for any positive integer n,

$$S_\delta^n(q)\ge\sum_{B\in\mathcal{B}_\delta^n}\mu(B)^q\ge\delta^{q(\alpha+\epsilon)}\#\mathcal{B}_\delta^n.$$

Taking supremum over all such $\mathcal{B}_\delta^n$ yields

$$S_\delta^n(q)\ge\delta^{q(\alpha+\epsilon)}\widetilde{N}_\delta^n(\alpha+\epsilon). \tag{3.4}$$

By (3.2), there exists $\delta_\epsilon(n)>0$, depending on n, such that for all $0<\delta<\delta_\epsilon(n)$,

$$S_\delta^n(q)\le\delta^{\tau(q)-\xi\epsilon}. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain, for all $0<\delta<\delta_\epsilon(n)$,

$$\widetilde{N}_\delta^n(\alpha+\epsilon)\le\delta^{-q(\alpha+\epsilon)}\delta^{\tau(q)-\xi\epsilon}=\delta^{-\tau^*(\alpha)-(\xi+q)\epsilon}.$$

Similarly, we can obtain the stated result for $\widetilde{N}_\delta^n(\alpha-\epsilon)$. □

The following Theorem proves one inequality of equation (1.4) in Theorem 1.2.

Theorem 3.3. *Let τ be the L^q-spectrum of a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) such that each bounded subset of $\mathbb{R}^d$ has finite measure.*

(a) *If $\alpha_{\min}<\alpha<\alpha_0^+$, then*

$$\lim_{\epsilon\to0^+}\lim_{n\to\infty}\overline{\lim_{\delta\to0^+}}\frac{\ln N_\delta^n(\alpha+\epsilon)}{-\ln\delta}\le\tau^*(\alpha).$$

(b) *If $\alpha_0^+\le\alpha<\alpha_{\max}$, then*

$$\lim_{\epsilon\to0^+}\lim_{n\to\infty}\overline{\lim_{\delta\to0^+}}\frac{\ln \widetilde{N}_\delta^n(\alpha-\epsilon)}{-\ln\delta}\le\tau^*(\alpha).$$

Consequently, for any $\alpha \in (\operatorname{Dom}\tau^*)^\circ$,

$$\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\varlimsup_{\delta\to 0^+}\frac{\ln N_\delta^n(\alpha-\epsilon,\alpha+\epsilon)}{-\ln\delta}\le \tau^*(\alpha).$$

Proof. (a) Let $\alpha_{\min} < \alpha < \alpha_0^+$, $q \in \partial\tau^*(\alpha)$, and $n \in \mathbb{N}$. Then for any $\epsilon > 0$, Lemma 3.1 (with $\xi = 1$) implies that there exists $\delta_\epsilon(n) > 0$ such that for all $0 < \delta < \delta_\epsilon(n)$,

$$N_\delta^n(\alpha+\epsilon)\le \delta^{-\tau^*(\alpha)-(1+q)\epsilon},$$

which implies that

$$\varlimsup_{\delta\to 0^+}\frac{\ln N_\delta^n(\alpha+\epsilon)}{-\ln\delta}\le \tau^*(\alpha)+(1+q)\epsilon.$$

Note that $N_\delta^n(\alpha+\epsilon)$ is an increasing sequence of n. Hence by letting $n\to\infty$ we obtain

$$\lim_{n\to\infty}\varlimsup_{\delta\to 0^+}\frac{\ln N_\delta^n(\alpha+\epsilon)}{-\ln\delta}\le \tau^*(\alpha)+(1+q)\epsilon.$$

The left side is a decreasing function of ϵ. Part (a) now follows by letting $\epsilon\to 0^+$.

(b) Let $\alpha_0^+ \le \alpha < \alpha_{\max}$, $q \in \partial\tau^*(\alpha)$, and $n \in \mathbb{N}$. Then for any $\epsilon > 0$, Lemma 3.2 (with $\xi = 1$) implies that there exists $\delta_\epsilon(n) > 0$ such that for all $0 < \delta < \delta_\epsilon(n)$,

$$\widetilde{N}_\delta^n(\alpha-\epsilon)\le \delta^{-\tau^*(\alpha)-(1-q)\epsilon},$$

and hence

$$\varlimsup_{\delta\to 0^+}\frac{\ln \widetilde{N}_\delta^n(\alpha-\epsilon)}{-\ln\delta}\le \tau^*(\alpha)+(1-q)\epsilon.$$

Part (b) now follows by letting $n\to\infty$, followed by $\epsilon\to 0^+$. The last assertion follows by combining the results in (a) and (b). □

Proof of Theorem 1.1. Part (a) follows by applying Lemma 3.1 and a similar argument as that of the proof of [LN2, Theorem 4.1(i)]. We will focus on the proof of (b). Let $\alpha_0^+ \le \alpha < \alpha_{\max}$. Let $n_0 \in \mathbb{N}$ be sufficiently large so that $B_{n_0}^* \ne \emptyset$. For each positive integer $n \ge n_0$, we define

$$\underline{K}^n(\alpha) := \underline{K}(\alpha)\cap B_n(0) = \left\{x\in B_n^* : \lim_{\delta\to 0^+}\frac{\ln\mu(B_\delta(x))}{\ln\delta}=\alpha\right\},$$
$$\underline{L}^n(\alpha) := \underline{L}(\alpha)\cap B_n(0) = \left\{x\in B_n^* : \varliminf_{\delta\to 0^+}\frac{\ln\mu(B_\delta(x))}{\ln\delta}\ge\alpha\right\}.$$

Clearly, $\underline{K}(\alpha)=\bigcup_{n=n_0}^\infty \underline{K}^n(\alpha)$ and $\underline{L}(\alpha)=\bigcup_{n=n_0}^\infty \underline{L}^n(\alpha)$. By the countable stability of the Hausdorff dimension,

$$\dim_{\rm H}\underline{K}(\alpha)=\lim_{n\to\infty}\dim_{\rm H}\underline{K}^n(\alpha)\qquad\text{and}\qquad \dim_{\rm H}\underline{L}(\alpha)=\lim_{n\to\infty}\dim_{\rm H}\underline{L}^n(\alpha).$$

Hence the theorem will follow if we can prove that for all $n \geq n_0$,

$$\dim_{\mathrm{H}} \underline{L}^n(\alpha) \leq \tau^*(\alpha).$$

Fix $n \geq n_0$. Let $q \in \partial\tau^*(\alpha)$ and let $\epsilon > 0$ be arbitrary. Note that $q \leq 0$. By applying Lemma 3.2 with $\xi = 1$, we can find a positive integer $k_\epsilon(n)$ such that for all $k \geq k_\epsilon(n)$,

$$\widetilde{N}^n_{2^{-k}}(\alpha - \epsilon) \leq 2^{k(\tau^*(\alpha)+\eta)}, \tag{3.6}$$

where $\eta = (1-q)\epsilon$.

For each $x \in \underline{L}^n(\alpha)$, we choose $k_x \in \mathbb{N}$ sufficiently large so that for all integers $k \geq k_x$,

$$\mu(B_{2^{-k}}(x)) < 2^{-k(\alpha-\epsilon)}. \tag{3.7}$$

For each $0 < \delta < 2^{-k_\epsilon}$, define

$$\mathcal{G}^n_\delta := \left\{B_{2^{-k}}(x) : x \in \underline{L}^n(\alpha),\ k \geq k_x,\ 2^{-k} < \delta/2\right\}.$$

Observe that $\mathcal{G}^n_\delta$ is a Vitali cover of $\underline{L}^n(\alpha)$. Let $s = \tau^*(\alpha) + 2\eta$. By the Vitali covering theorem (see, e.g., [F1]), there exists a disjoint subcollection $\{B_i\}$ of $\mathcal{G}^n_\delta$ such that

$$\text{either} \quad \sum_i |B_i|^s = \infty \qquad \text{or} \qquad \mathcal{H}^s\Big(\underline{L}^n(\alpha) \setminus \bigcup_i B_i\Big) = 0,$$

where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure on $\mathbb{R}^d$ and $|E|$ denotes the diameter of a set E. The former case is impossible since

$$\begin{aligned}
\sum_i |B_i|^s &= \sum_{2^{-k+1}<\delta} \ \sum_{|B_i|=2^{-k+1}} |B_i|^s \\
&\leq \sum_{2^{-k+1}<\delta} 2^{k(\tau^*(\alpha)+\eta)} \cdot 2^{(-k+1)s} \qquad \text{(by (3.6) and (3.7))} \\
&\leq 2^s \sum_{2^{-k+1}<\delta} 2^{k(\tau^*(\alpha)+\eta)} \cdot 2^{-k(\tau^*(\alpha)+2\eta)} \\
&\leq \frac{2^s}{2^\eta - 1} =: C < \infty.
\end{aligned}$$

It follows that the latter case must hold and therefore

$$\begin{aligned}
\mathcal{H}^s_\delta(\underline{L}^n(\alpha)) &= \mathcal{H}^s_\delta\Big(\underline{L}^n(\alpha) \cap \bigcup_i B_i\Big) + \mathcal{H}^s_\delta\Big(\underline{L}^n(\alpha) \setminus \bigcup_i B_i\Big) \\
&\leq \mathcal{H}^s_\delta\Big(\bigcup_i B_i\Big) + 0 \\
&\leq \sum_i |B_i|^s \leq C < \infty.
\end{aligned}$$

It follows that

$$\dim_{\mathrm{H}} \underline{L}^n(\alpha) \leq s = \tau^*(\alpha) + 2\eta.$$

Letting $\eta \to 0^+$ yields $\dim_{\mathrm{H}} \underline{L}^n(\alpha) \leq \tau^*(\alpha)$ and this completes the proof. □

4. Proof of Theorem 1.2

We devote this section to the proof of Theorem 1.2. Throughout this section we assume that μ is a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets, and we fix the admissible sequence $\{B_n(0)\}$.

Lemma 4.1. *For any $\eta > 0$, there exists $N = N(\eta) \in \mathbb{N}$ such that for all $n \geq N$, there exists some $\delta(n,\eta) > 0$ such that for all $0 < \delta < \delta(n,\eta)$,*

$$\delta^{\alpha_{\max}+\eta} < \mu(B_\delta(x)) < \delta^{\alpha_{\min}-\eta} \quad \textit{for all } x \in B_n^*.$$

Consequently, for any $x \in \operatorname{supp}(\mu)$,

$$\alpha_{\min} \leq \varliminf_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \varlimsup_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha_{\max}.$$

Proof. It follows from Propositions 2.4 and 2.8 that

$$\alpha_{\min} = \underline{\dim}_\infty(\mu) = \lim_{n\to\infty} \underline{\dim}^{(n)}_\infty(\mu) = \lim_{n\to\infty} \varliminf_{\delta\to 0^+} \frac{\ln(\sup_{x\in B_n^*} \mu(B_\delta(x)))}{\ln \delta} \quad \text{and}$$

$$\alpha_{\max} = \overline{\dim}_{-\infty}(\mu) = \lim_{n\to\infty} \underline{\dim}^{(n)}_{-\infty}(\mu) = \lim_{n\to\infty} \varlimsup_{\delta\to 0^+} \frac{\ln(\inf_{x\in B_n^*} \mu(B_\delta(x)))}{\ln \delta},$$

from which the stated inequalities follow directly. □

Proof of Theorem 1.2. Since one inequality has already been established in Theorem 3.3, we only need to prove the reverse inequality.

Let $q \in \partial\tau^*(\alpha)$ satisfy the following inequality for strict concavity (see (1.3)):

$$\tau^*(\xi) < \tau^*(\alpha) + q(\xi - \alpha) \quad \text{for all } \xi \neq \alpha.$$

According to the properties of strictly concave functions (see [Ro], [LN2]), for any $\epsilon > 0$ sufficiently small so that

$$\alpha_{\min} < \alpha - \epsilon < \alpha < \alpha + \epsilon < \alpha_{\max},$$

there exists $\eta = \eta(\epsilon) > 0$, with $0 < \eta < \epsilon/2$, such that whenever $|\xi - \alpha| \geq \epsilon/2$,

$$q\xi - \tau^*(\xi) \geq q\alpha - \tau^*(\alpha) + (1 + 2|q|)\eta = \tau(q) + (1 + 2|q|)\eta. \tag{4.1}$$

By Lemma 4.1, there exists $N_1 = N_1(\eta) \in \mathbb{N}$ such that for all $n \geq N_1$, there exists $\delta(n,\eta) > 0$ such that for all $0 < \delta < \delta(n,\eta)$,

$$\delta^{\alpha_{\max}+\eta/2} < \mu(B_\delta(x)) < \delta^{\alpha_{\min}-\eta/2} \quad \text{for all } x \in B_n^*. \tag{4.2}$$

Let $N_2 \geq N_1$ be a positive integer such that $B_n^* \neq \emptyset$ for all $n \geq N_2$.

We now divide the proof into four cases, according to the value of α.

Case 1. $\alpha_{\min} < \alpha < \alpha_0^+ < \infty$.

Note that in this case, $q \geq 0$. We assume $\epsilon > 0$ is sufficiently small so that

$$\alpha_{\min} < \alpha - \epsilon < \alpha < \alpha + \epsilon < \alpha_0^+.$$

Let $\mathcal{P} = \{\alpha_j\}_{j=1}^{\ell+1}$ be an approximate η-partition of $[\alpha_{\min} - \eta/2, \alpha_0^+] \setminus (\alpha - \epsilon, \alpha + \epsilon)$ satisfying

$$\alpha_1 = \alpha_{\min} - \frac{\eta}{2}, \quad \alpha_{j+1} = \alpha_j + \eta, \quad \alpha_{\ell+1} = \alpha_0^+, \quad \text{and}$$
$$(\alpha_j, \alpha_{j+1}) \cap \big([\alpha_{\min} - \eta/2, \alpha_0^+] \setminus (\alpha-\epsilon, \alpha+\epsilon)\big) \neq \emptyset, \quad \text{for all } j = 1, \ldots, \ell.$$

Now fix $n \geq N_3$ (to be determined below) and let $\delta > 0$ be sufficiently small (depending on η and n) so that (4.2) and conditions (i), (ii), and (iii) below hold.

(i) $(\ell - 1)\delta^{\eta/4} \leq 1$.

(ii) For all $j = 1, \ldots, \ell - 1$ and $q_{j+2} \in \partial\tau^*(\alpha_{j+2})$,

$$N_\delta^n(\alpha_{j+1}) \leq \delta^{-\tau^*(\alpha_{j+2})-(1/4-q_{j+2})\eta}.$$

Moreover,

$$N_\delta^n(\alpha_{\ell+1}) \leq \delta^{-\tau^*(\alpha_\ell)-(1/4+q_\ell)\eta}.$$

((ii) follows by putting $\xi = 1/4$ in Lemma 3.1.) Also, since $\tau^*(\alpha_0^+) = -\tau(0) = \overline{\dim}_B(\operatorname{supp}(\mu))$ (see Proposition 2.6), there exists a positive integer $N_3 \geq N_2$ such that for all $n \geq N_3$ and $\delta > 0$ sufficiently small,

(iii) $\#\mathcal{B}_\delta^n \leq \delta^{-\tau^*(\alpha_0^+)-\eta/2}$ for any $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$.

For any $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$, by using (4.2), we can write

$$\sum_{B \in \mathcal{B}_\delta^n} \mu(B)^q = \left(\sum_I + \sum_{II} + \sum_{III}\right) \mu(B)^q, \tag{4.3}$$

where $\sum_I$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying $\delta^{\alpha+\epsilon} \leq \mu(B) < \delta^{\alpha-\epsilon}$, $\sum_{II}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying

$$\delta^{\alpha_0^+} \leq \mu(B) < \delta^{\alpha+\epsilon} \qquad \text{or} \qquad \delta^{\alpha-\epsilon} \leq \mu(B) < \delta^{\alpha_{\min}-\eta/2},$$

and $\sum_{III}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying $\mu(B) < \delta^{\alpha_0^+}$.

We first estimate $\sum_{II}$. Ignoring the rightmost subinterval for the moment, we have

$$\begin{aligned}
\sum_{II} \mu(B)^q &= \sum_{j=1}^{\ell-1} \sum \{\mu(B)^q : \delta^{\alpha_{j+1}} \le \mu(B) < \delta^{\alpha_j}\} \\
&\le \sum_{j=1}^{\ell-1} N_\delta^n(\alpha_{j+1})\delta^{q\alpha_j} \\
&\le \sum_{j=1}^{\ell-1} \delta^{-\tau^*(\alpha_{j+2})-(1/4-q_{j+2})\eta+q\alpha_{j+2}-2q\eta} \qquad \text{(by (ii))} \\
&\le \sum_{j=1}^{\ell-1} \delta^{\tau(q)+(1+2q)\eta-(1/4-q_{j+2})\eta-2q\eta} \qquad \text{(by (4.1))} \\
&\le \sum_{j=1}^{\ell-1} \delta^{\tau(q)+3\eta/4} \quad (\text{since } q_{j+2}\eta \ge 0) \\
&\le \delta^{\tau(q)+\eta/2} \qquad \text{(by (i))}.
\end{aligned}$$

For the rightmost subinterval in the partition, the above estimation does not hold because $\alpha_{\ell+1} + \eta > \alpha_0^+$; we use a different estimation.

$$\begin{aligned}
\sum\{\mu(B)^q : \delta^{\alpha_{\ell+1}} \le \mu(B) < \delta^{\alpha_\ell}\} &\le N_\delta^n(\alpha_{\ell+1})\delta^{q\alpha_\ell} \\
&\le \delta^{-\tau^*(\alpha_\ell)-(1/4+q_\ell)\eta+q\alpha_\ell} \qquad \text{(by (ii))} \\
&\le \delta^{\tau(q)+(1+2q)\eta-(1/4+q_\ell)\eta} \qquad \text{(by (4.1))} \\
&\le \delta^{\tau(q)+3\eta/4} \qquad (\text{since } (2q-q_\ell)\eta \ge 0).
\end{aligned}$$

Thus,

$$\sum_{II} \mu(B)^q \le 2\delta^{\tau(q)+\eta/2}. \tag{4.4}$$

$\sum_{III}$ can be estimated as follows:

$$\begin{aligned}
\sum_{III} \mu(B)^q &\le \delta^{q\alpha_0^+} \# \mathcal{B}_\delta^n \\
&\le \delta^{q\alpha_0^+ - \tau^*(\alpha_0^+)-\eta/2} \qquad \text{(by (iii))} \\
&\le \delta^{\tau(q)+(1+2q)\eta-\eta/2} \qquad \text{(by (4.1))} \\
&\le \delta^{\tau(q)+\eta/2} \qquad (\text{since } 2q\eta \ge 0).
\end{aligned}$$

Combining this with the estimation in (4.4) and taking supremum over all $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$, we have

$$S_\delta^n(q) \le \delta^{q(\alpha-\epsilon)} N_\delta^n(\alpha-\epsilon, \alpha+\epsilon) + 3\delta^{\tau(q)+\eta/2}. \tag{4.5}$$

Let $N_4 \geq N_3$ be a positive integer such that for all $n \geq N_4$,

$$\tau_n(q) \leq \tau(q) + \frac{\eta}{8}. \tag{4.6}$$

Since

$$\varliminf_{\delta \to 0^+} \frac{\ln S_\delta^n(q)}{\ln \delta} = \tau_n(q),$$

for each $n \geq N_4$, there exists a decreasing sequence $\delta_k = \delta_k(n)$, depending on n and tending to 0, such that

$$\frac{\ln S_{\delta_k}^n(q)}{\ln \delta_k} \leq \tau_n(q) + \frac{\eta}{8} \leq \tau(q) + \frac{\eta}{4}$$

(the second inequality follows from (4.6)), and thus for all k sufficiently large,

$$S_{\delta_k}^n(q) \geq 4\delta_k^{\tau(q)+\eta/2}. \tag{4.7}$$

Combining (4.5) and (4.7) gives

$$4\delta_k^{\tau(q)+\eta/2} \leq \delta_k^{q(\alpha-\epsilon)} N_{\delta_k}^n(\alpha-\epsilon, \alpha+\epsilon) + 3\delta_k^{\tau(q)+\eta/2},$$

and thus

$$\frac{\ln N_{\delta_k}^n(\alpha-\epsilon, \alpha+\epsilon)}{-\ln \delta_k} \geq \tau^*(\alpha) - q\epsilon - \frac{\eta}{2}.$$

(Keep in mind that for each n, there is a different sequence δ_k.) It follows that

$$\lim_{n\to\infty} \varlimsup_{\delta \to 0^+} \frac{\ln N_\delta^n(\alpha-\epsilon, \alpha+\epsilon)}{-\ln \delta} \geq \tau^*(\alpha) - q\epsilon - \frac{\eta}{2}.$$

Since $0 < \eta < \epsilon/2$ and the above inequality holds for all $\epsilon > 0$ sufficiently small, we have

$$\tau^*(\alpha) \leq \lim_{\epsilon\to 0^+} \lim_{n\to\infty} \varlimsup_{\delta \to 0^+} \frac{\ln N_\delta^n(\alpha-\epsilon, \alpha+\epsilon)}{-\ln \delta}.$$

This proves the theorem for Case 1.

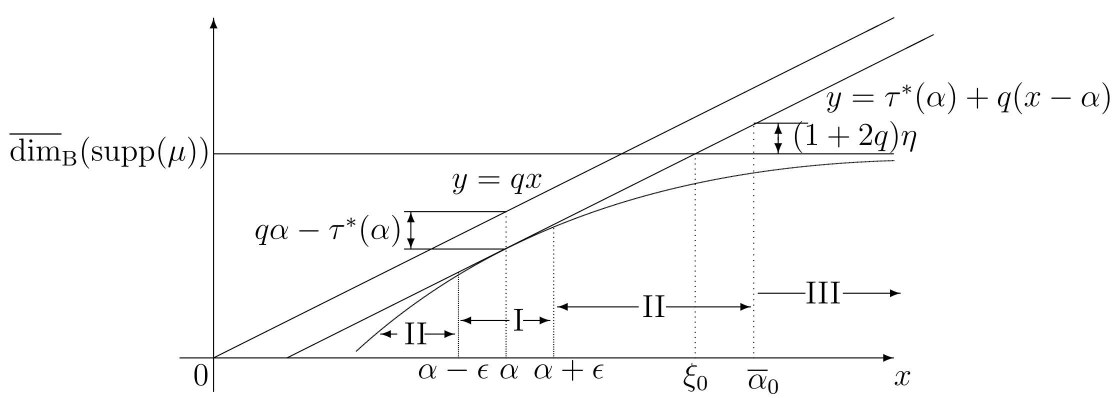

FIGURE 1. Illustration for Case 2.

Case 2. $\alpha_{\min} < \alpha < \alpha_0^+ = \infty$. Note that Example 2.9 corresponds to this case.

Again, $q \geq 0$. In this case the interval $[\alpha - \epsilon, \alpha_0^+)$ is unbounded and hence the estimation of $\sum_{II}$ in Case 1 fails. We need a different approach.

By the strict concavity of $\tau^*(\alpha)$ at α, we can choose $q > 0$ in $\partial\tau^*(\alpha)$. Fix such a q and let $n \geq N_2$.

The graph of τ^* tends to the horizontal asymptote $y = \overline{\dim}_{\mathrm{B}}(\mathrm{supp}(\mu))$ (see Figure 1). The intersection of this horizontal asymptote and the line passing through the point $(\alpha, \tau^*(\alpha))$ and with slope q is the point $(\xi_0, \overline{\dim}_{\mathrm{B}}(\mathrm{supp}(\mu)))$, where

$$\xi_0 = (\overline{\dim}_{\mathrm{B}}(\mathrm{supp}(\mu)) - \tau^*(\alpha) + q\alpha)/q.$$

Define

$$\overline{\alpha}_0 := \max\left\{\xi_0 + \frac{(1+2q)\eta}{q}, \alpha + \epsilon\right\}.$$

Then for all $\xi > \overline{\alpha}_0$,

$$q\xi - \tau^*(\xi) \geq q\alpha - \tau^*(\alpha) + (1+2q)\eta + q(\xi - \overline{\alpha}_0) = \tau(q) + (1+2q)\eta + q(\xi - \overline{\alpha}_0). \tag{4.8}$$

Construct an approximate η-partition $\mathcal{P} = \{\alpha_i\}_{i=1}^{\ell+1}$ of $[\alpha_{\min} - \eta/2, \overline{\alpha}_0] \setminus (\alpha - \epsilon, \alpha + \epsilon)$ as in Case 1 and satisfying

$$\alpha_1 = \alpha_{\min} - \eta/2 \qquad \text{and} \qquad \alpha_{i+1} = \alpha_i + \eta \qquad \text{for all } i = 1, \ldots, \ell.$$

Also, construct an η-partition $\widetilde{\mathcal{P}} = \{\beta_j\}_{j=1}^{\infty}$ of $[\overline{\alpha}_0, \infty)$ with

$$\beta_1 = \overline{\alpha}_0 \qquad \text{and} \qquad \beta_{j+1} = \beta_j + \eta \qquad \text{for all } j \geq 1.$$

Fix $n \geq N_2$ and let $\delta > 0$ be sufficiently small so that (4.2) and the following conditions hold:

(i′) $\ell\delta^{\eta/4} \leq 1$.

(ii′) For all $j = 1, \ldots, \ell - 1$ and $q_{j+2} \in \partial\tau^*(\alpha_{j+2})$,

$$N_\delta^n(\alpha_{j+1}) \leq \delta^{-\tau^*(\alpha_{j+2}) - (1/4 - q_{j+2})\eta}.$$

Also, for all $j \geq 0$ and $q_{j+2} \in \partial\tau^*(\beta_{j+2})$,

$$N_\delta^n(\beta_{j+1}) \leq \delta^{-\tau^*(\beta_{j+2}) - (1/4 - q_{j+2})\eta}.$$

(iii′) $\dfrac{\delta^{(1/4+2q)\eta}}{1 - \delta^{q\eta}} \leq 1.$

We now proceed as in Case 1 and obtain the decomposition (4.3), where $\sum_I$ is the same as in Case 1, $\sum_{II}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying

$$\delta^{\alpha - \epsilon} \leq \mu(B) \qquad \text{or} \qquad \delta^{\overline{\alpha}_0} \leq \mu(B) < \delta^{\alpha+\epsilon},$$

and $\sum_{III}$ is the sum over all $B \in \mathcal{B}_\delta^n$ such that $\mu(B) < \delta^{\overline{\alpha}_0}$. It follows by a similar argument as in Case 1 that

$$\sum_{II} \mu(B)^q \le \delta^{\tau+\eta/2}.$$

$\sum_{III}$ can be estimated as follows:

$$\begin{aligned}
\sum_{III} \mu(B)^q &= \sum_{j=1}^{\infty} \sum \{\mu(B)^q : \delta^{\beta_{j+1}} \le \mu(B) < \delta^{\beta_j}\} \\
&\le \sum_{j=1}^{\infty} N_\delta^n(\beta_{j+1}) \delta^{q\beta_j} \\
&\le \sum_{j=1}^{\infty} \delta^{-\tau^*(\beta_{j+2})-(1/4-q_{j+2})\eta+q\beta_{j+2}-2q\eta} \qquad \text{(by (ii}'))\\
&\le \sum_{j=1}^{\infty} \delta^{\tau(q)+(1+2q)\eta+q(\beta_{j+2}-\overline{\alpha}_0)-(1/4-q_{j+2})\eta-2q\eta} \qquad \text{(by (4.8))}\\
&\le \sum_{j=1}^{\infty} \delta^{\tau(q)+3\eta/4+q_{j+2}\eta+q(j+1)\eta} \qquad (\beta_{j+2}-\overline{\alpha}_0 = (j+1)\eta)\\
&\le \sum_{j=1}^{\infty} \delta^{\tau(q)+3\eta/4+q(j+1)\eta} \qquad (\text{since } q_{j+2}\eta \ge 0)\\
&\le \delta^{\tau(q)+3\eta/4} \cdot \frac{\delta^{2q\eta}}{1-\delta^{q\eta}} \\
&\le \delta^{\tau(q)+\eta/2} \qquad \text{(by (iii}')).
\end{aligned}$$

The rest of the proof is similar to that of Case 1. This proves the theorem for Case 2.

We now turn to the case $\alpha_0^+ \le \alpha < \alpha_{\max}$. In this case, $\alpha_{\max} < \infty$. This is because the inequality $\alpha_0^+ < \alpha_{\max}$ forces $\operatorname{Dom}\tau = \mathbb{R}$ and this holds if and only if $\overline{\dim}_{-\infty}(\mu) < \infty$ (Proposition 2.7). We also know that $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu)$ (Proposition 2.8). We divide the proof into two cases.

Case 3. $\alpha_0^+ < \alpha < \alpha_{\max}$. Note that in this case, $q \le 0$ for any $q \in \partial\tau^*(\alpha)$.

Let $\epsilon > 0$ be sufficiently small so that

$$\alpha_0^+ < \alpha - \epsilon < \alpha < \alpha + \epsilon < \alpha_{\max}.$$

Let $\mathcal{P} = \{\alpha_k\}_{k=1}^{\ell+1}$ where $\alpha_1 = \alpha_{\min} - \eta/2$ and $\alpha_{\ell+1} = \alpha_0^+$, be an approximate η-partition of $[\alpha_{\min} - \eta/2, \alpha_0^+]$ such that all subintervals have length η, except possibly the one containing the end-point α_0^+. The exceptional subinterval has length $\le \eta$. Similarly, let $\widetilde{P} = \{\widetilde{\alpha}_j\}_{j=1}^{\tilde{\ell}+1}$, where $\widetilde{\alpha}_1 = \alpha_0^+$ and $\widetilde{\alpha}_{\ell+1} = \alpha_{\max} + \eta/2$, be an approximate η-partition of $[\alpha_0^+, \alpha_{\max} + \eta/2] \setminus (\alpha - \epsilon, \alpha + \epsilon)$.

Let $\delta > 0$ be sufficiently small (depending on η and n) such that the following are satisfied:

(i$''$) $(\ell - 1)\delta^{\eta/4} \leq 1$ and $\tilde{\ell}\delta^{\eta/4} \leq 1$;

(ii$''$) For all $j = 1, \ldots, \ell - 1$ and $q_{j+2} \in \partial\tau^*(\alpha_{j+2})$,
$$N_\delta^n(\alpha_{j+1}) \leq \delta^{-\tau^*(\alpha_{j+2})-(1/4-q_{j+2})\eta}.$$
Also, for all $j = 2, \ldots, \tilde{\ell}$ and $\tilde{q}_{j-1} \in \partial\tau^*(\widetilde{\alpha}_{j-1})$,
$$\widetilde{N}_\delta^n(\widetilde{\alpha}_j) \leq \delta^{-\tau^*(\widetilde{\alpha}_{j-1})-(1/4+\tilde{q}_{j-1})\eta}.$$

(iii$''$) $\epsilon(> \eta) > 0$ is sufficiently small so that $\tilde{q}_2 > -1/4$ and $q_\ell < 1/4$.

Now assume that $\delta > 0$ is sufficiently small so that (i$''$)–(iii$''$) hold. For any $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$, obtain a decomposition as in (4.3), where $\sum_I$ is as in Case 1, $\sum_{II}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying $\mu(B) \geq \delta^{\alpha_0^+}$, and $\sum_{III}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying
$$\mu(B) < \delta^{\alpha+\epsilon} \qquad \text{or} \qquad \delta^{\alpha-\epsilon} \leq \mu(B) < \delta^{\alpha_0^+}.$$
The rest of the proof is similar to that of Case 1.

Case 4. $\alpha_0^+ = \alpha < \alpha_{\max}$. Again, $q \leq 0$ for $q \in \partial\tau^*(\alpha)$.

Let $\epsilon > 0$ be sufficiently small so that
$$\alpha_{\min} < \alpha - \epsilon < \alpha < \alpha + \epsilon < \alpha_{\max}.$$
Let $\mathcal{P} = \{\alpha_i\}_{i=1}^{\ell+1}$, where $\alpha_1 = \alpha_{\min} - \eta/2$ and $\alpha_{i+1} = \alpha_i + \eta$ and $\alpha_\ell \leq \alpha - \epsilon < \alpha_{\ell+1}$. Let $\widetilde{\mathcal{P}} = \{\widetilde{\alpha}_i\}_{i=0}^{\tilde{\ell}}$ be an approximate η-partition of $[\alpha + \epsilon - \eta, \alpha_{\max}]$, with $\widetilde{\alpha}_0 = \alpha + \epsilon - \eta$.

Now assume that $\delta > 0$ is sufficiently small so that (i$''$), (ii$''$), and the following analogue of (iii$''$) hold:

(iii$'''$) $\epsilon(> \eta) > 0$ is sufficiently small so that $q_\ell < 1/4$.

For any $\mathcal{B}_\delta^n \in \mathcal{P}_\delta(B_n^*)$, obtain a decomposition as in (4.3), where $\sum_I$ is as before, $\sum_{II}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying $\mu(B) \leq \delta^{\alpha+\epsilon}$, and $\sum_{III}$ is the sum over all $B \in \mathcal{B}_\delta^n$ satisfying $\mu(B) \geq \delta^{\alpha-\epsilon}$. This case is slightly easier because $\alpha - \epsilon < \alpha_0^+$. We can now proceed as in Case 1 and this completes the proof of the theorem. □

5. Dimension of the measure

In this section we will prove Theorem 1.3. We begin by extending the definition of the Hausdorff dimension of a measure to measures we study in this paper.

Definition 5.1. *Let μ be a positive Borel measure on $\mathbb{R}^d$ which is finite on bounded sets. For each measurable subset $E \subseteq \mathbb{R}^d$ with $\mu(E) < \infty$, let $\dim_{\mathrm{H}}(\mu|_E)$ be the Hausdorff dimension of restriction of μ to E, i.e.,*

$$\dim_{\mathrm{H}}(\mu|_E) := \inf\{\dim_{\mathrm{H}}(A) : A \subseteq E,\ \mu(A) = \mu(E)\}.$$

Let $\{E_n\}_{n=1}^{\infty}$ be an admissible sequence of measurable subsets of $\mathbb{R}^d$. Define the Hausdorff dimension *of μ as*

$$\dim_{\mathrm{H}}(\mu) := \lim_{n\to\infty} \dim_{\mathrm{H}}(\mu|_{E_n}). \tag{5.1}$$

It is clear that if μ is a finite measure with compact support, then above definition coincides with the usual one. The following proposition justifies the above definition.

Proposition 5.1. *Let μ be a Borel measure on $\mathbb{R}^d$ which is finite on bounded sets.*

(a) $\dim_{\mathrm{H}}(\mu|_E)$ *is a monotone increasing function of E and is bounded above by d, i.e., if E and F are measurable subsets of $\mathbb{R}^d$ of finite measure such that $E \subseteq F$, then* $\dim_{\mathrm{H}}(\mu|_E) \le \dim_{\mathrm{H}}(\mu|_F) \le d$. *In particular, the limit in Definition 5.1 exists and* $\dim_{\mathrm{H}}(\mu) \le d$.

(b) *The definition of* $\dim_{\mathrm{H}}(\mu)$ *is independent of the choice of the admissible sequence $\{E_n\}$.*

Proof. (a) follows directly from definition.

(b) Let $\{E_n\}$ and $\{F_n\}$ be two admissible sequences of measurable subsets, and denote by $\dim^1_{\mathrm{H}}(\mu)$, and $\dim^2_{\mathrm{H}}(\mu)$ the Hausdorff dimension of μ defined by using $\{E_n\}$ and $\{F_n\}$, respectively. Since F_n is admissible, for each $n \in \mathbb{N}$, there exists some $k_n \in \mathbb{N}$ such that $E_n \subseteq F_{k_n}$. Thus, by (a) above,

$$\dim_{\mathrm{H}}(\mu|_{E_n}) \le \dim_{\mathrm{H}}(\mu|_{F_{k_n}}) \le \dim^2_{\mathrm{H}}(\mu).$$

Consequently, $\dim^1_{\mathrm{H}}(\mu) \le \dim^2_{\mathrm{H}}(\mu)$. The reverse inequality follows by symmetry. □

As an illustration, we will compute the Hausdorff dimension of the measure μ in Example 5.2.

Example 5.2. *Let μ be the measure defined in Example 2.9. Then* $\dim_{\mathrm{H}}(\mu) = \ln 2/\ln 3$.

Proof. For all $n \in \mathbb{N}$, it is well-known (see [KP], [S]) that

$$\dim_{\mathrm{H}}(\mu_n) = \frac{(1/2^n)\ln(1/2^n) + ((2^n-1)/2^n)\ln((2^n-1)/2^n)}{-\ln 3}.$$

Using the fact that the function $g(x) := x\ln x + (1-x)\ln(1-x)$ is strictly increasing on $(0, 1/2)$, we see that $\dim_{\mathrm{H}}(\mu_n)$ is a strictly decreasing sequence as $n \to \infty$. So by definition,

$$\dim_{\mathrm{H}}(\mu|_{B_n}) = \dim_{\mathrm{H}}(\mu_1) = \frac{\ln 2}{\ln 3}.$$

and thus $\dim_{\mathrm{H}}(\mu) = \ln 2/\ln 3$. □

Lemma 5.3. *Let μ be a positive Borel measure on $\mathbb{R}^d$ which is finite on bounded sets. Let*

$$\mu = \mu_{\mathrm{c}} + \mu_{\mathrm{at}},$$

where μ_{c} is continuous and μ_{at} is the sum of all atoms. Then

$$\dim_{\mathrm{H}}(\mu) = \dim_{\mathrm{H}}(\mu_{\mathrm{c}}) \qquad \text{and} \qquad \dim_{\mathrm{H}}(\mu_{\mathrm{at}}) = 0.$$

Proof. To prove the first equality, let

$$S := \{x \in \mathbb{R}^d : \mu_{\mathrm{at}}(\{x\}) > 0\} = \{x \in \mathbb{R}^d : \mu(\{x\}) > 0\}$$

be the set of all atoms. Fix an admissible sequence $\{E_n\}$ of measurable subsets, fix an $n \in \mathbb{N}$, and let $\epsilon > 0$ be arbitrary. Then there exists some $A \subseteq E_n$ such that $\mu(A) = \mu(E_n)$ and

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}}(\mu|_{E_n}) + \epsilon. \tag{5.2}$$

Since $\mu(A) = \mu(E_n)$, all atoms in E_n must belong to A. Hence $\mu_{\mathrm{at}}(A) = \mu_{\mathrm{at}}(E_n)$ and thus $\mu_{\mathrm{c}}(A) = \mu_{\mathrm{c}}(E_n)$. This, together with (5.2), gives

$$\dim_{\mathrm{H}}(\mu_{\mathrm{c}}|_{E_n}) \le \dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}}(\mu|_{E_n}) + \epsilon.$$

Letting $\epsilon \to 0^+$ and $n \to \infty$ yields

$$\dim_{\mathrm{H}}(\mu_{\mathrm{c}}) \le \dim_{\mathrm{H}}(\mu).$$

To prove the reverse inequality we again fix $n \in \mathbb{N}$ and let $\epsilon > 0$ be arbitrary. Then there exists $A \subseteq E_n$ with $\mu_{\mathrm{c}}(A) = \mu_{\mathrm{c}}(E_n)$ such that

$$\dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}}(\mu_{\mathrm{c}}|_{E_n}) + \epsilon. \tag{5.3}$$

Let $S_n := S \cap E_n$. Note that $\mu_{\mathrm{c}}(A \cup S_n) = \mu_{\mathrm{c}}(A) = \mu_{\mathrm{c}}(E_n)$ and $\mu_{\mathrm{at}}(A \cup S_n) = \mu_{\mathrm{at}}(E_n)$. Thus,

$$\mu(A \cup S_n) = \mu_{\mathrm{c}}(A \cup S_n) + \mu_{\mathrm{at}}(A \cup S_n) = \mu_{\mathrm{c}}(E_n) + \mu_{\mathrm{at}}(E_n) = \mu(E_n).$$

Therefore, using (5.3) and the fact that S_n is countable, we have

$$\dim_{\mathrm{H}}(\mu|_{E_n}) \le \dim_{\mathrm{H}}(A \cup S_n) = \dim_{\mathrm{H}}(A) < \dim_{\mathrm{H}}(\mu_{\mathrm{c}}|_{E_n}) + \epsilon$$

Letting $\epsilon \to 0^+$ and $n \to \infty$ yields

$$\dim_{\mathrm{H}}(\mu) \le \dim_{\mathrm{H}}(\mu_{\mathrm{c}}),$$

and this completes the proof of the first equality. The second equality is obvious since S is countable. □

We now turn to the proof of Theorem 1.3. Throughout the rest of this section we fix the admissible sequence $\{B_n(0)\}$.

Lemma 5.4. *Let μ be a positive Borel measure on $\mathbb{R}^d$ (with possibly unbounded support) which is finite on bounded sets. Let $\alpha \in (\alpha_{\min}, \alpha_{\max})$.*

(a) *Suppose $\alpha < \alpha_1^-$ or $\alpha > \alpha_1^+$. Then $\alpha > \tau^*(\alpha)$.*
(b) *Suppose $\alpha_{\min} < \alpha_1^-$. Then $\alpha_{\min} > \tau^*(\alpha_{\min})$.*

Proof. (a) Let $\alpha < \alpha_1^-$ and $q \in \partial\tau^*(\alpha)$. Then $q > 1$. Since $\tau(q)$ is increasing, concave, and $\tau(1) = 0$, we have $\tau(q)/(q-1) \ge \tau'_-(q)$, and equality holds if and only if $\alpha_1^- = \tau'_-(q)$. Thus either

$$\frac{\tau(q)}{q-1} > \tau'_-(q) \ge \alpha \qquad \text{or} \qquad \frac{\tau(q)}{q-1} = \tau'_-(q) = \alpha_1^- > \alpha.$$

In either case we have $\tau(q) - (q-1)\alpha > 0$, i.e., $\alpha - \tau^*(\alpha) > 0$.

Now let $\alpha > \alpha_1^+$ and $q \in \partial\tau^*(\alpha)$ and thus $q < 1$. By the same reason as above, we have $\tau(q)/(q-1) \le \tau'_+(q)$, and equality holds if and only if $\tau'_+(q) = \alpha_1^+$. Thus either

$$\frac{\tau(q)}{q-1} < \tau'_+(q) \le \alpha \qquad \text{or} \qquad \frac{\tau(q)}{q-1} = \tau'_+(q) = \alpha_1^+ < \alpha,$$

from which the result follows.

(b) The proof is similar to that of [N, Proposition 2.1]; we omit the details. □

Lemma 5.5. *Let μ be as in Lemma 5.4. Then*

(a) $\mu\left\{x \in \operatorname{supp}(\mu) : \alpha_{\min} \le \varliminf_{\delta\to 0^+} \dfrac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0;$
(b) $\mu\left\{x \in \operatorname{supp}(\mu) : \alpha_1^+ < \varlimsup_{\delta\to 0^+} \dfrac{\ln \mu(B_\delta(x))}{\ln \delta}\right\} = 0.$

Proof. (a) We divide the proof into two parts.

Part 1. We first show

$$\mu\left\{x \in \operatorname{supp}(\mu) : \alpha_{\min} < \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_1^-\right\} = 0.$$

Let $\alpha_{\min} < \alpha < \alpha_1^-$ and $q \in \partial\tau^*(\alpha)$. Then $q > 1$. Fix $\epsilon > 0$ sufficiently small so that

$$0 < \sigma := \frac{\alpha - \tau^*(\alpha)}{2} \leq \alpha - \tau^*(\alpha) - (2+q)\epsilon. \tag{5.4}$$

Let

$$L_\epsilon(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \alpha - \frac{\epsilon}{3} \leq \underline{\lim}_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3} \right\}. \tag{5.5}$$

Let $n_0 \in \mathbb{N}$ be sufficiently large so that $B_n^* \neq \emptyset$ for all $n \geq n_0$. For each $n \geq n_0$, define

$$L_{n,\epsilon}(\alpha) := \left\{ x \in B_n^* : \alpha - \frac{\epsilon}{3} \leq \underline{\lim}_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha + \frac{\epsilon}{3} \right\}. \tag{5.6}$$

We will show that $\mu(L_\epsilon(\alpha)) = 0$ by first showing that $\mu(L_{n,\epsilon}(\alpha)) = 0$.

Putting $\xi = 1$ in Lemma 3.1(a), we obtain that there exists $\delta_\epsilon(n) > 0$ such that for all $0 < \delta < \delta_\epsilon(n)$,

$$N_\delta^n(\alpha + \epsilon) \leq \delta^{-\tau^*(\alpha) - (1+q)\epsilon}. \tag{5.7}$$

Fix $m \in \mathbb{N}$ that satisfies

$$2^{-m} < \delta_\epsilon(n) \qquad \text{and} \qquad m \geq \frac{3\alpha}{\epsilon} + 2. \tag{5.8}$$

For each $x \in L_{n,\epsilon}(\alpha)$ we let $k_x \in \mathbb{N}$ be the smallest integer such that $k_x \geq m$ and the following two conditions hold:

(I) $\mu(B_\delta(x)) < \delta^{\alpha - \epsilon}$ for all $0 < \delta \leq 2^{-(k_x - 2)}$;
(II) there exists $\delta_x > 0$ such that

$$2^{-(k_x+1)} < \delta_x \leq 2^{-k_x} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) > \delta_x^{\alpha + 2\epsilon/3}.$$

Since k_x is uniquely determined by x, we can partition $L_{n,\epsilon}(\alpha)$ into a countable disjoint union of subsets $L_{n,\epsilon}^k(\alpha)$, where

$$L_{n,\epsilon}^k(\alpha) := \{x \in L_{n,\epsilon}(\alpha) : k_x = k\}.$$

Then

$$L_{n,\epsilon}(\alpha) = \bigcup_{k=m}^{\infty} L_{n,\epsilon}^k(\alpha). \tag{5.9}$$

Clearly for each $k \geq m$,

$$L_{n,\epsilon}^k(\alpha) \subseteq \bigcup_{x \in L_{n,\epsilon}^k(\alpha)} B_{2^{-k}}(x).$$

For each $x \in L_{n,\epsilon}^k(\alpha)$, (II) and (5.8) imply that

$$\mu(B_{2^{-k}}(x)) > 2^{-(k+1)(\alpha + 2\epsilon/3)} \geq 2^{-k(\alpha+\epsilon)}. \tag{5.10}$$

Since $L_{n,\epsilon}^k(\alpha)$ is bounded, by a standard covering lemma (see [F2, Lemma 4.8]), there exists a finite sequence $\{x_i\}_{i=1}^{\ell}$ in $L_{n,\epsilon}^k(\alpha)$ such that $\{B_{2^{-k}}(x_i)\}_{i=1}^{\ell}$ is a disjoint family and

$$L_{n,\epsilon}^k(\alpha) \subseteq \bigcup_{i=1}^{\ell} B_{2^{-(k-2)}}(x_i). \tag{5.11}$$

By (5.7) and (5.10),

$$\ell \le 2^{-k(-\tau^*(\alpha)-(1+q)\epsilon)}. \tag{5.12}$$

Combining (I), (5.4), (5.11) and (5.12) we have

$$\begin{aligned}\mu(L_{n,\epsilon}^k(\alpha)) &\le \sum_{i=1}^{\ell} \mu(B_{2^{-(k-2)}}(x_i)) \\ &\le 2^{-(k-2)(\alpha-\epsilon)} \cdot 2^{-k(-\tau^*(\alpha)-(1+q)\epsilon)} \\ &= C \cdot 2^{-k(\alpha-\tau^*(\alpha)-(2+q)\epsilon)} \\ &\le C \cdot 2^{-k\sigma},\end{aligned}$$

where C is a constant independent of k. Using this and (5.9) we have

$$\mu(L_{n,\epsilon}(\alpha)) \le \sum_{k=m}^{\infty} \mu(L_{n,\epsilon}^k(\alpha)) \le C \sum_{k=m}^{\infty} 2^{-k\sigma} = C \cdot \frac{2^{-\sigma m}}{1-2^{-\sigma}}.$$

Letting $m \to \infty$ yields $\mu(L_{n,\epsilon}(\alpha)) = 0$. Since $L_\epsilon(\alpha) = \bigcup_{n=n_0}^{\infty} L_{n,\epsilon}(\alpha)$, we have $\mu(L_\epsilon(\alpha)) = \lim_{n\to\infty} \mu(L_{n,\epsilon}(\alpha)) = 0$.

Part 2. We show that if $\alpha_{\min} < \alpha_1^-$ then

$$\mu\left\{x \in \operatorname{supp}(\mu) : \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha_{\min}\right\} = 0.$$

By Lemma 5.4(b), we can choose $\epsilon > 0$ sufficiently small and $\alpha \in (\operatorname{Dom}\tau^*)^\circ$ sufficiently close to $\alpha_{\min}$ such that

$$0 < \sigma := (\alpha_{\min} - \tau^*(\alpha))/2 \le \alpha_{\min} - \tau^*(\alpha) - (2+q)\epsilon,$$

where $q \in \partial\tau^*(\alpha)$. Replace (5.6) by

$$L_{n,\epsilon} := \left\{x \in B_n^* : \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha_{\min}\right\}.$$

Also, replace the α in (5.8), (I), and (II) by $\alpha_{\min}$. The rest of the proof is similar.

(b) Again we divide the proof into two parts.

Part 3. We show that

$$\mu\left\{x \in \operatorname{supp}(\mu) : \alpha_1^+ < \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha_0^+\right\} = 0.$$

Replace (5.5) and (5.6) respectively by

$$U_\epsilon(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \alpha - \frac{\epsilon}{3} \le \overline{\lim_{\delta\to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \alpha + \frac{\epsilon}{3} \right\}$$
$$U_{n,\epsilon}(\alpha) := \left\{ x \in B_n^* : \alpha - \frac{\epsilon}{3} \le \overline{\lim_{\delta\to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \le \alpha + \frac{\epsilon}{3} \right\}.$$

Let $\delta_\epsilon(n)$ be chosen as in the proof of (a) and fix $m \in \mathbb{N}$ that satisfies

$$2^{-m} < \delta_\epsilon(n).$$

Replace conditions (I) and (II) by:

(I′) $\mu(B_\delta(x)) \ge \delta^{\alpha+\epsilon}$ for all $0 < \delta \le 2^{-(k_x-1)}$;

(II′) there exists $\delta_x > 0$ such that

$$2^{-(k_x-1)} < \delta_x \le 2^{-(k_x-2)} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) \le \delta_x^{\alpha-\epsilon}.$$

Then proceed as above.

Part 4. We will show that if $\alpha > \alpha_1^+$ and $\alpha_0^+ \le \alpha \le \alpha_{\max}$, then

$$\mu\left\{ x \in \operatorname{supp}(\mu) : \overline{\lim_{\delta\to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} > \alpha \right\} = 0.$$

Note that in this case $q \le 0$. Replace (5.4) by

$$0 < \sigma := (\alpha - \tau^*(\alpha))/2 \le \alpha - \tau^*(\alpha) - (5/3 - q)\epsilon.$$

Replace (5.5), (5.6), (5.7), and (5.8) respectively by

$$\widetilde{U}_\epsilon(\alpha) := \left\{ x \in \operatorname{supp}(\mu) : \alpha - \frac{\epsilon}{3} \le \overline{\lim_{\delta\to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \right\}$$
$$\widetilde{U}_{n,\epsilon}(\alpha) := \left\{ x \in B_n^* : \alpha - \frac{\epsilon}{3} \le \overline{\lim_{\delta\to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \right\}$$
$$\widetilde{N}_\delta^n(\alpha - \epsilon) \le \delta^{-\tau^*(\alpha)-(1-q)\epsilon}$$
$$2^{-m} < \delta_\epsilon(n) \qquad \text{and} \qquad m \ge \frac{9\alpha}{\epsilon} - 6.$$

Also, replace conditions (I) and (II) by the single condition below:

(II″) there exists $\delta_x > 0$ such that

$$2^{-(k_x-2)} < \delta_x \le 2^{-(k_x-3)} \qquad \text{and} \qquad \mu(B_{\delta_x}(x)) \le \delta_x^{\alpha-2\epsilon/3}.$$

Then proceed as in (a). This completes the proof. □

Proof of Theorem 1.3. (a) By Lemma 4.1, for μ a.e. $x \in \operatorname{supp}(\mu)$,

$$\alpha_{\min} \leq \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \alpha_{\max}.$$

Thus (1.5) follows from Lemma 5.5.

(b) To show that if $\tau(q)$ is differentiable at $q = 1$, then $\dim_{\mathrm{H}}(\mu) = \tau'(1)$, we consider the following three cases:

Case 1. Assume that μ is nonatomic. Let $\mu(B_n^*) > 0$ and let $C_n \subseteq B_n^*$ such that $\mu(C_n) = \mu(B_n^*)$. Then by part (a), there exists $A_n \subseteq C_n$ such that $\mu(A_n) = \mu(C_n) = \mu(B_n^*) > 0$ and for all $x \in A_n$,

$$\tau'_+(1) \leq \varliminf_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \varlimsup_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leq \tau'_-(1).$$

By [Yo, Proposition 2.1],

$$\tau'_+(1) \leq \dim_{\mathrm{H}}(A_n) \leq \tau'_-(1).$$

Thus

$$\dim_{\mathrm{H}}(\mu|_{B_n^*}) \leq \dim_{\mathrm{H}}(A_n) \leq \tau'_-(1).$$

On the other hand,

$$\tau'_+(1) \leq \inf\{\dim_{\mathrm{H}}(C_n) : C_n \subseteq B_n^*,\ \mu(C_n) = \mu(B_n^*)\} = \dim_{\mathrm{H}}(\mu|_{B_n^*}).$$

Combining the above inequalities, we get

$$\tau'_+(1) \leq \dim_{\mathrm{H}}(\mu|_{B_n^*}) \leq \tau'_-(1).$$

Letting $n \to \infty$ yields $\tau'_+(1) \leq \dim_{\mathrm{H}}(\mu) \leq \tau'_-(1)$ and the result follows.

Case 2. Assume that μ is atomic but $\mu_{\mathrm{c}}(B_n^*) > 0$ for all n sufficiently large. By Lemma 5.3 and Case 1, we obtain

$$\tau'_+(1) \leq \dim_{\mathrm{H}}(\mu) = \dim_{\mathrm{H}}(\mu_{\mathrm{c}}) \leq \tau'_-(1).$$

Case 3. Lastly, we assume that μ is purely atomic, i.e., $\mu = \mu_{\mathrm{at}}$. In this case, it is clear that $\dim_{\mathrm{H}}(\mu) = 0$. On the other hand, let S be the set of all atoms. Then for all $x \in S$,

$$\lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = 0.$$

It follows from part (a) that $\tau'_+(1) = 0$ and thus the result follows. This completes the proof. □

6. Comments

In this paper we have provided a framework for studying the multifractal structure of noncompactly support infinite measures, and recovered some basic results that hold for compactly supported finite measures. However, the exact range of validity of the multifractal formalism, that is, $f(\alpha) = \tau^*(\alpha)$, is far from being understood. It is of interest to study the multifractal formalism for noncompactly supported self-similar measures that are studied in [BE]. As for compactly supported self-similar measures, certain separation condition, such as the open set condition or the weak separation property ([LN2]) may be required.

We have also proved that if the L^q-spectrum $\tau(q)$ is differentiable at $q = 1$, then $\tau'(1)$ is the Hausdorff dimension of the measure. The noncompactly supported self-similar measures in [BE] are defined by IFSs that satisfy the so-called *average-contractivity condition*. It can be shown an IFS satisfying the average-contractivity condition and defines a noncompactly supported measure must have overlaps, i.e., the open set condition is not satisfied. It is often difficult to compute $\tau(q)$ if the corresponding IFS has overlaps. For compactly supported self-similar measures with overlaps, $\tau(q)$ for $q > 0$ has been obtained for the infinite Bernoulli convolution associated with the golden ratio ([LN1]) and a family of IFSs on $\mathbb{R}$ that includes the 3-fold convolution of the Cantor measure ([LN3]). For these measures, it is known that for $q > 0$, $\tau(q)$ is differentiable; moreover $\tau'(1)$ gives a closed formula for the Hausdorff dimension of the measure. It is of interest to compute $\tau(q)$ for noncompactly supported self-similar measures, such as those defined by the IFS in (1.1).

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Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA.

E-mail address: smngai@georgiasouthern.edu