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Normal and Half-Normal Probability Plots

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Abstract

Normal and half-normal probability plots have often been used to analyze data from an unreplicated two level factorial and fractional factorial designs. Interpreting the data for active effects is subjective on the users part. A method is given for estimating (via simulation) limits (critical values) for the plotted points that can be used to determine objectively if an effect is active. Recommendations are given for selecting plotting positions. A second method, also based on simulation, estimates the maximum probability that the observed estimated effects provide as much evidence or more against the assumption they are estimating inactive effects. Examples are provided.

1 Introduction

Normal probability plots are often designed as simple graphical techniques (1) to assess if the data are being generated from a distribution that has an (approximate) normal distribution; and (2) to analyze the data from an unreplicated factorial design. In the first case, the linearity of the plot is associated with a normal distribution. In the latter case, the analysis is made under the assumption that the means of the estimators of the main effects and interactions are zero. More specifically, if $Y_1, \ldots, Y_n$ are the estimators of the effects of a 2^{m-q} $(0 \leq q < m)$ (fractional, if $0 < q < m$) factorial design, the null hypothesis is $E(Y_1) = \ldots = E(Y_n) = 0$. Since these estimators are linear combinations of the data, it is accepted that the assumption of normality (approximately) holds by the Central Limit Theorem. The two plots that we will examine for assessing evidence in these data against the null hypothesis are the normal and half-normal probability plots designed under the assumption the $E(Y_i) = 0$ for $i = 1, \ldots, n$. A normal probability plot plots the points $(x_{n:i}, Y_{n:i})$, for $i = 1, \ldots, n$, where the $x_{n:i}$'s are given "plotting positions," the $Y_{n:i}$'s are the ordered estimators of the effects, and $n = 2^{m-q} - 1$. The half-normal is a plot of the points $(x_{n:i}, R_{n:i})$, where $R_{n:i}$ is the ith order statistic of the collection of statistics $R_1 = |Y_1|, \ldots, R_n = |Y_n|$. In the next section, we discuss the selection of plotting positions $x_{n:i}$ for both the normal and half-normal probability plots. We examine in the third and fourth sections the addition of critical values, respectively, to the normal probability and half-normal plots that provide an objective way of examining the data to determine which of the effects (main and interactions) are different from zero. Atkinson (1981) introduces a similar method for analyzing the residuals of regression models. A method is given in the fifth section for measuring of the strength of the evidence of the estimated effects against the hypotheses that the effects are zero. Examples are provided in the sixth section to illustrate the use of critical values. Recommendations are given in the final section.

2 Plotting Positions

A probability plot is a plot of the points $(x_{n:1}, Y_{n:1}), \ldots, (x_{n:n}, Y_{n:n})$, where $x_{n:1} < \ldots < x_{n:n}$ are known and $Y_{n:1} \leq \ldots \leq Y_{n:n}$ are the order statistics of a random sample drawn from a distribution of interest. Various sets of plotting positions $\{x_{n:i}\}$ have been recommended in the literature for use under the assumption the random sample is from a normal distribution. The following table, from Looney and Gulledge (1985), lists some of the commonly used plotting positions and the corresponding references.

Table 1. Plotting Positions

Case	$x_{n:i}$	Reference
1	$\Phi^{-1}\left(\frac{i-0.5}{n}\right)$	Hazen(1914)
2	$\Phi^{-1}\left(\frac{i}{n+1}\right)$	Weibull(1939)
3	$\delta_{n:i}$	Kimball(1960), Barnett(1975), Cunnane(1978)
4	$\widetilde{\delta}_{n:i}$	Bernard and Bos-Levenbach(1953), Filliben(1975)
5	$\Phi^{-1}\left(\frac{i-0.375}{n+0.25}\right)$	Blom(1958)
6	$\Phi^{-1}\left(\frac{i-0.3}{n+0.4}\right)$	Bernard and Bos-Levenbach(1953)
7	$\Phi^{-1}\left(\frac{i-0.4}{n+0.2}\right)$	Cunnane(1978)
8	$\Phi^{-1}\left(\frac{i}{n}\right)$	Mage(1982)

with $\Phi(z)$ is the cumulative distribution function of a standard normal distribution. Here $\delta_{n:i}$ and $\widetilde{\delta}_{n:i}$ are the mean and median of the distribution of the ith order statistic $Z_{n:i}$ of a random sample from a standard normal distribution. Except for the case in which $i = (n+1)/2$ with n odd, the distribution of $Z_{n:i}$ is not a symmetric distribution; hence, $\delta_{n:i} \neq \widetilde{\delta}_{n:i}$. We note that in case 8, the point $(x_{n:n}, Y_{n:n})$ cannot be plotted. We will not consider this case any further.

For cases 1-7,

$$\textstyle\sum_{i=1}^{n} x_{n:i} = 0 \text{ and } x_{n:n-i+1} = -x_{n:i}.$$

We see this by observing for cases 1,2, 5, 6, and 7 that

$$\begin{aligned}
&(1)\text{: } 1 - \frac{i-0.5}{n} = \frac{n}{n} - \frac{i-0.5}{n} = \frac{(n-i+1)-0.5}{n} \\
&(2)\text{: } 1 - \frac{i}{n+1} = \frac{n+1}{n+1} - \frac{i}{n+1} = \frac{n-i+1}{n+1} \\
&(5)\text{: } 1 - \frac{i-0.375}{n+0.25} = \frac{n+0.25}{n+0.25} - \frac{i-0.375}{n+0.25} = \frac{(n-i+1)-0.375}{n+0.25} \\
&(6)\text{: } 1 - \frac{i-0.3}{n+0.4} = \frac{n+0.4}{n+0.4} - \frac{i-0.3}{n+0.4} = \frac{(n-i+1)-0.3}{n+0.4} \\
&(7)\text{: } 1 - \frac{i-0.4}{n+0.2} = \frac{n+0.2}{n+0.2} - \frac{i-0.4}{n+0.2} = \frac{(n-i+1)-0.4}{n+0.2}
\end{aligned}$$

For case 3, first observe $\phi_{n:n-i+1}(z) = \phi_{n:i}(-z)$, where $\phi_{n:j}(z)$ is the density of $Z_{n:i}$. We now see that

$$\delta_{n:n-i+1} = \int_{-\infty}^{\infty} z\phi_{n:n-i+1}(z)\,dz = \int_{-\infty}^{\infty} z\phi_{n:i}(-z)\,dz = -\int_{-\infty}^{\infty} z\phi_{n:i}(z)\,dz = -\delta_{n:i}.$$

The median $\widetilde{\delta}_{n:i}$ is defined such that

$$0.5 = \int_{-\infty}^{\widetilde{\delta}_{n:n-i+1}} \phi_{n:n-i+1}(z)\,dz \Rightarrow 0.5 = \int_{-\infty}^{\widetilde{\delta}_{n:n-i+1}} \phi_{n:i}(-z)\,dz = -\int_{-\widetilde{\delta}_{n:n-i+1}}^{\infty} \phi_{n:i}(z)\,dz.$$

Thus, $-\widetilde{\delta}_{n:n-i+1}$ is the median of the distribution of $Z_{n:i}$. Hence, $\widetilde{\delta}_{n:n-i+1} = -\widetilde{\delta}_{n:i}$. For Cases 1-7, Table 2 gives the values for $x_{n:i}$ for $n = 7$ and $i = 4$, 5, 6, and 7 and Table 2 (b) gives values for $x_{n:i}$ for $n = 15$ and $i = 8$, 9, 10, 11, 12, 13, 14, and 15.

Table 2 (a): Cases 1-7 Plotting Positions for $n = 7$

$i \backslash x_{7:i}$	$\Phi^{-1}\left(\frac{i-0.5}{7}\right)$	$\Phi^{-1}\left(\frac{i}{7+1}\right)$	$\delta_{7:i}$	$\widetilde{\delta}_{7:i}$	$\Phi^{-1}\left(\frac{i-0.375}{7+0.25}\right)$	$\Phi^{-1}\left(\frac{i-0.3}{7+0.4}\right)$	$\Phi^{-1}\left(\frac{i-0.4}{7+0.2}\right)$
7	1.46523	1.15035	1.35218	1.31487	1.36449	1.31298	1.38299
6	0.79164	0.67449	0.75737	0.74383	0.75829	0.73974	0.76471
5	0.36611	0.31864	0.35271	0.34748	0.35293	0.34549	0.35549
4	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 2 (b): Cases 1-7 Plotting Positions for $n = 15$

$i \backslash x_{15:i}$	$\Phi^{-1}\left(\frac{i-0.5}{15}\right)$	$\Phi^{-1}\left(\frac{i}{15+1}\right)$	$\delta_{15:i}$	$\widetilde{\delta}_{15:i}$	$\Phi^{-1}\left(\frac{i-0.375}{15+0.25}\right)$	$\Phi^{-1}\left(\frac{i-0.3}{15+0.4}\right)$	$\Phi^{-1}\left(\frac{i-0.4}{15+0.2}\right)$
15	1.83391	1.53412	1.73591	1.69373	1.73938	1.69062	1.75683
14	1.28155	1.15035	1.24794	1.22975	1.24505	1.22446	1.25212
13	0.96742	0.88715	0.94769	0.93723	0.94578	0.93333	0.95001
12	0.72791	0.67449	0.71488	0.70826	0.71370	0.70547	0.71650
11	0.52440	0.48878	0.51570	0.51144	0.51499	0.50952	0.51685
10	0.34069	0.31864	0.33530	0.33273	0.33489	0.33151	0.33604
9	0.16789	0.15731	0.16530	0.16408	0.16512	0.16349	0.16566
8	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

The values given in Table 2 for Cases 1, 2, 4, 5, 7, and 8 were obtained by using the *TI*-83 calculator function INVNORM with

$$\widetilde{\delta}_{n:i} = \Phi^{-1}\left(\beta_{i,n-i+1,0.50}\right),$$

where $\beta_{i,n-i+1,0.50}$ is the median of a beta distribution with parameters i and $n-i+1$. The values for Case 3 are from Harter (1961). It is easy to see that for $n = 7$ and 15

$$\delta_{n:i} \approx \Phi^{-1}\left(\frac{i-0.375}{n+0.25}\right).$$

Let us assume that we will have the ordered statistics $Y_{n:1}, \ldots, Y_{n:n}$ of a random sample from a $N\left(\mu, \sigma^2\right)$ distribution. It is easy to see that

$$\mu_{n:i} = E\left(Y_{n:i}\right) = \mu + \sigma\delta_{n:i},$$

for $i = 1, \ldots, n$. We could also write

$$Y_{n:i} = \mu + \sigma\delta_{n:i} + \epsilon_{n:i},$$

where $\epsilon_{n:i}$ is a random variable with mean 0 and variance $\sigma_{n:i}^2 = V\left(Y_{n:i}\right)$. This simply illustrates that the points $\left(\delta_{n:i}, Y_{n:i}\right)$ will plot randomly about the line $\mu_{n:i} = \mu + \sigma\delta_{n:i}$.

Suppose now that the means of the order statistics are linearly related to a given set $\{x_{n:i}\}$ of plotting points such that

$$\textstyle\sum_{i=1}^{n} x_{n:i} = 0 \text{ and } x_{n:n-i+1} = -x_{n:i}.$$

It follows there exist values β_0 and β_1 such that

$$\mu_{n:i} = \beta_0 + \beta_1 x_{n:i}.$$

Since $\mu_{n:i} = \mu + \sigma\delta_{n:i}$, then

$$0 = (\mu - \beta_0) + (\sigma\delta_{n:i} - \beta_1 x_{n:i})$$

for all $i = 1, \ldots, n$. Hence, we have that for $j \neq i$,

$$0 = \sigma\left(\delta_{n:i} - \delta_{n:j}\right) - \beta_1\left(x_{n:i} - x_{n:j}\right).$$

In particular, for $j = n - i + 1$, we know that $\delta_{n:n-i+1} = -\delta_{n:i}$ and we are assuming that $x_{n:n-i+1} = -x_{n:i}$ for all i. Hence,

$$\lambda = \frac{\sigma}{\beta_1} = \frac{x_{n:i}}{\delta_{n:i}},$$

for all $i = 1, \ldots, n$. This requires that $x_{n:i} = \lambda\delta_{n:i}$ for all i. Now letting $x_{n:i} = \lambda\delta_{n:i}$ and noting that $\sigma = \lambda\beta_1$, then

$$0 = (\mu - \beta_0) + (\sigma\delta_{n:i} - \beta_1\lambda\delta_{n:i}) = \mu - \beta_0.$$

This implies that $\beta_0 = \mu$. Therefore, the set of plotting positions $\{x_{n:i}\}$ for a normal probability plot must have the form $\{\lambda\delta_{n:i}\}$ for any nonzero real number λ for the linear relationship $\mu_{n:i} = \beta_0 + \beta_1 x_{n:i}$ to hold. It is not difficult to show that this does not hold for the plotting positions in Cases 1, 2, 4, 5, 6, and 7.

Under the model, the Y_i's are independent and identically distributed $N\left(0, \sigma^2\right)$, we are assuming that $\mu_{n:i} = \beta_1 x_{n:i}$. Thus, for a given set $\{\lambda\delta_{n:i}\}$ of plotting positions the generalized least squares estimator for β_1 is given by

$$\widehat{\beta}_1 = \left[(\lambda\delta)^{\mathbf{T}}\mathbf{B}^{-1}(\lambda\delta)\right]^{-1}(\lambda\delta)^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{Y} = \lambda^{-1}\widehat{\sigma}, \tag{1}$$

where $\mathbf{B} = cov\left(\mathbf{Z}\right)$ and

$$\delta = \left[\delta_{n:1}, \ldots, \delta_{n:n}\right]^{\mathbf{T}} \text{ and } \widehat{\sigma} = \left(\delta^{\mathbf{T}}\mathbf{B}^{-1}\delta\right)^{-1}\delta^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{Y}. \tag{2}$$

Values of the vector δ and the matrix $\mathbf{B}$ can be found in Teichroew (1956) and Tietjen, Kahaner, and Beckman (1977). Note that $\beta_1\left(\lambda\delta_{n:i}\right) = \sigma\delta_{n:i}$. It is easy to show that $\widehat{\sigma}$ is an unbiased estimator for σ. Thus, regardless of the choice of $\lambda \neq 0$, we obtain the same unbiased estimator of σ. The estimated regression line is now given by

$$\widehat{\mu}_{n:i} = \widehat{\beta}_1\left(\lambda\delta_{n:i}\right)$$

for the set of plotting points $\{\lambda\delta_{n:i}\}$. Specifically, we will be interested in a plot of the points $\left(\lambda\delta_{n:i}, \widehat{\beta}_1\left(\lambda\delta_{n:i}\right)\right)$ on this line for $i = 1, \ldots, n$.

For each of the Cases 1, 2, 4, 5, 6, and 7, using the respective plotting positions $\{x_{n:i}\}$ and attempting to fit a line to the data $(x_{n:i}, Y_{n:i})$ would give an estimate of the slope as

$$\widehat{\beta}_1 = \left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{x}\right)^{-1}\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{Y}.$$

Observing that

$$E\left(\widehat{\beta}_1\right) = \left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{x}\right)^{-1}\left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\delta\right)\sigma,$$

an unbiased estimator for σ is thus given by

$$\widehat{\sigma}_{\{x_{n:i}\}} = \left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\delta\right)^{-1}\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{Y}$$

The variance of this estimator is determined as

$$V\left(\widehat{\sigma}_{\{x_{n:i}\}}\right) = \left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\delta\right)^{-2}\left(\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{x}\right)\sigma^2.$$

To compare the two variances, $V(\widehat{\sigma})$ with $V\left(\widehat{\sigma}_{\{x_{n:i}\}}\right)$, we examine the ratio

$$\frac{V\left(\widehat{\sigma}_{\{x_{n:i}\}}\right)}{V(\widehat{\sigma})} = \frac{\left[\delta^{\mathbf{T}}\mathbf{B}^{-1}\delta\right]\left[\delta^{\mathbf{T}}\mathbf{\Lambda}\mathbf{B}^{-1}\mathbf{\Lambda}\delta\right]}{\left[\delta^{\mathbf{T}}\mathbf{\Lambda}\mathbf{B}^{-1}\delta\right]^2}.$$

For $n = 7$ and the case in which $x_{7:i} = z_{1-i/(7+1)}$ (Weibull (1939), see Tables 1 and 2), we have that

$$\mathbf{x} = \begin{bmatrix} -z_{0.125} \\ -z_{0.250} \\ -z_{0.375} \\ z_{0.500} \\ z_{0.375} \\ z_{0.250} \\ z_{0.125} \end{bmatrix} = \begin{bmatrix} -1.15035 \\ -0.67449 \\ -0.31864 \\ 0.0 \\ 0.31864 \\ 0.67449 \\ 1.15035 \end{bmatrix}.$$

It follows that

$$\frac{V\left(\widehat{\sigma}_{\left\{z_{1-i/(7+1)}\right\}}\right)}{V(\widehat{\sigma})} = \frac{\left[\delta^{\mathbf{T}}\mathbf{B}^{-1}\delta\right]\left[\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\mathbf{x}\right]}{\left[\mathbf{x}^{\mathbf{T}}\mathbf{B}^{-1}\delta\right]^2} = 1.00174.$$

We see that the estimator $\widehat{\sigma}$ has a smaller variance than the estimator $\widehat{\sigma}_{\left\{z_{1-i/(7+1)}\right\}}$ based on the plotting positions in Case 2 for $n = 7$. This also holds for the other sets of plotting positions given in Cases 1, 4, 5, 6, and 7 and for other values of n. Hence, when using any set of plotting positions of the form $\{\lambda\delta_{n:i}\}$, the unbiased linear estimator $\widehat{\sigma}$ has smaller variance than the estimators of σ in Cases 1, 2, 4, 5, 6, and 7. Since

$\mu_{n:i}$ is linearly related to plotting positions of the form $\{\lambda\delta_{n:i}\}$ and when using these plotting positions an estimator of σ can be obtained that is unbiased and has minimum variance when compared to the estimators of σ in Cases 1,2, and 4-7, we recommend a set of plotting positions of the form $\{\lambda\delta_{n:i}\}$ for $\lambda \neq 0$ be used for a normal probability plot. In what follows, only sets of plotting positions of the form $\{\lambda\delta_{n:i}\}$ will be considered with $\lambda \neq 0$.

3 Normal Probability Plots

A normal probability plot plots the ordered observations from a set of independent and identically distributed normal variates against a set of plotting positions. The typical plotting positions are given in the previous section. As seen if the data are from a normal distribution, then the points will plot about a line if a set of plotting positions of the form $\{\lambda\delta_{n:i}\}$ is used with $\lambda \neq 0$. For the other sets of plotting positions listed in Table 1, the relationship is not linear. Thus, at best, the plot of the order statistics versus these plotting positions will only appear to plot about a line.

We are interested in the case in which we assume we will have n (an odd number of) observations $Y_1, \ldots, Y_n$ that are independent and identically distributed $N\left(0, \sigma^2\right)$. For the unreplicated two level (fractional) factorial design, this is equivalent to assuming that measurements are normally distributed with common variance and under the null hypothesis the effects are all zero. In a test of hypothesis framework, we will be testing

$$H_0 : E\left(Y_1\right) = \ldots = E\left(Y_n\right) = 0 \text{ versus } H_a : \text{At least one of the } E\left(Y_i\right) \neq 0.$$

To analyze these data, we will use a normal probability plot of the points $\left(\lambda\delta_{n:i}, Y_{n:i}\right)$ for some given value of λ. Also, we plot the points

$$\left(\lambda\delta_{n:i}, \widehat{\mu}_{n:i}\right) = \left(\lambda\delta_{n:i}, \widehat{\beta}_1\left(\lambda\delta_{n:i}\right)\right) = \left(\lambda\delta_{n:i}, \widehat{\sigma}\delta_{n:i}\right),$$

where $\widehat{\sigma}$ is given by Equation (2). Further, we plot the critical points $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,1-\alpha}\right)$ for $i = 1, \ldots, (n+1)/2$ and $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,\alpha}\right)$ for $i = (n+1)/2, \ldots, n$. In this section, we will give a method for determining the critical points $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,1-\alpha}\right)$ and $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,\alpha}\right)$.

It is convenient for us to define the random variable $T_{n:i}$ as

$$T_{n:i} = \frac{Y_{n:i}}{\widehat{\sigma}} = \frac{Y_{n:i}}{\mathbf{b}^{\mathbf{T}}\mathbf{Y}},$$

where

$$\mathbf{b}^{\mathbf{T}} = \left(\delta^{\mathbf{T}}\mathbf{B}^{-1}\delta\right)^{-1}\delta^{\mathbf{T}}\mathbf{B}^{-1}. \tag{3}$$

For $n = 7$ and $n = 15$, the vector **b** with each component rounded to five decimal places is given in Table 3.

Table 3.Vector **b**

$n = 7$	$n = 15$
−0.27781	−0.14313
−0.13510	−0.09305
−0.06246	−0.07558
0	−0.04757
0.06246	−0.03787
0.13510	−0.02461
0.27781	−0.01213
	0
	0.01213
	0.02461
	0.03787
	0.04757
	0.07558
	0.09305
	0.14313

It is easy to see under the null hypothesis that

$$T_{n:i} = \frac{Z_{n:i}}{\mathbf{b}^{\mathbf{T}}\mathbf{Z}},$$

where $\mathbf{Z} = [Z_{n:1}, \ldots, Z_{n:n}]^{\mathbf{T}}$ is the vector of order statistics of a random sample from a standard normal distribution. Clearly the distribution of $T_{n:i}$ under the null hypothesis does not depend on any parameters and we can, at least in theory, determine any $100\,(1-\alpha)$th percentile $t_{n:i,\alpha}$ of its distribution. We see that

$$P\left[T_{n:i} \leq t_{n:i,\alpha}\right] = P\left[Y_{n:i} \leq \widehat{\sigma} t_{n:i,\alpha}\right].$$

It is desirable to select the value of α such that the probability of the event

$$\bigcap\nolimits_{i=1}^{h-1}\left\{Y_{n:i} \geq \widehat{\sigma} t_{n:i,1-\alpha}\right\} \bigcap \left\{\widehat{\sigma} t_{n:h,1-\alpha} \leq Y_{n:h} \leq \widehat{\sigma} t_{n:h,\alpha}\right\} \bigcap\nolimits_{i=h+1}^{n}\left\{Y_{n:i} \leq \widehat{\sigma} t_{n:i,\alpha}\right\}$$

is a given value $1-\alpha_0$, where $h=(n+1)/2$. Unfortunately, the distributional results are not available for determining (at least numerically) the values of the $t_{n:i,\alpha}$'s that meet this criterion. As this is the case, we will use simulation to estimate the desired $t_{n:i,\alpha}$'s.

There are several ways to obtain a sample counterpart $\widehat{t}_{n:i,\alpha}$ for $t_{n:i,\alpha}$. One such method uses the estimator

$$\widehat{t}_{n:i,\alpha}=r\left(T_{n:i}\right)_{N:k}+(1-r)\left(T_{n:i}\right)_{N:k+1},$$

where $\left(T_{n:i}\right)_{N:k}$ is the kth order statistics from a random sample of size N from the distribution of $T_{n:i}$. Here k and r are selected by setting

$$k=\lfloor (N+1)(1-\alpha)\rfloor \text{ and } r=1-(N+1)(1-\alpha)+k. \tag{4}$$

The symbol $\lfloor x\rfloor$ represents the largest integer less than or equal to the real number x. This method of estimating a percentile of a distribution is based on the well known fact that

$$P\left[T_{n:i}\leq\left(T_{n:i}\right)_{N:k}\right]=\frac{k}{N+1}$$

and linear interpolation. The value of k chosen such that

$$\frac{k}{N+1}\leq 1-\alpha<\frac{k+1}{N+1}.$$

This gives the value of k in Equations (4). Assuming that $t_{n:i,\alpha}$ is between $\left(T_{n:i}\right)_{N:k}$ and $\left(T_{n:i}\right)_{N:k+1}$, linear interpolation is then used to obtain the estimate $\widehat{t}_{n:i,\alpha}$ for $t_{n:i,\alpha}$. Table 4 list the estimated values of $\widehat{t}_{7:i,1-\alpha}$ and $\widehat{t}_{7:i,\alpha}$, where $\alpha=1-(1-0.078)^{1/7}$. This value of α was chosen such that the event

$$\bigcap_{i=1}^{3}\left\{Y_{7:i}\geq\widehat{\sigma}\widehat{t}_{7:i,1-\alpha}\right\}\bigcap\left\{\widehat{\sigma}\widehat{t}_{7:4,1-\alpha}\leq Y_{7:4}\leq\widehat{\sigma}\widehat{t}_{7:4,\alpha}\right\}\bigcap_{i=4}^{7}\left\{Y_{7:i}\leq\widehat{\sigma}\widehat{t}_{7:i,\alpha}\right\}$$

has an approximate $100\left(1-\alpha_0\right)\%=95\%$ chance of occurring under the null hypothesis.

Table 4. Estimated Percentiles

i	$\widehat{t}_{7:i,1-\alpha}$	i	$\widehat{t}_{7:i,\alpha}$
1	-2.64053	4	1.44814
2	-1.95281	5	1.56575
3	-1.56575	6	1.95281
4	-1.44814	7	2.64053

For the case in which $n = 2^4 - 1 = 15$, the values for $\widehat{t}_{15:i,1-\alpha}$ and $\widehat{t}_{15:i,\alpha}$ are given in Table 5 with $\alpha = 1 - (1 - 0.089)^{1/15}$ such that the event

$$\bigcap\nolimits_{i=1}^{7} \left\{Y_{15:i} \geq \widehat{\sigma}\widehat{t}_{15:i,1-\alpha}\right\} \bigcap \left\{\widehat{\sigma}\widehat{t}_{15:8,1-\alpha} \leq Y_{15:8} \leq \widehat{\sigma}\widehat{t}_{15:8,\alpha}\right\} \bigcap\nolimits_{i=9}^{15} \left\{Y_{15:i} \leq \widehat{\sigma}\widehat{t}_{15:i,\alpha}\right\}$$

has an approximate 95% chance of occurring under the null hypothesis.

Table 5. Estimated Percentiles

i	$\widehat{t}_{15:i,1-\alpha}$	i	$\widehat{t}_{15:i,\alpha}$
1	-0.65958	8	0.20771
2	-0.47800	9	0.22364
3	-0.40390	10	0.26204
4	-0.34644	11	0.29864
5	-0.29864	12	0.34644
6	-0.26204	13	0.40390
7	-0.22364	14	0.47800
8	-0.20772	15	0.65958

A FORTRAN program used to simulate the values in Tables 4 and 5 is available from the authors.

In summary, a normal probability plot for examining the data to see if there is evidence against

$$H_0 : E\left(Y_1\right) = \ldots = E\left(Y_n\right) = 0 \text{ in favor of } H_a : \text{At least one of the } E\left(Y_i\right) \neq 0,$$

plots the points $\left(\lambda\delta_{n:i}, Y_{n:i}\right)$ and $\left(\lambda\delta_{n:i}, \widehat{\sigma}\delta_{n:i}\right)$ for $i = 1, \ldots, n$; the points $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,1-\alpha}\right)$, $i = 1, \ldots, (n+1)/2$; and the points $\left(\lambda\delta_{n:i}, \widehat{\sigma}\widehat{t}_{n:i,\alpha}\right)$, $i = (n+1)/2, \ldots, n$. Since the second coordinates of each of these points does not depend on $\lambda \neq 0$, then a point plotting inside its critical limit will plot inside the critical limit regardless of the choice of λ. This also holds for a point that plots outside their respective critical limits. Essentially choices of $\lambda > 1$ stretches the plot horizontally while choices of $\lambda < 1$ compacts the graph horizontally with respect to a choice of $\lambda = 1$. For convenience, we will select $\lambda = 1$ and consider the plot based on the plotting positions $\{\delta_{n:i}\}$ as the normal probability plot.

4 Half-Normal Plots

The half-normal plot was first introduced by Daniel (1959). He (Daniel (1976)) later argued in favor of the normal probability plot over the half-normal plot. This plot was designed specifically for analyzing the data from an unreplicated two level (fractional) factorial design. The half-normal plot plots the points $(\eta_{n:i}, R_{n:i})$, where $R_{n:i}$ is the ith order statistic of the independent random variables $R_i = |Y_i|, ..., R_n = |Y_n|$ with $Y_i \sim N(\mu_i, \sigma^2)$ and $\eta_{n:i} = E(W_{n:i})$, where the $W_{n:i}$'s are the order statistics from a random sample from the distribution of $W = |Z|$ with $Z \sim N(0,1)$. If $E(Y_i) = 0$ for all $i = 1, \ldots, n$, then the plotted points should plot about the line passing through the points $(\eta_{n:i}, E[R_{n:i}])$. Also, included on this plot are the points $(\eta_{n:i}, \widehat{\sigma}^* \eta_{n:i})$ on the estimated regression line, where the estimator $\widehat{\sigma}^*$ of σ is defined in Equation (**??**). Further, so that the data can be assessed objectively, we add the critical points $(\eta_{n:i}, \widehat{\sigma}^* \widehat{v}_{n:i,\alpha})$, for $i = (n+1)/2, \ldots, n$. In what follows, we give a method for determining estimating $\widehat{v}_{n:i,\alpha}$ and selecting the value of α.

The method for estimating $\widehat{v}_{n:i,\alpha}$ is similar to estimating $\widehat{t}_{n:i,\alpha}$ in the previous section. We first define

$$V_{n:i} = \frac{R_{n:i}}{\widehat{\sigma}^*}.$$

We have under the null hypothesis $R_{n:i} = \sigma W_{n:i}$ and hence

$$V_{n:i} = \frac{W_{n:i}}{\mathbf{c^T W}},$$

where $\mathbf{W} = [W_{n:1}, \ldots, W_{n:n}]^{\mathbf{T}}$, $\mathbf{C} = cov(\mathbf{W})$, and $\mathbf{c^T} = \left(\eta^{\mathbf{T}} \mathbf{C}^{-1} \eta\right)^{-1} \eta^{\mathbf{T}} \mathbf{C}^{-1}$. The values of the vector η and the matrix $\mathbf{C}$ can be found in Govindarajulu and Eisenstat (1965). Values for the vector $\mathbf{c}$ are listed in

Table 6 for $n = 7$ and $n = 15$.

Table 6. Vector **c**

i	$n = 7$	$n = 15$
1	0.03528	0.00891
2	0.05593	0.01381
3	0.07977	0.01845
4	0.10726	0.02368
5	0.14063	0.02924
6	0.18573	0.03591
7	0.28995	0.03862
8		0.04862
9		0.05433
10		0.06245
11		0.07087
12		0.08149
13		0.09381
14		0.11162
15		0.15493

Thus, we see that the distribution of $V_{n:i}$ does not dependent on any unknown parameters. Further, we note that

$$P\left[V_{n:i} \leq v_{n:i,\alpha}\right] = P\left[R_{n:i} \leq \widehat{\sigma}^{*} v_{n:i,\alpha}\right].$$

The distributional results for determining the joint distribution of $V_{n:i}$ are not yet available, hence we used simulation to estimate $v_{n:i,\alpha}$. Table 7 gives the estimates of $\widehat{v}_{7:i,\alpha}$ for $\alpha = 1 - (1 - 0.089)^{1/7}$.

Table 7. Estimated Percentiles

i	$\widehat{v}_{7:i,\alpha}$
1	0.54522
2	0.74657
3	0.93398
4	1. 12449
5	1.33409
6	1.65346
7	2.43717

This value of α was selected such that the probability of the event

$$\bigcap_{i=1}^{7} \{V_{7:i} \geq \widehat{v}_{7:i,\alpha}\}$$

is approximately 95%. For the case in which $n = 2^4 - 1$, the estimated values of $\{\widehat{v}_{15:i,\alpha}\}$ for $\alpha = 1 - (1 - 0.0959)^{1/15}$ are given in Table 8. The values in both Tables 7 and 8 are based on $20,000$ simulated results.

Table 8. Estimated Percentiles

i	$\widehat{v}_{15:i,\alpha}$
1	0.35010
2	0.47131
3	0.57841
4	0.66300
5	0.75350
6	0.85010
7	0.93444
8	1.01393
9	1.10987
10	1.21375
11	1.33652
12	1.50275
13	1.72557
14	2.11135
15	2.91150

The simulation program written in FORTRAN that was used to simulate the values in Tables 6 and 7 is available from the authors.

5 Joint Observed Significance Levels

The methods presented in the previous two sections determine if an effect is "active" if its estimated value exceeds the associated "joint" percentile of the distribution of the respective order statistic. This is an objective method that has a fixed but small probability of saying there is one or more active effects when in fact there are none. In this section, we present a method for estimating the "joint" observed significance levels (or p-values) for each of the n ordered estimated effects. A researcher having a set of "joint" levels of significance could then decide objectively which effects are active.

Let $Y_{n:1}, \ldots, Y_{n:n}$ be the ordered estimators of the effects. We have defined

$$T_{n:i} = \frac{Y_{n:i}}{\mathbf{b^T Y}},$$

where $\mathbf{Y} = [Y_{n:1}, \ldots, Y_{n:n}]^{\mathbf{T}}$ and $\mathbf{b}$ as defined in Equation (3). If all the effects are zero, then $T_{n:i}$ has the same distribution as $Z_{n:i} / \left(\mathbf{b^T Z}\right)$, where the vector $\mathbf{Z} = [Z_{n:1}, \ldots, Z_{n:n}]^{\mathbf{T}}$ is the vector of order statistics of a random sample from a standard normal distribution. We define

$$P_{n:i} = \begin{cases} P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \leq T_{n:i}\right), & \text{for } i = 1, \ldots, (n-1)/2; \\ P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \geq |T_{n:i}|\right), & \text{for } i = (n+1)/2; \\ P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \geq T_{n:i}\right), & \text{for } i = (n+3)/2, \ldots, n. \end{cases}$$

The probabilities $P_{n:1}, \ldots, P_{n:n}$ will be called the joint significant levels (JSL). Clearly these values are random variables before the sample is observed. They have a joint distribution as well as marginal distributions. The observed JSLs (or joint p-values), $p_{n:1}, \ldots, p_{n:n}$, can each be used as measures of the evidence in the sample against the hypothesis that the corresponding effects are zero, where

$$p_{n:i} = \begin{cases} P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \leq t_{n:i}\right), & \text{for } i = 1, \ldots, (n-1)/2; \\ P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \leq t_{n:i}\right), & \text{for } i = (n+1)/2; \\ P\left(\frac{Z_{n:i}}{\mathbf{b^T Z}} \geq t_{n:i}\right), & \text{for } i = (n+3)/2, \ldots, n; \end{cases}$$

with $t_{n:i}$ the observed value of $T_{n:i}$. The minimum of these joint (OSLs) would be a single measure of the presents of at least one nonzero effect.

Determing the values $p_{n:1}, \ldots, p_{n:n}$ given the sample requires determining and evaluating the joint distribution of $\mathbf{Z} / \left(\mathbf{b^T Z}\right)$ from which the marginal distributions of $Z_{n:1} / \left(\mathbf{b^T Z}\right), \ldots, Z_{n:n} / \left(\mathbf{b^T Z}\right)$ can be determined. These distributions are not readily obtainable so the joint OSLs will be determined using simulation under the assumed normal model. The estimated values are denoted by $\widehat{p}_{n:1}, \ldots, \widehat{p}_{n:n}$. For convenience, we define the random variables $\left(A_{n:i}\right)_l = 1$ if the lth simulated value of $Z_{n:i} / \mathbf{b^T Z} \leq t_{n:i}$ for $i = 1, \ldots, (n-1)/2$ or if the lth simulated value of $\left|Z_{n:(n+1)/2} / \mathbf{b^T Z}\right| \geq \left|t_{n:(n+1)/2}\right|$ or the lth simulated value of $Z_{n:i} / \mathbf{b^T Z} \geq t_{n:i}$ for $i = (n+3)/2, \ldots, n$. For N (number of simulations) independent samples each of size n from a standard normal distribution, it is easy to see that for each i, $\left(A_{n:i}\right)_1, \ldots, \left(A_{n:i}\right)_N$, is a random sample from a Bernoulli distribution with parameter $p_{n:i}$. Defining $\widehat{p}_{n:i}$ by

$$\widehat{p}_{n:i} = \frac{1}{N} \textstyle\sum_{l=1}^{N} \left(A_{n:i}\right)_l,$$

we have an unbiased estimator for $p_{n:i}$ with variance $p_{n:i}\left(1-p_{n:i}\right)/N$. Further, the $\widehat{p}_{n:i}$'s are correlated. Similar values can be computed for the absolute values of the estimated effects that are plotted on the half-normal plot. In this case, we define

$$p_{n:i}=P\left(\frac{W_{n:i}}{\mathbf{c^{T}W}}\geq v_{n:i}\right)$$

for $i=1,\ldots,n$. Presently the authors are examining their proposed method for obtaining estimated joint OSLs.

6 Examples

In this section, we illustrate the use of normal and half-normal plots for unreplicated 2^3 and a 2^4 factorial designs as well as an unreplicated 2^{7-3} fractional factorial design. The data for the following unreplicated 2^3example is taken from Daniel (1976), page 54. In this experiment the response variable is thickening time, in minutes, to achieve a certain degree of hardness of a given type of cement. There are three factors of interest: Factor 1 – time of stirring, Factor 2 – temperature, and Factor 3 – pressure. The design model has the form

$$E\left(\mathbf{X}\right)=\begin{bmatrix}E\left(X_{1,1,1}\right)\\E\left(X_{2,1,1}\right)\\E\left(X_{1,2,1}\right)\\E\left(X_{2,2,1}\right)\\E\left(X_{1,1,2}\right)\\E\left(X_{2,1,2}\right)\\E\left(X_{1,2,2}\right)\\E\left(X_{2,2,2}\right)\end{bmatrix}=\begin{bmatrix}1&-1&-1&-1&1&1&1&-1\\1&1&-1&-1&-1&-1&1&1\\1&-1&1&-1&-1&1&-1&1\\1&1&1&-1&1&-1&-1&-1\\1&-1&-1&1&1&-1&-1&1\\1&1&-1&1&-1&1&-1&-1\\1&-1&1&1&-1&-1&1&-1\\1&1&1&1&1&1&1&1\end{bmatrix}\begin{bmatrix}\mu\\\alpha\\\beta\\\gamma\\(\alpha\beta)\\(\alpha\gamma)\\(\beta\gamma)\\(\alpha\beta\gamma)\end{bmatrix}=\mathbf{A}\theta,$$

where $\mathbf{X}$ is the response matrix with $X_{i,j,k}$ is the response when Factors A, B, and C are at levels i, j, k respectively with $i,j,k=1,2$. The matrix $\mathbf{A}$ is the design matrix and the vector θ is the parameter with the parameter μ representing the overall mean, the main effects of Factors A, B, and C are respectively α, β, and γ with two and three factors effects $(\alpha\beta)$, $(\alpha\gamma)$, $(\beta\gamma)$, and $(\alpha\beta\gamma)$. Using the Table 5.1, p. 54 of Daniel (1959), we obtain the data vector $\mathbf{X}$ for this experiment, calculate the vector of estimated parameters, and the ordered estimated effects. These vectors are given as

$$\mathbf{X} = \begin{bmatrix} 297 \\ 300 \\ 106 \\ 131 \\ 177 \\ 178 \\ 76 \\ 109 \end{bmatrix}, \widehat{\theta} = \left(\mathbf{A^T A}\right)^{-1} \mathbf{A^T X} = \begin{bmatrix} 171.75 \\ 7.75 \\ -66.25 \\ -36.75 \\ 6.75 \\ 0.75 \\ 23.75 \\ 1.25 \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} -66.25 \\ -36.75 \\ 0.75 \\ 1.25 \\ 6.75 \\ 7.75 \\ 23.75 \end{bmatrix}.$$

Here $\widehat{\gamma} = 7.75$, $\widehat{\beta} = -66.25$, and $\widehat{\alpha} = -36.75$ are estimates of the parameters associated with main effects, $\widehat{(\beta\gamma)} = 6.75$, $\widehat{(\alpha\gamma)} = 0.75$, and $\widehat{(\alpha\beta)} = 23.75$ are the estimates of the parameters associated with the two factor interactions, and $\widehat{(\alpha\beta\gamma)} = 1.25$ is the estimate of the parameter associated with the three factor interaction. The ordered values of these estimated "effects" are the components of the vector $\mathbf{Y}$.

Our estimate for σ is

$$\widehat{\sigma} = \mathbf{b^T Y} = 31.38956,$$

where the 7×1 vector $\mathbf{b}$ is given in Table 3. The normal probability plot for this example is the plot of the set of points

$$\{(\delta_{7:i}, y_{7:i})\,|i = 1, \ldots, 7\} = \left\{ \begin{array}{c} (-1.35218, -66.25) \\ (-0.75737, -36.75) \\ (-0.35271, 0.75) \\ (0.0, 1.25) \\ (0.35271, 6.75) \\ (0.75737, 7.75) \\ (1.35218, 23.75) \end{array} \right\}.$$

This plot is given in Figure 1.

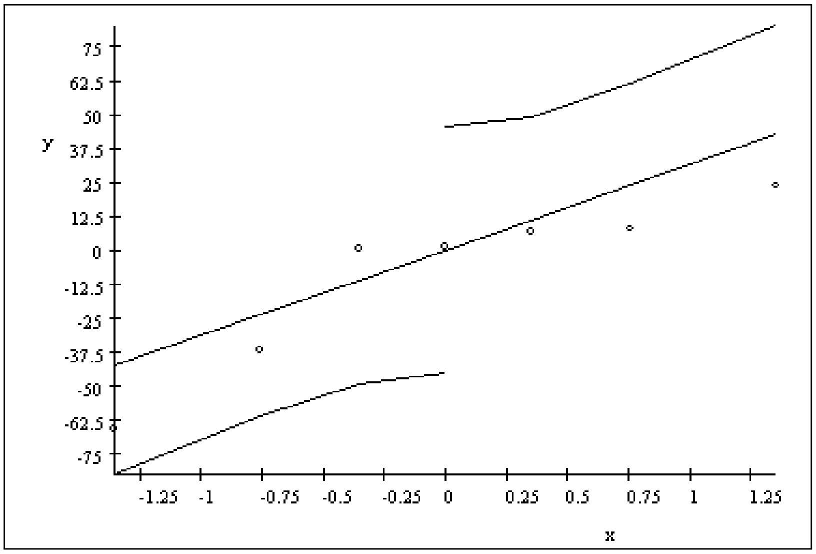

Figure 1. Normal Probability Plot for Data from Daniel (1976)

Also on this plot is the least squares regression line on which the following points fall.

$$\left\{(\delta_{7:i}, \widehat{\sigma}\delta_{7:i})\,|i = 1, \ldots, 7\right\} = \left\{\begin{array}{c} (-1.35218, -42.44430) \\ (-0.75737, -23.77364) \\ (-0.35271, -11.07132) \\ (0, 0) \\ (0.35271, 11.07132) \\ (0.75737, 23.77365) \\ (1.35218, 42.44430) \end{array}\right\}$$

Further, we include the critical points

$$\left\{\left(\delta_{7:i}, \widehat{\sigma}\widehat{t}_{7:i,1-\alpha}\right)|i = 1, 2, 3, 4\right\} = \left\{\begin{array}{c} (-1.35218, -82.88502) \\ (-0.75737, -61.29774) \\ (-0.35271, -49.14818) \\ (0, -45.45644) \end{array}\right\}$$

$$\left\{\left(\delta_{7:i}, \widehat{\sigma}\widehat{t}_{7:i,\alpha}\right)|i = 4, 5, 6, 7\right\} = \left\{\begin{array}{c} (0, 45.45644) \\ (0.35271, 49.14818) \\ (0.75737, 61.29774) \\ (1.35218, 82.88502) \end{array}\right\}$$

in which the lower ones are joined by line segments as are the upper ones. Since none of the points fall outside their respective critical values, there is no evidence that any of the effects are active.

The half-normal plot of these data is

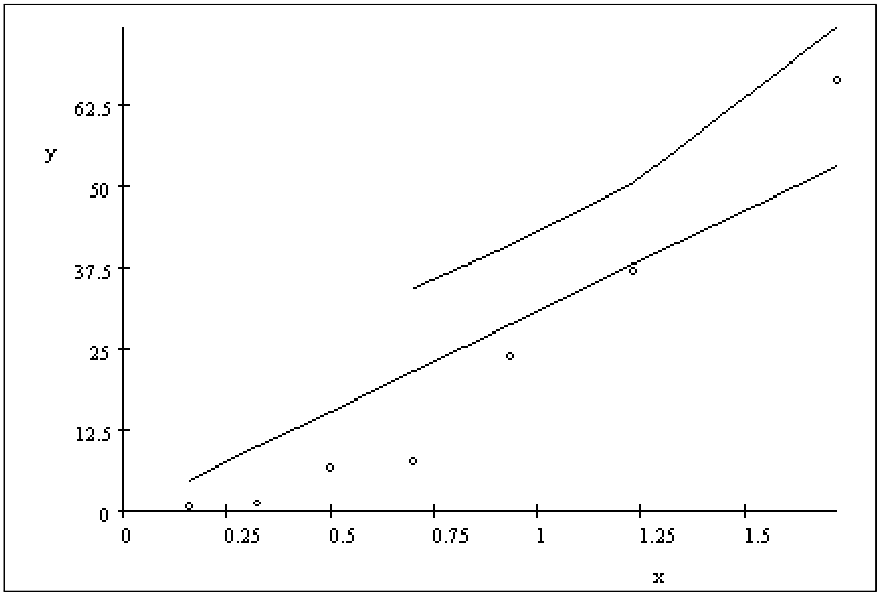

Figure 2. Half-Normal Probability Plot of Data from Daniel (1976)

The estimate for σ is $\widehat{\sigma}^* = \left(\eta^{\mathbf{T}}\eta\right)^{-1}\eta^{\mathbf{T}}\mathbf{R} = 30.57734$. On this plot there is no indication that any of the effects are non-zero. Daniel (1976) gives no analysis of these data indicating which, if any, of the effects are non-zero.

Box, Hunter, and Hunter (1978), p. 324, provide an example of the unreplicated 2^4 factorial designs used on a process development study. Table 9 list of the factors and their two levels are given

Table 9. Factors - Process Development Study

	Factor	Low	High
A	Catalyst change (lbs)	10	15
B	Temperature (^{o}C)	220	240
C	Pressure (psi)	50	80
D	Concentration (%)	10	12

The design of the experiments as well as the observed responses for each treatment are given in the Table 10. [This table comes from Box, Hunter, and Hunter (1978) Table 10.6, page 325. Virtually no information is given as to the mean of the factors used or the response variable.(conversion)]

Table 10 Data - Process Development Study, Yates Order

Obs #	A	B	C	D	Conv.(%)	order of run
1	-1	-1	-1	-1	71	8
2	1	-1	-1	-1	61	5
3	-1	1	-1	-1	90	10
4	1	1	-1	-1	82	4
5	-1	-1	1	-1	68	15
6	1	-1	1	-1	61	9
7	-1	1	1	-1	87	1
8	1	1	1	-1	80	13
9	-1	-1	-1	1	61	16
10	1	-1	-1	1	50	5
11	-1	1	-1	1	89	11
12	1	1	-1	1	83	14
13	-1	-1	1	1	59	3
14	1	-1	1	1	51	12
15	-1	1	1	1	85	6
16	1	1	1	1	78	7

The data vector **X**, the vector of estimated effects Using the Table 5.1, p. 54 of Daniel (1959), we obtain the design matrix **A** and corresponding data vector **X** for this experiment, and the vector **Y** of ordered estimated effects are given, respectively, as

$$\mathbf{X} = \begin{bmatrix} 71 \\ 61 \\ 90 \\ 82 \\ 68 \\ 61 \\ 87 \\ 80 \\ 61 \\ 50 \\ 89 \\ 83 \\ 59 \\ 51 \\ 85 \\ 78 \end{bmatrix}; \widehat{\theta} = \begin{bmatrix} \widehat{\mu} \\ \widehat{\alpha} \\ \widehat{\beta} \\ \widehat{\gamma} \\ \widehat{\tau} \\ \widehat{(\alpha\beta)} \\ \widehat{(\alpha\gamma)} \\ \widehat{(\alpha\tau)} \\ \widehat{(\beta\gamma)} \\ \widehat{(\beta\tau)} \\ \widehat{(\gamma\tau)} \\ \widehat{(\alpha\beta\gamma)} \\ \widehat{(\alpha\beta\tau)} \\ \widehat{(\alpha\gamma\tau)} \\ \widehat{(\beta\gamma\tau)} \\ \widehat{(\alpha\beta\gamma\tau)} \end{bmatrix} = \begin{bmatrix} 72.\,25 \\ -4.0 \\ 12.0 \\ -1.125 \\ -2.75 \\ 0.5 \\ 0.375 \\ 0 \\ -0.625 \\ 2.25 \\ -0.125 \\ -0.375 \\ 0.25 \\ -0.125 \\ -0.375 \\ -0.125 \end{bmatrix}; \text{and } \mathbf{Y} = \begin{bmatrix} -4.0 \\ -2.75 \\ -1.125 \\ -0.625 \\ -0.375 \\ -0.375 \\ -0.125 \\ -0.125 \\ -0.125 \\ 0.0 \\ 0.25 \\ 0.375 \\ 0.5 \\ 2.25 \\ 12.0 \end{bmatrix}.$$

The normal probability plot of these data with critical limits is given by

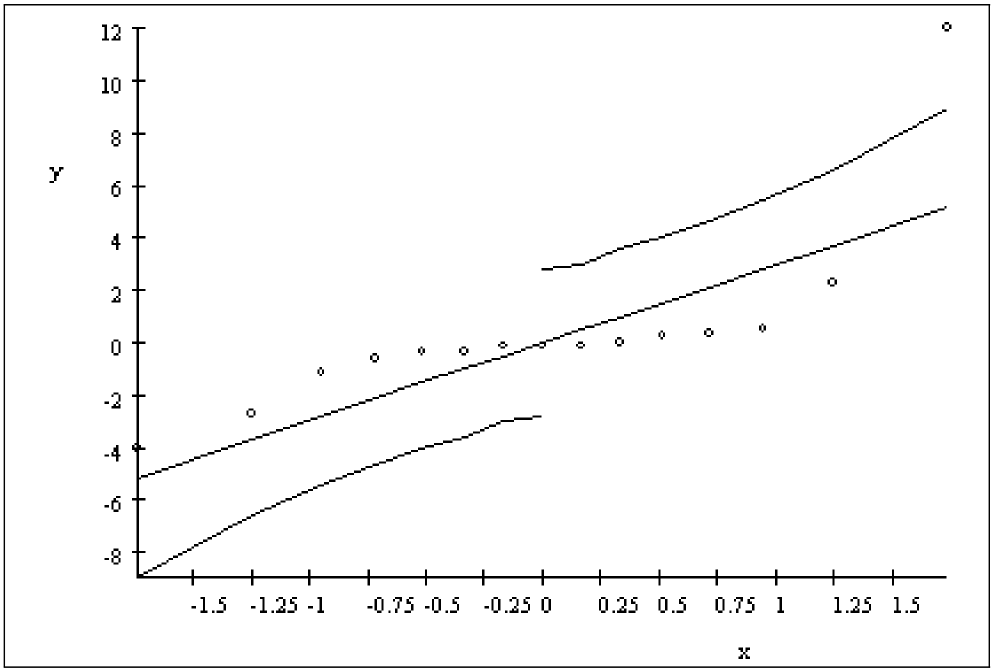

Figure 3. Normal Probability Plot of Data from Box, Hunter, and Hunter (1978)

The normal probability plot reveals that the only effect that is active (significantly different from zero) is the main effect associated with Factor B. This is the same conclusion given by Box, Hunter, and Hunter (1978). We note here that the normal and half-normal plots are only being used to locate potential main effects and interactions. There appears to be some evidence of a pattern in the plot in Figure 3. It appears that further investigation of these data is needed; but this is outside the scope of this paper.

The half-normal plot of these data is given in Figure 4.

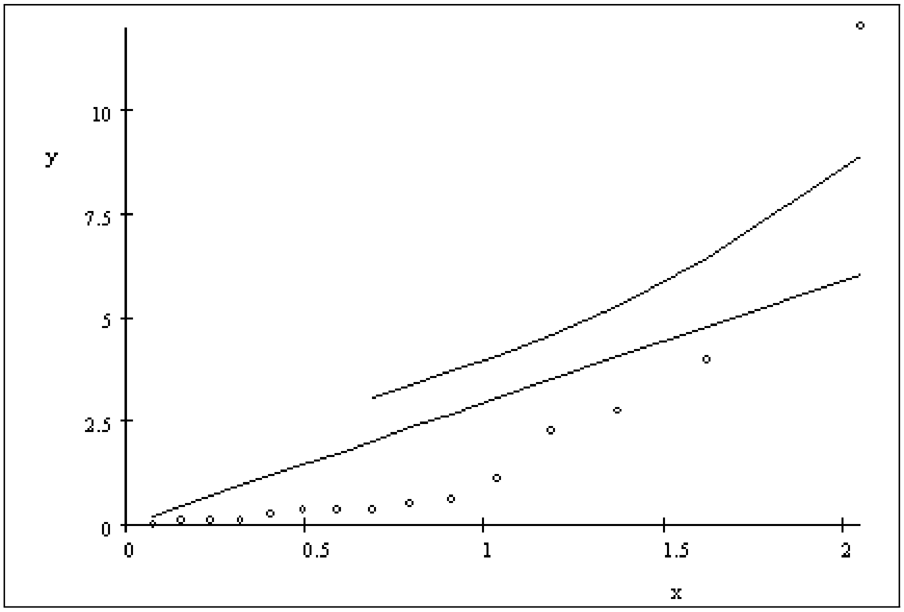

Figure 4. Half-Normal Plot of Data from Box, Hunter, and Hunter (1978)

This plot suggests that the only significant effect is the main effect associated with Factor B. The half-normal plot agrees with the normal probability plot.

Koita, Rawizza, and Staelin (1995) provide an example of an unreplicated 2^{7-3} fractional factorial design. Virtually no information is given as to the meaning of the factors and the response variable. The design matrix **A** for this experiment is given as

$$
\mathbf{A} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}
$$

The data vector $\mathbf{X}$ and the vector $\mathbf{Y}$ of ordered effects are

$$\mathbf{X} = \begin{bmatrix} 28.84 \\ 46.44 \\ 24.08 \\ 23.28 \\ 23.22 \\ 36.50 \\ 19.98 \\ 21.34 \\ 18.52 \\ 34.88 \\ 27.84 \\ 26.12 \\ -1.90 \\ 17.18 \\ 5.98 \\ 7.86 \end{bmatrix} \text{ and } \mathbf{Y} = \begin{bmatrix} -4.19 \\ -4.10 \\ -3.54 \\ -0.53 \\ -0.46 \\ -0.35 \\ 0.26 \\ 0.26 \\ 0.31 \\ 0.47 \\ 0.72 \\ 2.84 \\ 2.95 \\ 5.45 \\ 6.246 \end{bmatrix}.$$

Our estimate for σ is

$$\widehat{\sigma} = \mathbf{b}^{\mathbf{T}}\mathbf{Y} = 3.09862.$$

The normal probability plot of these data with estimated critical limits is given in Figure 5.

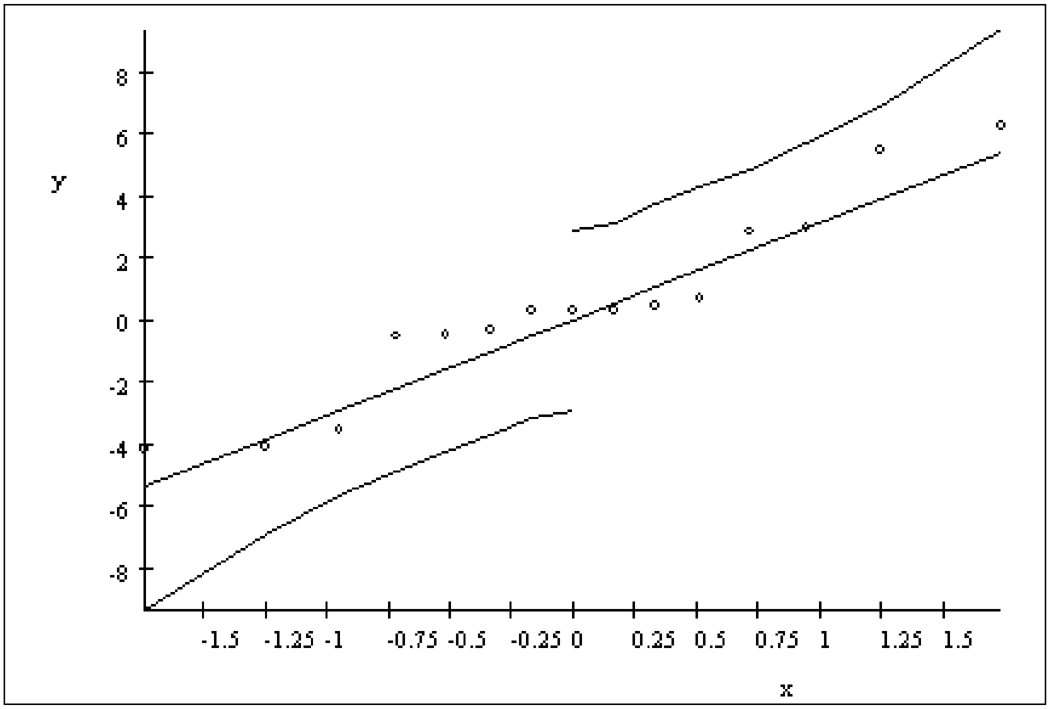

Figure 5. Normal Probability Plot of Data from Koita, Rawizza, and Staelin (1995)

This plots shows no evidence that the parameters being estimated are significantly different from zero. [Note that the aforementioned parameters are not the main effects and interactions (effects) since there is confounding.]

The half-normal plot, Figure 6 does not indicate that any of the estimated parameters are different from zero. Using Lenth's (1989) algorithm, Koita, Rawizza, and Staelin (1995) determined that parameters associated with the estimated values 5.45, 6.24, 2.95, −4.19, −3.54, 2.84, are −4.10 are significantly different from zero.

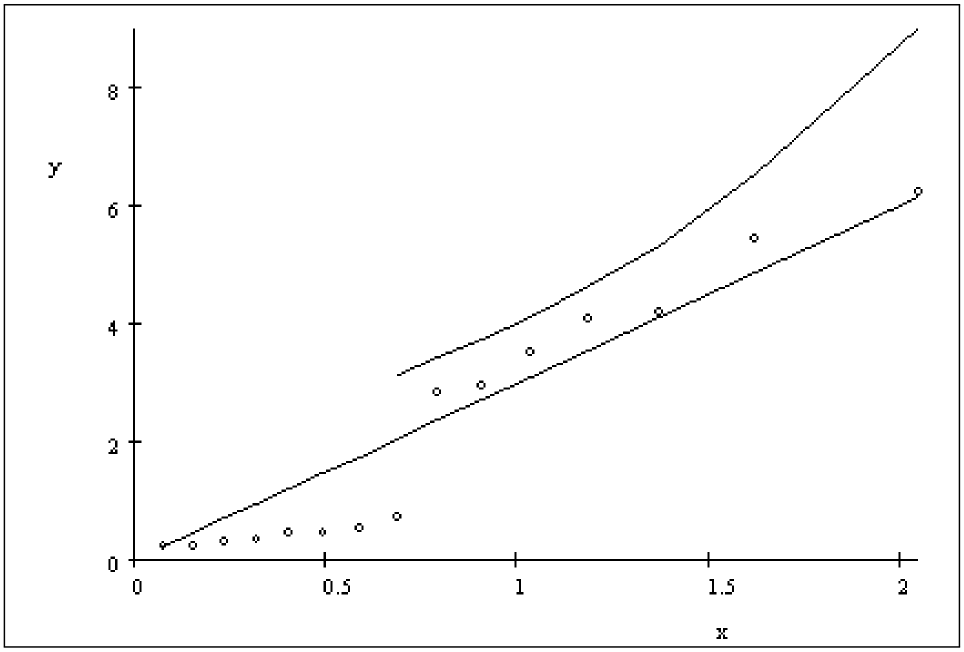

Figure 6. Half-Normal Plot of Data from Koita, Rawizza, and Staelin (1995)

7 Conclusion

The normal and half-normal probability plots were discussed for their use in analyzing unreplicated two level factorial and fractional factorial designs. A method was given for adding critical points to the plot so that one can objectively decide if an effect is non-zero. Based on our analysis of the methods for constructing probability plots, we recommend the set of plotting positions $\{\delta_{n:i}\}$ for the normal probability plot. Although no similar analysis was done for the half-normal plots, the analysis presented for the normal probability plots suggest that the set of plotting positions $\{\eta_{n:i}\}$ should be used in constructing a half-normal plot. Joint observed significance levels were introduced as simple numerical values a researcher could use to aid in the assessment of active effects. Examples were given to illustrate the graphical methods presented.

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