DEPARTMENT OF MATHEMATICAL SCIENCES

TECHNICAL REPORT SERIES

When Is the Isbell Topology a Group Topology?

by

Szymon Dolecki
Institut de Mathématiques de Bourgogne
Université de Bourgogne, B.P. 47870
21078 Dijon, France

Frédéric Mynard
Department of Mathmatical Sciences
Georgia Southern University, Statesboro, GA 30460-8093

Number 2008-003
Submitted: September 16, 2008

WHEN IS THE ISBELL TOPOLOGY A GROUP TOPOLOGY?

SZYMON DOLECKI AND FRÉDÉRIC MYNARD

Abstract. It is shown that the Isbell topology on $C(X,\mathbb{R})$ is always completely regular but not always a group topology. In particular, it is not always translation invariant and addition is not always jointly continuous at the zero function. These two properties are moreover independent. However, in the case of a countable space X with only one non-isolated point and finite compact subsets, the Isbell topology on $C(X,\mathbb{R})$ is translation invariant but is a group topology if and only if X is consonant.

1. Introduction

In [7] and [8] Isbell introduced and studied a topology on the space $C(X,Z)$ of continuous functions, defined in terms of (what is now called) *compact families* of subsets of X and open subsets of Z. The *Isbell topology* is finer than the *compact-open* topology and coarser than the *natural topology* (that is, the topological reflection of the *natural convergence*, most often called *continuous convergence*). Recently Jordan introduced in [10] several intermediate topologies, finer than the Isbell and coarser than the natural topology, that turn out to be instrumental in understanding function spaces. One of them is the so-called *fine Isbell topology.*

The Isbell topology on $C(X,\mathbb{R})$ coincides with the compact-open topology if and only X is *consonant*, that is, if each compact family on X is compactly generated. There are consonant examples (e.g., [5, Example 5.12], [6]) of spaces, for which the Isbell topology is strictly coarser than the natural topology, but to our knowledge there is so far no characterization of X for which the Isbell topology and the natural topology coincide on $C(X,\mathbb{R})$.

The natural convergence is always a group convergence, in particular, it is invariant under translations[1], hence the natural topology is also invariant under translations as the topological reflection of the natural convergence (see [4]), but need not be a group topology, e.g. [9]. In [12], B. Papadopoulos proposes a sufficient condition on a topological space X for the Isbell topological space $C_\kappa(X,\mathbb{R})$ to be a vector space topology. However, it seems that no example has been known so far of a space X, for which $C_\kappa(X,\mathbb{R})$ is *not* a vector space topology.

In this note, we investigate under what conditions the Isbell topology is a group topology.

In general, a topology on an abelian group is a group topology if and only if it is invariant under translations and if the group operation is (jointly) continuous at the neutral group element. Therefore we are confronted with two quests about the Isbell topology on $C(X,\mathbb{R})$:

(1) invariance by translations, and

Date: September 16, 2008.

[1] Actually, the natural convergence is a convergence vector space.

(2) continuity of the addition at the zero function $\overline{0}$, that is, the property

$$\mathcal{N}_\kappa(\overline{0}) + \mathcal{N}_\kappa(\overline{0}) \geq \mathcal{N}_\kappa(\overline{0}). \tag{1.1}$$

More specifically, we show that the space $C_\kappa(X,\mathbb{R})$ of real-valued continuous functions on X endowed with the Isbell topology is invariant under translations if and only if the Isbell and fine Isbell topologies coincide. In [10], Jordan provides an example of a topological space X, for which the Isbell and fine Isbell topologies on $C(X,\mathbb{R})$ do *not* coincide. This shows that there exist X for which $C_\kappa(X,\mathbb{R})$ is not invariant by translations.

We characterize spaces X for which (1.1) holds. It turns out that $C_\kappa(X,\mathbb{R})$ may be translation-invariant without satisfying (1.1). However if a space X is *prime* (called also *atomic*), that is, with at most one non-isolated point, then $C_\kappa(X,\mathbb{R})$ is a topological group if and only if the addition is continuous at $\overline{0}$ if and only if X is consonant.

Finally, we provide a positive answer to [13, Problem 61] by showing that $C_\kappa(X,Z)$ is completely regular whenever Z is.

2. Generalities

If $\mathcal{A}$ is a family of subsets of a topological space X then $\mathcal{O}_X(\mathcal{A})$ denotes the family of open subsets of X containing an element of $\mathcal{A}$. In particular, if $A \subset X$ then $\mathcal{O}_X(A)$ denotes the family of open subsets of X containing an element of A. For two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X we denote $\mathcal{A} \wedge \mathcal{B}$ the family $\mathcal{O}_X(\{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\})$.

A family $\mathcal{A}$ is *compact* if $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ and whenever $\bigcup_{i\in I} O_i \in \mathcal{A}$ where each O_i is open, there is a finite subsets F of I such that $\bigcup_{i\in F} O_i \in \mathcal{A}$. We denote by $\kappa(X)$ the collection of compact families on X. Of course, for each compact subset K of X, the family $\mathcal{O}_X(K)$ is compact. On the other hand, the family $\kappa(X)$ of subsets of $\mathcal{O}_X$ (the set of open subsets of X), is that of open subsets of the *Scott topology*; in particular, every union of compact families is compact. A topological space is called *consonant* if every compact family $\mathcal{A}$ is *compactly generated*, that is, there is a family $(K_i)_{i\in I}$ of compact subsets of X such that $\mathcal{A} = \bigcup_{i\in I} \mathcal{O}_X(K_i)$.

The *Isbell topology* on $C(X,Z)$ can be defined by the following subbase of open sets

$$[\mathcal{A}, U] := \{f \in C(X,Z) : \exists A \in \mathcal{A}, f(A) \subseteq U\},$$

where $\mathcal{A}$ ranges over $\kappa(X)$ and U ranges over open subsets of Z. We write $C_\kappa(X,Z)$ for the set $C(X,Z)$ endowed with the Isbell topology. It is immediate from the definitions that

Proposition 2.1. *X is consonant if and only if the Isbell topology and the compact-open topology on $C(X,Z)$ coincide for every Z. In particular, if X is consonant, then $C_\kappa(X,\mathbb{R})$ is a topological vector space.*

Papadopoulos says that a space X has *property* (A^*) if whenever $\mathcal{A} \in \kappa(X)$ and A_1 and A_2 are open subsets of X such that $A_1 \cup A_2 \in \mathcal{A}$, there exists filters $\mathcal{F}_i$ such that $A_i \in \mathcal{F}_i$, $i = 1, 2$ such that $\mathcal{O}_X(\mathcal{F}_i) \in \kappa(X)$ and $\mathcal{O}(\mathcal{F}_1) \wedge \mathcal{O}(\mathcal{F}_2) \subseteq \mathcal{A}$. The main result of [12] is that property (A^*) is sufficient for the Isbell topology on $C(X,\mathbb{R})$ to be a vector space topology. If X is Hausdorff, this result follows immediately from Proposition 2.1 because of the following:

Proposition 2.2. *Let X be a Hausdorff topological space. Then X is consonant if and only if X has property (A^*).*

Proof. Assume that X is consonant and that $A_1 \cup A_2 \in \mathcal{A}$ where $\mathcal{A} \in \kappa(X)$. Because X is consonant, there is a compact set $K \subseteq A_1 \cup A_2$ such that $\mathcal{O}(K) \subseteq \mathcal{A}$. Then there exist compact subsets K_1 of A_1 and K_2 of A_2 such that $K = K_1 \cup K_2$, so that $\mathcal{O}(K_1)$ and $\mathcal{O}(K_2)$ are the sought compact filters. Conversely, if X satisfies (A^*) then for every $A \in \mathcal{A}$ there is a compact filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{O}(\mathcal{F}) \subseteq \mathcal{A}$. Because $\mathcal{F}$ is a compact filter in a Hausdorff space, $\mathcal{O}(\mathcal{F}) = \mathcal{O}(\operatorname{adh} \mathcal{F})$ and $\operatorname{adh} \mathcal{F}$ is compact (e.g. [3, Proposition 2.2]). Therefore, $\mathcal{A}$ is compactly generated and X is consonant. □

The *grill of* a family $\mathcal{A}$ of subsets of X is the family $\mathcal{A}^{\#} := \{B \subseteq X : \forall A \in \mathcal{A}, A \cap B \neq \varnothing\}$. Note that if $\mathcal{A} = \mathcal{O}(\mathcal{A})$, then

$$A \in \mathcal{A} \Longleftrightarrow A^c \notin \mathcal{A}^{\#}.$$

Lemma 2.3. [2] *If $\mathcal{A} \in \kappa(X)$ and C is a closed subset of X such that $C \in \mathcal{A}^{\#}$ then*

$$\mathcal{A} \vee C := \mathcal{O}\left(\{A \cap C : A \in \mathcal{A}\}\right)$$

is a compact family on X.

Lemma 2.4. *If $\mathcal{A} \in \kappa(X)$ and $A_0 \in \mathcal{A}$ then*

$$\mathcal{A} \downarrow A_0 := \mathcal{O}\left(\{A \in \mathcal{A} : A \subseteq A_0\}\right)$$

is a compact family on X.

Proof. If $\bigcup_{i \in I} O_i \in \mathcal{A} \downarrow A_0$ then there is $A \in \mathcal{A}$ such that $A \subseteq A_0$ and $A \subseteq \bigcup_{i \in I} O_i$ so that $A \subseteq \bigcup_{i \in I} (O_i \cap A_0)$. By compactness of $\mathcal{A}$ there is a finite subset F of I such that $\bigcup_{i \in F} (O_i \cap A_0) \in \mathcal{A}$. But $\bigcup_{i \in F} (O_i \cap A_0) \subseteq A_0$ so that $\bigcup_{i \in F} (O_i \cap A_0) \in \mathcal{A} \downarrow A_0$ and $\bigcup_{i \in F} O_i \in \mathcal{A} \downarrow A_0$. □

In completely regular spaces, a closed and a compact disjoint sets can be functionally separated: *if A is a compact subset of a completely regular space X and F is a closed subset of X such that $A \cap F = \varnothing$, then there exists $h \in C(X, [0,1])$ such that $h(A) = \{0\}$ and $h(F) = \{1\}$.* We extend this well-known fact to a closed set and a compact family.

Lemma 2.5. *If $\mathcal{A} = \mathcal{O}(\mathcal{A})$ is a compact family of subsets of a completely regular topological space X, and F is a closed subset of X with $F^c \in \mathcal{A}$, then there is $A \in \mathcal{A}$ and $h \in C(X, [0,1])$ such that $h(A) = \{0\}$ and $h(F) = \{1\}$.*

Proof. By complete regularity, for every $x \notin F$, there is an open neighborhood O_x of x and $h_x \in C(X, [0,1])$ such that $h_x(O_x) = \{0\}$ and $h_x(F) = \{1\}$. Therefore $F^c = \bigcup_{x \notin F} O_x \in \mathcal{A}$, so that by the compactness of $\mathcal{A}$ there is $n < \omega$ and $x_1, \ldots, x_n \notin F$ such that $A = \bigcup_{1 \le i \le n} O_{x_i} \in \mathcal{A}$. The continuous function $\min_{1 \le i \le n} h_{x_i}$ is 0 on A and 1 on F. □

3. Structure of $C_\kappa(X, \mathbb{R})$ at the zero function

As usual, if A and B are subsets of a group, $A + B := \{a + b : a \in A, b \in B\}$ and if $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets, $\mathcal{A} + \mathcal{B} := \{A + B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

As we have mentioned, a topology on an abelian group is a group topology if and only if translations are continuous and $\mathcal{N}(0)+\mathcal{N}(0)\geq\mathcal{N}(0)$. In this subsection, we investigate the latter property, that is,

$$\mathcal{N}_\kappa(\overline{0})+\mathcal{N}_\kappa(\overline{0})\geq\mathcal{N}_\kappa(\overline{0}), \tag{3.1}$$

for the space $C_\kappa(X,\mathbb{R})$. If (p_n) is a decreasing sequence of positive numbers that tends to zero, then

$$\left[\bigcap_{i=1}^{n}\mathcal{A}_i,(-\max_{i=1}^{n}p_i,\max_{i=1}^{n}p_i)\right]\subseteq\bigcap_{i=1}^{n}\left[\mathcal{A}_i,(-p_i,p_i)\right],$$

and thus $\mathcal{N}_\kappa(\overline{0})$ has a filter base of the form

$$\left\{[\mathcal{A},(-p_n,p_n)]:\mathcal{A}\in\kappa(X),n\in\mathbb{N}\right\}, \tag{3.2}$$

because a finite intersection of compact families is compact.

We call a topological space X *weakly consonant* if for every compact family $\mathcal{A}$ on X there is a compact family $\mathcal{B}$ such that $\mathcal{B}\vee\mathcal{B}=\{B\cap C:B\in\mathcal{B},C\in\mathcal{B}\}$ is a (not necessarily compact) subfamily of $\mathcal{A}$. Note that if X is consonant then every compact family includes a compact filter of the form $\mathcal{O}(K)$ for a compact set K. Taking $\mathcal{B}=\mathcal{O}(K)$ gives weak consonance, so that every consonant space is weakly consonant.

Theorem 3.1. *Let $(G,+)$ be an abelian topological group. If X is weakly consonant, then the addition is continuous at $\overline{0}$ in $C_\kappa(X,G)$. Moreover if X is completely regular, then the addition is continuous at $\overline{0}$ in $C_\kappa(X,\mathbb{R})$ if and only if X is weakly consonant.*

Proof. Assume that X is weakly consonant. Let $\mathcal{A}\in\kappa(X)$ and $V\in\mathcal{N}_G(0)$. By weak consonance, there exist a compact subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}\vee\mathcal{B}\subseteq\mathcal{A}$. If $W\in\mathcal{N}_G(0)$ such that $W+W\subseteq V$, then $[\mathcal{B},W]+[\mathcal{B},W]\subseteq[\mathcal{A},V]$, which proves (3.1).

Conversely, assume that X is not weakly consonant. Let $\mathcal{A}$ be a compact family witnessing the definition of non weak consonance. Note that $\mathcal{B}\vee\mathcal{C}\nsubseteq\mathcal{A}$ for every pair of compact families $\mathcal{B}$ and $\mathcal{C}$ for otherwise $\mathcal{D}=\mathcal{B}\cap\mathcal{C}$ would be a compact subfamily of $\mathcal{A}$ such that $\mathcal{D}\vee\mathcal{D}\subseteq\mathcal{A}$. Let $V=\left(-\frac{1}{2},\frac{1}{2}\right)$. We claim that for any pair $(\mathcal{B},\mathcal{C})$ of compact families and any pair (U,W) of $\mathbb{R}$-neighborhood of 0, $[\mathcal{B},U]+[\mathcal{C},W]\nsubseteq[\mathcal{A},V]$. Indeed, there exist $B\in\mathcal{B}$ and $C\in\mathcal{C}$ such that $B\cap C\notin\mathcal{A}$. Then $B^c\cup C^c\in\mathcal{A}^\#$. Moreover, $B^c\notin\mathcal{B}^\#$ so that by Lemma 2.5, there exist $B_1\in\mathcal{B}$ and $f\in C(X,\mathbb{R})$ such that $f(B_2)=\{0\}$ and $f(B^c)=\{1\}$. Similarly, $C^c\notin\mathcal{C}$ so that there exist $C_1\in\mathcal{C}$ and $g\in C(X,\mathbb{R})$ such that $g(C_1)=\{0\}$ and $g(C^c)=\{1\}$. Then $f+g\in[\mathcal{B},U]+[\mathcal{C},W]$ but $1\in(f+g)(A)$ for all $A\in\mathcal{A}$ so that $f+g\notin[\mathcal{A},V]$. □

Let $\mathcal{O}_S(X)$ be the lattice of open subsets of X endowed with the Scott topology, in which open sets are exactly the compact families of X. Dually, the set of closed subsets of X endowed with the *upper Kuratowski topology,* in which $\mathcal{F}$ is open if the family $\mathcal{F}_c=\{X\setminus F:F\in\mathcal{F}\}$ is compact, is denoted $\mathcal{C}_{uK}(X)$. The following observation was prompted by a conversation with Ahmed Bouziad (University of Rouen).

Proposition 3.2. *The following are equivalent:*

(1) *X is weakly consonant;*

(2) *The map* $M:\mathcal{O}_S(X)\times\mathcal{O}_S(X)\to\mathcal{O}_S(X)$ *defined by* $M(A,B)=A\cap B$ *is (jointly) continuous for the Scott topology;*
(3) *The map* $J:\mathcal{C}_{uK}(X)\times\mathcal{C}_{uK}(X)\to\mathcal{C}_{uK}(X)$ *defined by* $J(A,B)=A\cup B$ *is (jointly) continuous for the upper Kuratowski topology*

Proof. The equivalence between (2) and (3) is immediate. Assume X is weakly consonant and let U and V be two open subsets of X. Let $\mathcal{A}$ be a Scott open neighborhood of $M(U,V)$,i.e., a compact family containing $U\cap V$. In view of Lemma 2.4, $\mathcal{B}=\mathcal{A}\downarrow(U\cap V)$ is compact. Since X is weakly consonant, there is a compact family $\mathcal{C}$ such that

$$M(\mathcal{C},\mathcal{C})=\mathcal{C}\vee\mathcal{C}\subseteq\mathcal{B}\subseteq\mathcal{A}.$$

Note that $U\cap V\in\mathcal{C}$ so that $\mathcal{C}$ is a common Scott neighborhood of U and V and M is continuous.

Conversely, if M is continuous then $M^{-1}(\mathcal{A})$ is open in $\mathcal{O}_S(X)\times\mathcal{O}_S(X)$ for every compact family $\mathcal{A}$. In particular, $M^{-1}(\mathcal{A})$ has non empty interior and therefore contains $\mathcal{B}\times\mathcal{C}$ for some compact families $\mathcal{B}$ and $\mathcal{C}$. The compact family $\mathcal{D}=\mathcal{B}\vee\mathcal{C}$ then satisfies $\mathcal{D}\vee\mathcal{D}\subseteq\mathcal{A}$ so that X is weakly consonant. ☐

The equivalence (1) and (2) in the following theorem is due to Bouziad [1, Proposition 2.2].

Theorem 3.3. *Let X be a countable prime topological space with finite compact subsets and non-isolated point ∞. The following are equivalent:*

(1) *For every* $(A_n)_{n\in\omega}\subseteq\mathcal{N}(\infty)$ *such that for each finite subset F of X there is n such that $F\subset A_n$, there is an infinite subset J of ω such that* $\bigcap\limits_{n\in J}A_n\in\mathcal{N}(\infty)$*;*
(2) *X is consonant;*
(3) *X is weakly consonant.*

Proof. The equivalence between (1) and (2) is [1, Proposition 2.2]. (2) $\Longrightarrow$ (3) is clear. We show that if (1) is false, then X is not weakly consonant. Indeed, there exists $(A_n)_{n\in\omega}\subseteq\mathcal{N}(\infty)$ such that each finite subset of X is included in A_n for some n and $\bigcap\limits_{k\in J}A_k\notin\mathcal{N}(\infty)$ for every infinite subset J of ω. For each n, let $I_n=X\setminus A_n$. Note that $\infty\notin\operatorname{adh}I_n$ for each n and that for each $N\in\mathcal{N}(\infty)$, the set $J=\{n\in\omega:I_n\cap N=\varnothing\}$ is finite because $N\subseteq\bigcap\limits_{k\in J}A_k$. In particular, the family

$$\mathcal{D}=\{N\cup F:N\in\mathcal{N}(\infty),F\in[X]^{<\omega},\forall n\in\omega:(N\cup F)\cap I_n\neq\varnothing\}$$

is a non-empty compact family.

For each compact family $\mathcal{A}$, the set $K=\bigcap\mathcal{A}$ is compact, hence finite. Therefore there is n such that $K\subseteq A_n$ so that $I_n\cap K=\varnothing$. In particular, for each compact family $\mathcal{C}$ there is $n\in\omega$ and C_1,C_2 in $\mathcal{C}$ such that $C_1\cap C_2\cap I_n=\varnothing$ so that $C_1\cap C_2\notin\mathcal{D}$ and $\mathcal{C}\vee\mathcal{C}\nsubseteq\mathcal{D}$. Hence X is not weakly consonant. ☐

This theorem provides a number of examples of non weakly consonant spaces. For instance, Bouziad proved in [1, Proposition 2.3] that if $\mathcal{N}(\infty)=\mathcal{U}\wedge\{\infty\}$ for an ultrafilter $\mathcal{U}$ then the equivalent conditions of Theorem 3.3 hold if and only if $\mathcal{U}$ is a P-point. Therefore if X is not a P-point then $C_\kappa(X,\mathbb{R})$ is not a topological group. A. Bouziad pointed out to us that, in view of Proposition 3.2, Theorem 3.3

also shows that the assumption of separation is essential in the result of J. Lawson stating that a compact Hausdorff semitopological lattice is topological [11]. Indeed, if X is not weakly consonant, then $\mathcal{O}_S(X)$ is a T_0 compact semitopological lattice (i.e., J is separately continuous) which is not a topological lattice, because J is not jointly continuous.

However, we do not know if there are weakly consonant spaces that are not consonant.

4. Continuity of translations

We shall now prove that translations are always continuous for the fine Isbell topology. Moreover, we will see that the neighborhood filters at the zero function $\overline{0}$ for the Isbell and for the fine Isbell topologies coincide. Therefore translations are continuous for the Isbell topology if and only if it coincides with the fine Isbell topology.

As we mentioned, Francis Jordan introduced in [10] the *fine Isbell topology* on the set $C(X,Y)$ of continuous functions from X to Y. If N and M are two subsets of $X\times Y$, the set N is *buried in* M, in symbols $N\ll M$, if for every $x\in X$ there exists $V\in\mathcal{O}_X(x)$ and $W\in\mathcal{O}_Y(N(x))$ such that $V\times W\subseteq M$. If $f\in C(X,Y)$ and $A\subseteq X$, we denote by $f_{|A}$ the graph of the restriction of f to A. A subbase for the fine Isbell topologies is given by sets of the form:

$$\langle\mathcal{A},M\rangle:=\left\{f\in C(X,Y):\exists A\in\mathcal{A},f_{|A}\ll M\right\},$$

where $\mathcal{A}$ ranges over compact families of X and M ranges over open subsets of $X\times Y$. We denote by $C_{\overline{\kappa}}(X,Y)$ the set $C(X,Y)$ endowed with the fine Isbell topology. If $(G,+)$ is topological group, we denote by 0 its neutral element and by $\overline{0}$ the constant function zero of $C(X,G)$.

Theorem 4.1. *Let $(G,+)$ be a topological group. The fine Isbell topological space $C_{\overline{\kappa}}(X,G)$ is invariant by translations.*

The neighborhood filters at $\overline{0}$ for the fine Isbell and the Isbell topologies coincide.

Proof. 1. $\mathcal{N}_\kappa(\overline{0})\leq\mathcal{N}_{\overline{\kappa}}(\overline{0})$: is clear. Consider now $\langle\mathcal{A},M\rangle$ such that $\overline{0}\in\langle\mathcal{A},M\rangle$, $\mathcal{A}\in\kappa(X)$ and M is open in $X\times G$. There is $A\in\mathcal{A}$ such that for every $x\in A$, there is $V_x\in\mathcal{O}(x)$ and $W_x\in\mathcal{O}_G(0)$ such that $V_x\times W_x\subseteq M$. Since $\mathcal{A}$ is compact and $A=\bigcup_{x\in A}V_x$ there is a finite subset F of A such that $B=\bigcup_{x\in F}V_x\in\mathcal{A}$. But then $W=\bigcap_{x\in F}W_x\in\mathcal{O}_G(0)$ and $B\times W\subseteq M$ so that

$$\overline{0}\in[\mathcal{A}\downarrow B,W]\subseteq\langle\mathcal{A},M\rangle.$$

2. $\mathcal{N}_{\overline{\kappa}}(f)\geq f+\mathcal{N}_\kappa(\overline{0})$: Let $\mathcal{A}\in\kappa(X)$, $B\in\mathcal{O}_G(0)$. Consider $M:=\bigcup_{x\in X}\{x\}\times(f(x)+B)$. Then $f\in\langle\mathcal{A},M\rangle$ and $\langle\mathcal{A},M\rangle\subseteq f+[\mathcal{A},B]$. Indeed, if $h\in\langle\mathcal{A},M\rangle$ then there is $A\in\mathcal{A}$ such that for all $x\in A$, there is an open neighborhood V_x of x and an open neighborhood W_x of $h(x)$ such that $V_x\times W_x\subseteq M$. In particular, $\{x\}\times W_x\subseteq M$ so that $W_x\subseteq f(x)+B$ and $(h-f)(x)\in B$. Therefore $(h-f)(A)\subseteq B$.

3. $\mathcal{N}_{\overline{\kappa}}(f)\leq f+\mathcal{N}_\kappa(\overline{0})$: Let $\mathcal{A}\in\kappa(X)$ and let M be an open subset of $X\times G$ such that $f\in\langle\mathcal{A},M\rangle$, that is, there is $A\in\mathcal{A}$ such that for all $x\in A$, there

is an open neighborhood V_x of x and an open neighborhood $W_x = f(x) + B_x$ of $f(x)$, where $B_x \in \mathcal{O}_G(0)$ such that $V_x \times W_x \subseteq M$. By continuity of f we may assume that $f(V_x) \subseteq f(x) + B'_x \subseteq W_x$ for each x, where $B'_x \in \mathcal{O}_G(0)$ and $B'_x + B'_x \subseteq B_x$. Since $\mathcal{A}$ is compact and $A = \bigcup_{x \in A} V_x$ there is a finite subset F of A such that $A_1 = \bigcup_{x \in F} V_x \in \mathcal{A}$. Let $W \in \mathcal{O}_G(0)$ be such that $W = -W$ and $W \subseteq \bigcap_{x \in F} B'_x \in \mathcal{O}_G(0)$. Then $f + [\mathcal{A} \downarrow A_1, W] \subseteq \langle \mathcal{A}, M \rangle$. Indeed, if $h \in [\mathcal{A} \downarrow A_1, W]$ then there is $A_2 \in \mathcal{A}$, $A_2 \subseteq A_1$ such that $h(A_2) \subseteq W$. For each $x \in A_2$, there is $t_x \in F$ such that $x \in V_{t_x}$. Note that $V_{t_x} \times W_{t_x} \subseteq M$ and that $f(x) \in f(V_{t_x})$ and

$$f(V_{t_x}) + W \subseteq f(x) + B'_x + B'_x \subseteq W_{t_x}$$

so that $V_{t_x} \times (f(x) + W) \subseteq M$ which completes the proof because $f(x) + W \in \mathcal{O}\left((f+h)(x)\right)$ and $V_{t_x} \in \mathcal{O}(x)$. □

Corollary 4.2. *$C_\kappa(X,G)$ is invariant by translation if and only if $C_\kappa(X,G) = C_{\overline{\kappa}}(X,G)$.*

The result above also provides a more handy description of the fine Isbell topology on $C(X,G)$ when G is a topological group (for instance for $C_{\overline{\kappa}}(X,\mathbb{R})$) :

$$\mathcal{N}_{\overline{\kappa}}(f) = f + \{[\mathcal{A}, B] : \mathcal{A} \in \kappa(X), B \in \mathcal{O}_G(0)\}.$$

In [10, Example 1] Jordan shows that if X and Y are two completely regular consonant spaces such that the topological sum Z is not consonant (e.g., if X is the sequential fan and Y the space of rationals), then $C_\kappa(Z,\mathbb{R}) < C_{\overline{\kappa}}(Z,\mathbb{R})$. In view of Corollary 4.2, this proves that $C_\kappa(X,\mathbb{R})$ may fail to be invariant by translation.

5. When is $C_\kappa(X,\mathbb{R})$ a topological group for prime X?

As we have seen, translations in $C(X,\mathbb{R})$ are, in general, not continuous for the Isbell topology. They are however continuous if X is prime (that is, has at most one non-isolated point). More generally,

Proposition 5.1. *If X is atomic and G is an abelian consonant topological group, then the Isbell topology on $C(X,G)$ is translation invariant.*

Proof. If a family $\mathcal{A}$ is compact on X, then for each $x \in X$ the family $\mathcal{A}_x := \mathcal{O}_X(x) \cap \mathcal{A}$ is compact included in $\mathcal{A}$. Therefore it is enough to consider basic neighborhoods for the Isbell topology of the form $[\mathcal{A}_x, U]$ where U is an open subset of G.

Consider $f \mapsto g + f$ for $g \in C(X,G)$. Let $x \in X$ and $\mathcal{A}$ be a compact family included in $\mathcal{O}_X(x)$ and U is an open subset of G. Let $f_0 + g \in [\mathcal{A}, U]$.

The family $\mathcal{D} := \mathcal{O}_G((f_0 + g)(\mathcal{A}))$ is compact in a consonant space G, hence there is a compact set $K \subset U$ such that $\mathcal{O}_G(K) \subset \mathcal{D}$. Let $W = -W$ be a closed neighborhood of 0 in G such that $K + 3W \subset U$. Then there is $A \in \mathcal{A}$ such that $(f_0 + g)(A) \subset K + W$. Furthermore there exists $A_0 \in \mathcal{A}$ such that $A_0 \subset A$ and f_0 (A_0) is bounded and $(f_0 + g)(A_0) \subset K + W$.

Let V_0 be an element of $\mathcal{O}_X(x)$ included in A_0 such that f_0 $(V_0) \subset f_0$ $(x_\infty) + W$ and $g(V_0) \subset g(x_\infty) + W$.

Then there is a finite subset F of A_0 (disjoint from V_0) such that $A_1 := V_0 \cup F \in \mathcal{A}$. Then let $\mathcal{A}_1 := \mathcal{A} \downarrow A_1$. Of course, $f_0 + g \in [\mathcal{A}_1, K + W]$ and $\mathcal{A}_1 \in \kappa(X)$.

Then there is $n < \omega$ and finite sets $F_1, \ldots, F_n$ such that $f_0\,(F_k) -\, f_0\,(F_k) \subset W$ and $g(F_k) -\, g(F_k) \subset W$ for each $1 \leq k \leq n$, and moreover $F_1 \cup \ldots \cup F_n = F$. Finally, let $\mathcal{D}_0 := \mathcal{A}_1 \vee V_0$ and $\mathcal{D}_k := \mathcal{A}_1 \vee F_k$ for $1 \leq k \leq n$. Then $\mathcal{A}_1 = \bigcap_{k=0}^{n} \mathcal{D}_k$. On the other hand, there exist $x_k \in F_k$ for $1 \leq k \leq n$, such that

$$f_0 \in \bigcap_{k=0}^{n} [\mathcal{D}_k, f_0(x_k) + W],$$

where $x_0 := x_\infty$. If now $f \in \bigcap_{k=0}^{n} [\mathcal{D}_k, f_0(x_k) + 2W]$ then $f + g \in \bigcap_{k=0}^{n} [\mathcal{D}_k, f_0(x_k) + g(x_k) + 3W] \subset [\mathcal{A}_1, U] \subset [\mathcal{A}, U]$. □

Corollary 5.2. *If X be a prime topological space, then $C_\kappa(X, \mathbb{R})$ is translation invariant.*

Since, by Corollary 4.2, $C_\kappa(X, \mathbb{R})$ is translation invariant if and only if it coincides with $C_{\overline{\kappa}}(X, \mathbb{R})$, Corollary 5.2 implies a result [10, Theorem 18] of Jordan that the Isbell and the fine Isbell topologies coincide provided that the underlying topology is prime.

Corollary 5.2 combined with Theorem 3.1 and Corollary 4.2 leads to the following results.

Theorem 5.3. *If X is prime, then $C_\kappa(X, \mathbb{R})$ is a topological group if and only if X is weakly consonant.*

On recalling Theorem 3.3, we get

Theorem 5.4. *Let X be prime, countable and with finite compact subsets. Then $C_\kappa(X, \mathbb{R})$ is a topological group if and only if X is consonant.*

Of course, by Corollary 5.3, the Isbell space in Corollary 5.4 is translation invariant.

6. Complete regularity of the Isbell topology

In view of the previous section, the complete regularity of $C_\kappa(X, \mathbb{R})$ in general doesn't follow from the fact the the topology is a group topology. [13, Problem 61] asks whether $C_\kappa(X, Z)$ is completely regular when Z is. We provide a positive answer.

Theorem 6.1. *If Z is completely regular, then $C_\kappa(X, Z)$ is completely regular.*

Proof. Let $f \in [\mathcal{A}, O]$ where $\mathcal{A} \in \kappa(X)$ and O is Z-open. As $\mathcal{A}$ is compact and f continuous, $f(\mathcal{A})$ is a compact family in Z, hence $\mathcal{O}_Z(f(\mathcal{A}))$ is also compact, and since Z is completely regular, by Lemma 2.5, there is $A \in \mathcal{A}$ and $h \in C(Z, [0, 1])$ such that $h(f(A)) = \{0\}$ and $h(Z \backslash O) = \{1\}$. Define

$$F(g) := \inf_{A \in \mathcal{A}} \sup_{x \in A} h(g(x)) = \sup_{H \in \mathcal{A}^\#} \inf_{x \in H} h(g(x))$$

for each $g \in C(X, Z)$. Then $F(f) = 0$ and $F(g) = 1$ for each $g \notin [\mathcal{A}, O]$. Moreover, $F : C_\kappa(X, Z) \to [0, 1]$ is continuous. To see that $F^-\,([0, r))$ is open for each

$r \in [0,1]$, notice that $F(g) < r$ if and only if there is $A_r \in \mathcal{A}$ such that $g(A_r) \subset [0,r)$, that is, if and only if $g \in [\mathcal{A}, h^-([0,r))]$.

On the other hand, $g \in F^-((r,1])$ if and only if if $F(g) > r$, equivalently there is $r_1 > r$ such that for every $A \in \mathcal{A}$ there is $x \in A$ such that $h(g(x)) \geq r_1$. In other words, $K = g^-(h^-[r_1,1])$ is a closed subset of X meshing with $\mathcal{A}$ and $g \in [\mathcal{A} \vee K, h^-(r,1]]$. By Lemma 2.3, $\mathcal{A} \vee K$ is compact, thus $[\mathcal{A} \vee K, h^-(r,1]]$ is open. If now $b \in [\mathcal{A} \vee K, h^-(r,1]]$ then there is $A \in \mathcal{A}$ such that $h(b(A \cap K)) \subset (r,1]$, hence

$$r < \sup_{A \in \mathcal{A}} \inf_{x \in A \cap K} h(b(x)) \leq \inf_{A \in \mathcal{A}} \sup_{x \in A \cap K} h(b(x)) \leq \inf_{A \in \mathcal{A}} \sup_{x \in A} h(b(x)) = F(b).$$

□

REFERENCES

[1] A. Bouziad, *Filters, consonance and hereditary Baireness*, Topology Appl., **104**, 2000, pp. 27–38.

[2] S. Dolecki, *Properties transfer between topologies on function spaces, hyperspaces and underlying spaces*, Matematica Pannonica.

[3] S. Dolecki, G. H. Greco, and A. Lechicki, *When do the upper Kuratowski topology (homeomorphically, Scott topology) and the cocompact topology coincide?*, Trans. Amer. Math. Soc. **347** (1995), 2869–2884.

[4] S. Dolecki and F. Mynard, *Properties transfer between function spaces, hyperspaces and domains*, in preparation.

[5] M. Escardó, J. Lawson, and A. Simpson, *Comparing cartesian closed categories of (core) compactly generated spaces*, Topology Appl. **143** (2004), 105–145.

[6] D. N. Georgiou and S. D. Iliadis, *On the compact open and finest splitting topologies*, Topology Appl. **154** (2007), no. 10, 2110–2116.

[7] J.R. Isbell, *Function spaces and adjoints*, Math. Scandinavica **36** (1975), 317–339.

[8] ______, *Meet-continuous lattices*, Symposia Mathematica **16** (1975), 41–54.

[9] H. Jarchow, *Locally convex spaces*, Teubner, 1981.

[10] F. Jordan, *Coincidence of function space topologies*, submitted, 2008.

[11] J. Lawson, *Joint continuity in semitopological semigroups*, Illinois J. Math, **18** (1974), 275–285.

[12] B. Papadopoulos, *On the Isbell topology*, Mathematica Japonicae **35** (1990), no. 6, 1043–1046.

[13] Elliot Pearl (ed.), *Open problems in topology II*, Elsevier, 2007.

[14] Walter Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw Hill, 1991.