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Compactoid Relations and Product Theorems

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Abstract

Many classes of maps are characterized as (possibly multi-valued) maps preserving particular types of compactoidness. As a consequence, two results on product of compactoid filters are shown to be the common principle behind a surprisingly large number of theorems.

1 Introduction

A filter is compactoid if every finer ultrafilter is convergent. As a common generalization of compactness (in the case of a principal filter) and of convergence, it is not surprising that the notion turned out to be very useful in a variety of context (see for instance [6], [7], [3]). The purpose of this paper is two fold: firstly, building on the results of [6] and [7], I intend to show that a large number of classes of single and multi-valued maps classically used in topology, analysis and optimization are instances of *compactoid relation*, that is, relation preserving compactoidness. It is well known (see for instance [6]) that upper semi-continuous multivalued maps and compact valued upper semi-continuous maps are such instances. S. Dolecki showed [7] that closed, countably perfect, inversely Lindelöf and perfect maps are other examples. In this paper, it is shown that various types of quotient maps (hereditarily quotient, countably biquotient, biquotient) are also compactoid relations. This requires to work in the category of convergence spaces rather than in the category of topological spaces. Therefore, I recall basic facts on convergence spaces in the next section.

Secondly, I show how results on products of compactoid filters are the common principle behind a suprising number of product theorems. In particular, Section 4 presents how Kuratowski's theorem characterizing compact spaces in terms of closed projections, its variants characterizing countable compactness and Lindelöfness; Bourbaki's Theorem characterizing a perfect map in terms of its product with every identity map being closed and its variants; and the theorems characterizing various types of quotient maps in terms of their product with identity maps being hereditarily quotient are all instances of the same simple result on compactoid filters.

Finally, I present applications of the main result of [12] on product of compactoid filters to product theorems pertaining to stability under finite product of variants of compactness, of local topological properties like Fréchetness and strong Fréchetness, and of various types of maps, showing that a large collection of seemingly unrelated theorems are all instances of a single principle.

2 Terminology and basic facts

2.1 Convergence spaces

By a *convergence space* (X,ξ) I mean a set X endowed with a relation ξ between points of X and filters on X, denoted $x \in \lim_\xi \mathcal{F}$ or $\mathcal{F} \underset{\xi}{\rightarrow} x$, whenever x and $\mathcal{F}$ are in relation, and satisfying $\lim \mathcal{F} \subset \lim \mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$; $\{x\}^{\uparrow} \to x$ ([1]) for every $x \in X$ and $\lim(\mathcal{F} \wedge \mathcal{G}) = \lim \mathcal{F} \cap \lim \mathcal{G}$ for every filters $\mathcal{F}$ and $\mathcal{G}$ ([2]). A map $f : (X,\xi) \to (Y,\tau)$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$. If ξ and τ are two convergences on X, we say that ξ *is finer than* τ, in symbols $\xi \geq \tau$, if $Id_X : (X,\xi) \to (X,\tau)$ is continuous. The category **Conv** of convergence spaces and continuous maps is topological ([3]) and cartesian-closed ([4]).

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh,* in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A subset A of X is ξ*-closed* if $\lim_\xi \mathcal{F} \subset A$ whenever $\mathcal{F}\#A$. The family of ξ-closed sets defines a topology $T\xi$ on X called *topological modification* of ξ. The neighborhood filter of $x \in X$ for this topology is denoted $\mathcal{N}_\xi(x)$ and the closure operator for this topology is denoted cl_ξ. A convergence is a topology if $x \in \lim_\xi \mathcal{N}_\xi(x)$. By definition, the adherence of a filter (in a convergence space) is:

$$\mathrm{adh}_\xi \mathcal{F} = \bigcup_{\mathcal{G}\#\mathcal{F}} \lim_\xi \mathcal{G}. \tag{1}$$

In particular, the adherence of a subset A of X is the adherence of its *principal filter* $\{A\}^{\uparrow}$. The *vicinity filter* $\mathcal{V}_\xi(x)$ of x for ξ is the infimum of the filters

[1] If $\mathcal{A} \subset 2^X$, $\mathcal{A}^{\uparrow} = \{B \subset X : \exists A \in \mathcal{A},\ A \subset B\}$.

[2] Several different variants of these axioms have been used by various authors under the name *convergence space*.

[3] In other words, for every sink $(f_i : (X_i,\xi_i) \to X)_{i\in I}$, there exists a *final convergence* structure on X : the finest convergence on X making each f_i continuous. Equivalently, for every source $(f_i : X \to (Y_i,\tau_i))_{i\in I}$ there exists an *initial convergence*: the coarsest convergence on X making each f_i continuous.

[4] In other words, for any pair (X,ξ), (Y,τ) of convergence spaces, there exists the coarsest convergence $[\xi,\tau]$ -called *continuous convergence*- on the set $C(\xi,\tau)$ of continuous functions from X to Y making the evaluation map

$$ev : (X,\xi) \times (C(\xi,\tau),[\xi,\tau]) \to (Y,\tau)$$

(jointly) continuous.

converging to x for ξ. A convergence ξ is a *pretopology* if $x\in\lim_\xi \mathcal{V}_\xi(x)$. A convergence ξ is respectively a topology, a pretopology, a *paratopology,* a *pseudotopology* if $x\in\lim_\xi\mathcal{F}$ whenever $x\in\bigcap_{\mathbb{D}\ni\mathcal{D}\#\mathcal{F}}\operatorname{adh}_\xi\mathcal{D}$, where $\mathbb{D}$ is respectively, *the class* $\operatorname{cl}_\xi^\natural(\mathbb{F}_1)$ of principal filters of ξ-closed sets ([5]), *the class* $\mathbb{F}_1$ of principal filters, *the class* $\mathbb{F}_\omega$ of countably based filters, *the class* $\mathbb{F}$ of all filters. In other words, the map $\operatorname{Adh}_\mathbb{D}$ [5] defined by

$$\lim_{\operatorname{Adh}_\mathbb{D}\xi}\mathcal{F}=\bigcap_{\mathbb{D}\ni\mathcal{D}\#\mathcal{F}}\operatorname{adh}_\xi\mathcal{D} \tag{2}$$

is the (restriction to objects of the) reflector from **Conv** onto the full subcategory of respectively topological, pretopological, paratopological and pseudotopological spaces when $\mathbb{D}$ is respectively, the class $\operatorname{cl}_\xi^\natural(\mathbb{F}_1)$, $\mathbb{F}_1$, $\mathbb{F}_\omega$ and $\mathbb{F}$.

A class of filters $\mathbb{D}$ (under mild conditions on $\mathbb{D}$) defines a reflective subcategory of **Conv** (and the associated reflector) via (2). Dually, it also defines (under mild conditions on $\mathbb{D}$) the coreflective subcategory of **Conv** of $\mathbb{D}$-*based convergence spaces* [5], and the associated (restriction to objects of the) coreflector $\operatorname{Base}_\mathbb{D}$ is

$$\lim_{\operatorname{Base}_\mathbb{D}\xi}\mathcal{F}=\bigcup_{\mathbb{D}\ni\mathcal{D}\leq\mathcal{F}}\lim_\xi\mathcal{D}. \tag{3}$$

Let $\mathbb{D}$ and $\mathbb{J}$ be two classes of filters. A convergence space (X,ξ) is called $(\mathbb{J}/\mathbb{D})$-*accessible* if

$$\operatorname{adh}_\xi\mathcal{J}\subset\operatorname{adh}_{\operatorname{Base}_\mathbb{D}\xi}\mathcal{J},$$

for every $\mathcal{J}\in\mathbb{J}$. When $\mathbb{D}=\mathbb{F}_\omega$ and $\mathbb{J}$ is respectively the class $\mathbb{F}$, $\mathbb{F}_\omega$ and $\mathbb{F}_1$, then $(\mathbb{J}/\mathbb{D})$-accessible topological spaces are respectively bisequential, strongly Fréchet and Fréchet spaces. Analogously, if $\mathbb{D}$ is the class of filters generated by long sequences (of arbitrary length) and $\mathbb{J}=\mathbb{F}_1$ then $(\mathbb{J}/\mathbb{D})$-accessible topological spaces are radial spaces. We use the same names for these instances of $(\mathbb{J}/\mathbb{D})$-accessible *convergence spaces* (see [5] for details).

A filter $\mathcal{F}$ is called $\mathbb{J}$-*meshable to* $\mathbb{D}$-*refinable*, in symbol $\mathcal{F}\in(\mathbb{J}/\mathbb{D})_{\#\geq}$, if

$$\mathcal{J}\in\mathbb{J},\ \mathcal{J}\#\mathcal{F}\Longrightarrow\exists\mathcal{D}\in\mathbb{D},\ \mathcal{D}\#\mathcal{J}\text{ and }\mathcal{D}\geq\mathcal{F}.$$

It follows immediately from the definitions that a *topological* space is $(\mathbb{J}/\mathbb{D})$-accessible if and only if every neighborhood filter is $\mathbb{J}$-meshable to $\mathbb{D}$-refinable, and more generally that:

Theorem 1 *Let $\mathbb{D}$ and $\mathbb{J}$ be two classes of filters and let (X,ξ) be a convergence space.*

1. *(X,ξ) is $(\mathbb{J}/\mathbb{D})$-accessible if and only if $\xi\geq\operatorname{Adh}_\mathbb{J}\operatorname{Base}_\mathbb{D}\xi$;*
2. *If $\xi=\operatorname{Base}_{(\mathbb{J}/\mathbb{D})_{\#\geq}}\xi$, then ξ is $(\mathbb{J}/\mathbb{D})$-accessible. If moreover ξ is pretopological (in particular topological) then the converse is true.*

[5] More generally, if $o:2^X\longrightarrow 2^X$ and $\mathcal{F}\subset 2^X$ then $o^\natural\mathcal{F}$ denotes $\{o(F):F\in\mathcal{F}\}$ and if $\mathbb{D}$ is a class of filters (or of family of subsets) then $o^\natural(\mathbb{D})$ denotes $\{\mathcal{F}:\exists\mathcal{D}\in\mathbb{D},\ \mathcal{F}=o^\natural\mathcal{D}\}$.

The following gathers the most common cases of $(\mathbb{J}/\mathbb{D})$-accessible (topological) spaces and $(\mathbb{J}/\mathbb{D})_{\#\geq}$-filters when $\mathbb{D} = \mathbb{F}_\omega$. Denote by $\mathbb{F}_{\wedge\omega}$ the class of *countably deep* ([6]) filters. The names for $(\mathbb{J}/\mathbb{F}_\omega)_{\#\geq}$-filters come from the fact that a topological space is $(\mathbb{J}/\mathbb{F}_\omega)$-accessible if and only if every neighborhood filter is a $(\mathbb{J}/\mathbb{F}_\omega)_{\#\geq}$-filter.

class $\mathbb{J}$	$(\mathbb{J}/\mathbb{F}_\omega)$-accessible space	$(\mathbb{J}/\mathbb{F}_\omega)_{\#\geq}$-filter
$\mathbb{F}$	bisequential [19]	bisequential
$\mathbb{F}_\omega$	strongly Fréchet or countably bisequential [19]	strongly Fréchet
$(\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geq}$	productively Fréchet[15]	productively Fréchet
$\mathbb{F}_{\wedge\omega}$	weakly bisequential [1]	weakly bisequential
$\mathbb{F}_1$	Fréchet [19]	Fréchet

Table 1

2.2 Compactoidness

Let $\mathbb{D}$ be a class of filters on a convergence space (X, ξ) and let $\mathcal{A}$ be a family of subsets of X. A filter $\mathcal{F}$ is $\mathbb{D}$-*compactoid* at $\mathcal{A}$ (for ξ) if ([7])

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \operatorname{adh}_\xi \mathcal{D}\#\mathcal{A}. \tag{4}$$

Notice that a subset K of a convergence space X (in particular of a topological space) is respectively compact, countably compact, Lindelöf if $\{K\}^\uparrow$ is $\mathbb{D}$-compactoid at $\{K\}$ if $\mathbb{D}$ is respectively, the class $\mathbb{F}$ of all, $\mathbb{F}_\omega$ of countably based, $\mathbb{F}_{\wedge\omega}$ of countably deep filters. On the other hand

Theorem 2 *Let $\mathbb{D}$ be a class of filters.*
$x \in \lim_{\operatorname{Adh}_{\mathbb{D}} \xi} \mathcal{F}$ if and only if $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\{x\}$ for ξ.

In particular, if ξ is a topology, then $\mathcal{F} \to x$ if and only if $\mathcal{F}$ is compactoid at $\{x\}$ if and only if $\mathcal{F}$ is $\mathbb{F}_1$-compactoid at $\{x\}$.

For a topological space X, a subset K is compact if and only if every open cover of K has a finite subcover of K, if and only if every filter on K has adherent points in K. In contrast, for general convergence spaces, the definition of compactness in terms of cover (cover-compactoidness) and in terms of filters (compactoidness) are different. If (X, ξ) is a convergence space, a family $\mathcal{S} \subset 2^X$ is a *cover of* $K \subset X$ if every filter converging to a point of K contains an element of $\mathcal{S}$. Hence a subset K of a convergence space is called *cover-(countably) compact* if every (countable) cover of K has a finite subcover. It is easy to see that a cover-compact convergence is compact, but in general not conversely.

Notice that in this definition, we can assume the original cover $\mathcal{S}$ to be stable under finite union, in which case we call $\mathcal{S}$ an *ideal cover*. The family $\mathcal{S}_c$ of complements of elements of an ideal cover $\mathcal{S}$ is a filter-base on X with

[6] A filter $\mathcal{F}$ is *countably deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

[7] Notice that (4) makes sense not only for a filter but for a general family $\mathcal{F}$ of subsets of X. Such general compactoid families play an important role for instance in [8].

empty adherence. Hence K is cover-(countably) compact if every (countable) ideal cover of K has an elemet that is a cover of K, or equivalently, if every (countable) filter-base with no adherence point in K has an element with no adherence point in K. In other words K is cover-(countably) compact if every (countably based) filter whose every member has adherent points in K, has adherent points in K.

More generally, we will need the following characterization of cover-compactoidness in terms of filters [7]. Let $\mathbb{D}$ and $\mathbb{J}$ be two classes of filters. A filter $\mathcal{F}$ is $(\mathbb{D}/\mathbb{J})$-*compactoid at* $\mathcal{B}$ if

$$\mathcal{D}\in\mathbb{D},\ \forall\mathcal{J}\in\mathbb{J},\ \mathcal{J}\leq\mathcal{D},\ \operatorname{adh}\mathcal{J}\#\mathcal{F}\Longrightarrow\operatorname{adh}\mathcal{D}\#\mathcal{B}.$$

It is clear than if $\mathcal{F}$ is $(\mathbb{D}/\mathbb{F}_1)$-compactoid (at $\mathcal{B}$), then it is $\mathbb{D}$-compactoid (at $\mathcal{B}$). More precisely, we have the following relationship between $(\mathbb{D}/\mathbb{F}_1)$-compactoidness and $\mathbb{D}$-compactoidness (which could be deduced from the results of [7, section 8])

Proposition 3 *Let $\mathbb{D}$ be a class of filters on a convergence space (X,ξ). A filter $\mathcal{F}$ is $(\mathbb{D}/\mathbb{F}_1)$-compactoid at $\mathcal{B}$ if and only if $\mathcal{V}_\xi(\mathcal{F})=\bigcup_{F\in\mathcal{F}}\bigcap_{x\in F}\mathcal{V}_\xi(x)$ is $\mathbb{D}$-compactoid at $\mathcal{B}$.*

Proof. By definition, $\mathcal{F}$ is $(\mathbb{D}/\mathbb{F}_1)$-compactoid at $\mathcal{B}$ if and only if

$$\mathcal{D}\in\mathbb{D}:\left(\operatorname{adh}_\xi^\natural\mathcal{D}\right)\#\mathcal{F}\Longrightarrow\operatorname{adh}\mathcal{D}\#\mathcal{B}.$$

It is easy to verify that $\left(\operatorname{adh}_\xi^\natural\mathcal{D}\right)\#\mathcal{F}$ if and only if $\mathcal{D}\#\mathcal{V}_\xi(\mathcal{F})$, which concludes the proof. ■

Calling a convergence ξ *pretopologically diagonal*, or *P-diagonal*, if $\lim_\xi\mathcal{F}\subset\lim_\xi\mathcal{V}_\xi(\mathcal{F})$ for every filter $\mathcal{F}$, we obtain the following result, which is a particular case of a combination of Propositions 8.1 and 8.3 and of Theorem 8.2 in [7], even though the assumption that $\operatorname{adh}_\xi^\natural\mathbb{D}\subset\mathbb{D}$ seems to be erroneously missing in [7].

Corollary 4 *If ξ is P-diagonal (in particular if ξ is a topology) and if $\operatorname{adh}_\xi^\natural\mathbb{D}\subset\mathbb{D}$, then $(\mathbb{D}/\mathbb{F}_1)$-compactoidness amounts to $\mathbb{D}$-compactoidness for ξ.*

Proof. Assume that $\mathcal{F}$ is $\mathbb{D}$-compactoid (at $\mathcal{B}$). To show that it is $(\mathbb{D}/\mathbb{F}_1)$-compactoid (at $\mathcal{B}$), we only need to show that $\mathcal{V}_\xi(\mathcal{F})$ is $\mathbb{D}$-compactoid (at $\mathcal{B}$). But $\mathcal{D}\#\mathcal{V}_\xi(\mathcal{F})$ if and only if $\left(\operatorname{adh}_\xi^\natural\mathcal{D}\right)\#\mathcal{F}$. Therefore, $\operatorname{adh}_\xi(\operatorname{adh}_\xi^\natural\mathcal{D})\#\mathcal{B}$ because $\operatorname{adh}_\xi^\natural\mathcal{D}\in\mathbb{D}$. Now, $x\in\operatorname{adh}_\xi(\operatorname{adh}_\xi^\natural\mathcal{D})$ if there exists a filter $\mathcal{G}\#\operatorname{adh}_\xi^\natural\mathcal{D}$ with $x\in\lim_\xi\mathcal{G}$. Note that $\mathcal{V}_\xi(\mathcal{G})\#\mathcal{D}$ and that, by P-diagonality, $x\in\lim_\xi\mathcal{V}_\xi(\mathcal{G})$. Hence $\operatorname{adh}_\xi(\operatorname{adh}_\xi^\natural\mathcal{D})\subset\operatorname{adh}_\xi\mathcal{D}$ and $\operatorname{adh}_\xi\mathcal{D}\#\mathcal{B}$. ■

In some sense, the converse is true:

Proposition 5 *If $\xi=\operatorname{Adh}_\mathbb{D}\xi$ and $\mathbb{D}$-compactoidness implies $(\mathbb{D}/\mathbb{F}_1)$-compactoidness in ξ, then ξ is P-diagonal.*

Proof. If $x \in \lim_\xi \mathcal{F}$ then $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\{x\}$, hence $(\mathbb{D}/\mathbb{F}_1)$-compactoid at $\{x\}$. Since $\xi = \mathrm{Adh}_{\mathbb{D}}\, \xi$, we only need to show that $x \in \mathrm{adh}_\xi \mathcal{D}$ whenever $\mathcal{D}$ is a $\mathbb{D}$-filter meshing with $\mathcal{V}_\xi(\mathcal{F})$. For any such $\mathcal{D}$, we have $\mathrm{adh}_\xi^\natural \mathcal{D} \# \mathcal{F}$ so that $x \in \mathrm{adh}_\xi \mathcal{D}$ because $\mathcal{F}$ is $(\mathbb{D}/\mathbb{F}_1)$-compactoid at $\{x\}$. ■

2.3 Contour filters

If $\mathcal{F}$ is a filter on X and $\mathcal{G} : X \to \mathbb{F}X$ then the *contour of $\mathcal{G}$ along $\mathcal{F}$* is the filter on X defined by

$$\int_{\mathcal{F}} \mathcal{G} = \bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} \mathcal{G}(x).$$

This type of filters have been used in many situations, among others by Frolik under the name of sum of filters for a ZFC proof of the non-homogeneity of the remainder of $\beta\mathbb{N}$ [14], by C. H. Cook and H. R. Fisher [4] under the name of compression operator of $\mathcal{F}$ relative to $\mathcal{G}$, by H. J. Kowalsky [16] under the name of diagonal filter, and after them by many other authors to characterize topologicity and regularity of convergence spaces. To generalize this construction, I need to reproduce basic facts on cascades and multifilters. Detailed information on this topic can be found in [9].

If $(W, \sqsubseteq)$ is an ordered set, then we write

$$W(w) = \{x \in W : w \sqsubseteq x\}.$$

An ordered set $(W, \sqsubseteq)$ is *well-capped* if its every non empty subset has a maximal point ([8]). Each well-capped set admits the (upper) *rank* to the effect that $r(w) = 0$ if $w \in \max W$, and for $r(w) > 0$,

$$r(w) = r_W(w) = \sup_{v \sqsupset w} (r(v) + 1).$$

A well-capped tree with least element is called a *cascade*; the least element of a cascade V is denoted by $\varnothing = \varnothing_V$ and is called the *estuary* of V. It follows from the definition that each element of a cascade is of finite length. The *rank* of a cascade is by definition the rank of its estuary. A cascade is a *filter cascade* if its every (non maximal) element is a filter on the set of its immediate successors.

A map $\Phi : V \setminus \{\varnothing_V\} \to X$, where V is a cascade, is called a *multifilter on* X. We talk about a multifilter $\Phi : V \to X$ under the understanding that Φ is not defined at $\varnothing_V$.

A couple (V, Φ_0) where V is a cascade and $\Phi_0 : \max V \to A$ is a called a *perifilter on A*. In the sequel we will consider V implicitly talking about a perifilter Φ_0. If $\Phi|_{\max V} = \Phi_0$, then we say that the multifilter Φ is an *extension* of the perifilter Φ_0. The rank of a multifilter (perifilter) is, by definition, the rank of the corresponding cascade. If $\mathbb{D}$ is a class of filters, we call $\mathbb{D}$-*multifilter* a multifilter with a cascade of $\mathbb{D}$-filters as domain.

[8] In other words, a well-capped ordered set is a well-founded ordered set for the inverse order.

The contour of a multifilter $\Phi : V \to X$ depends entirely on the underlying cascade V and on the restriction of Φ to $\max V$, hence on the corresponding perifilter $(V, \Phi|_{\max V})$. Therefore we shall not distinguish between the contours of multifilters and of the corresponding perifilters. The *contour* of $\Phi : W \to X$ is defined by induction to the effect that $\int \Phi = \Phi_\natural(\varnothing_W)$ if $r(\Phi) = 1$, and ([9])

$$\int \Phi = \int_{\varnothing_W} \left(\int \Phi|_{W(.)} \right)$$

otherwise. With each class $\mathbb{D}$ of filters we associate the class $\int \mathbb{D}$ of all $\mathbb{D}$-contour filters, i. e., the contours of $\mathbb{D}$-multifilter.

Lemma 6 *Let $\mathbb{D}$ and $\mathbb{J}$ be two classes of filters. If $\mathbb{D}$ is a $\mathbb{J}$-composable class of filters, then $\int \mathbb{D}$ is also $\mathbb{J}$-composable.*

Proof. We proceed by induction on the rank of a $\mathbb{D}$-multifilter. The case of rank 1 is simply $\mathbb{J}$-composability of $\mathbb{D}$. Assume that for each $\mathbb{D}$-multifilter Φ on X of rank β smaller than α and each $\mathbb{J}$-filter $\mathcal{J}$ on $X \times Y$, the filter $\mathcal{J}(\int \Phi)$ is the contour of some $\mathbb{D}$-multifilter on Y. Consider now a $\mathbb{D}$-multifilter (Φ, V) on X of rank α and a $\mathbb{J}$-filter $\mathcal{J}$ on $X \times Y$. Then

$$\int \Phi = \int_{\varnothing_V} \left(\int \Phi|_{V(.)} \right) = \bigvee_{F \in \varnothing_V} \bigwedge_{v \in F} \int \Phi|_{V(v)},$$

and

$$\mathcal{J}\left(\int \Phi \right) = \bigvee_{F \in \varnothing_V} \bigwedge_{v \in F} \mathcal{J}\left(\int \Phi|_{V(v)} \right).$$

As each $\Phi|_{V(v)}$ is a multifilter of rank smaller than α, each $\mathcal{J}\left(\int \Phi|_{V(v)}\right)$ is a $\left(\int \mathbb{D}\right)$-filter. Moreover $\varnothing_V$ is a $\mathbb{D}$-filter, so that $\mathcal{J}\left(\int \Phi\right)$ is a contour of $\left(\int \mathbb{D}\right)$-filters along a $\mathbb{D}$-filter, hence a $\left(\int \mathbb{D}\right)$-filter. ■

3 Compactoid relations

A *relation* $R : (X, \xi) \rightrightarrows (Y, \tau)$ *is* $\mathbb{D}$-*compactoid* if for every subset A of X and every filter $\mathcal{F}$ that is $\mathbb{D}$-compactoid at A, the filter $R\mathcal{F}$ is $\mathbb{D}$-compactoid at RA.

If $\mathbb{D}$ and $\mathbb{J}$ are two classes of filters, we say that $\mathbb{J}$ is $\mathbb{D}$-*composable* if for every X and Y, the (possibly degenerate) filter $\mathcal{HF}=\{HF : H \in \mathcal{H}, F \in \mathcal{F}\}^{\uparrow}$ ([10]) belongs to $\mathbb{J}(Y)$ whenever $\mathcal{F} \in \mathbb{J}(X)$ and $\mathcal{H} \in \mathbb{D}(X \times Y)$, with the convention that every class of filters contains the degenerate filter. If a class $\mathbb{D}$ is $\mathbb{D}$-composable, we simply say that $\mathbb{D}$ is *composable*. Notice that

$$\mathcal{H}\# (\mathcal{F} \times \mathcal{G}) \iff \mathcal{HF}\#\mathcal{G} \iff \mathcal{H}^{-}\mathcal{G}\#\mathcal{F}, \tag{5}$$

where $\mathcal{H}^{-}\mathcal{G}=\{H^{-}G = \{x \in X : (x, y) \in H \text{ and } y \in G\} : H \in \mathcal{H}, G \in \mathcal{G}\}^{\uparrow}$.

[9] $\Phi(v)$ is the image by Φ of v treated as a point of V, while $\Phi_\natural(v)$ is the filter generated by $\{\Phi(F) : F \in v\}$.

[10] $HF = \{y \in Y : (x, y) \in H \text{ and } x \in F\}$.

Proposition 7 *If $\mathbb{D}$ is $\mathbb{F}_1$-composable, then $R:(X,\xi)\rightrightarrows(Y,\tau)$ is $\mathbb{D}$-compactoid if and only if $R\mathcal{F}$ is $\mathbb{D}$-compactoid at Rx whenever $x\in\lim_\xi\mathcal{F}$.*

Proof. Only the "if" part needs a proof, so assume that $R\mathcal{F}$ is $\mathbb{D}$-compactoid at Rx whenever $x\in\lim_\xi\mathcal{F}$, and consider a filter $\mathcal{G}$ on X which is $\mathbb{D}$-compactoid at A. Let $\mathcal{D}\#R\mathcal{G}$ be a $\mathbb{D}$-filter on Y. Then $R^-\mathcal{D}\#\mathcal{G}$ so that there exists $x\in A\cap\operatorname{adh}_\xi R^-\mathcal{D}$. Therefore, there exists $\mathcal{U}\#R^-\mathcal{D}$ such that $x\in\lim_\xi\mathcal{U}$. By assumption, $R\mathcal{U}$ is $\mathbb{D}$-compactoid at $Rx\subset RA$. Since $\mathcal{D}\#R\mathcal{U}$, the filter $\mathcal{D}$ has adherent points in Rx hence in RA. ■

Corollary 8 *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters and let $f:(X,\xi)\to(Y,\tau)$ with $\tau=\operatorname{Adh}_\mathbb{D}\tau$. The following are equivalent:*

1. *f is continuous;*
2. *f is a compactoid relation;*
3. *f is a $\mathbb{D}$-compactoid relation.*

Proof. $(1\Longrightarrow2)$.If $x\in\lim_\xi\mathcal{F}$, then $f(x)\in\lim_\tau f(\mathcal{F})$ so that $f(\mathcal{F})$ is compactoid at $f(x)$ and f is a compactoid relation by Proposition 7. $(2\Longrightarrow3)$ is obvious and $(3\Longrightarrow1)$ follows from Proposition 2. ■

In particular, $\mathbb{F}_1$-compactoid, equivalently compactoid, maps between pretopological spaces (in particular between topological spaces) are exactly the continuous ones.

Notice that when $\mathbb{D}$ contains the class of principal filters, then a $\mathbb{D}$-compactoid relation R is $\mathbb{F}_1$-compactoid and Rx is $\mathbb{D}$-compactoid for each x in the domain of R, because $\{x\}^\uparrow$ is $\mathbb{D}$-compactoid at $\{x\}$. When the cover and filter versions of compactoidness coincide (in particular, in a topological space), the converse is true:

Proposition 9 *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters. If $R:(X,\xi)\rightrightarrows(Y,\tau)$ is an $\mathbb{F}_1$-compactoid relation and if Rx is $(\mathbb{D}/\mathbb{F}_1)$-compact in τ for every $x\in X$, then R is $\mathbb{D}$-compactoid.*

Proof. Using Proposition 7, we need to show that $R\mathcal{F}$ is $\mathbb{D}$-compactoid at Rx whenever $x\in\lim_\xi\mathcal{F}$. Consider a $\mathbb{D}$-filter $\mathcal{D}\#R\mathcal{F}$. Then, $\operatorname{adh}_\tau D\#Rx$ for every $D\in\mathcal{D}$ so that $\operatorname{adh}_\tau\mathcal{D}\#Rx$, because Rx is $\frac{\mathbb{D}}{\mathbb{F}_1}$-compactoid. ■

In view of Corollary 4, we obtain:

Corollary 10 *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters such that $\operatorname{adh}^\natural_\tau\mathbb{D}(\tau)\subset\mathbb{D}(\tau)$ and let τ be a P-diagonal convergence (for instance a topology). Then $R:(X,\xi)\rightrightarrows(Y,\tau)$ is $\mathbb{D}$-compactoid if and only if it is $\mathbb{F}_1$-compactoid and Rx is $\mathbb{D}$-compact in τ for every $x\in X$.*

An immediate corollary of [9, Theorem 8.1] is that for a topology, $\mathbb{D}$-compactness amounts to $\left(\int\mathbb{D}\right)$-compactness, provided that $\mathbb{D}$ is a composable class of filters. However, the proof of [9, Theorem 8.1] only uses $\mathbb{F}_1$-composability of $\mathbb{D}$. Consequently,

Corollary 11 *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters and let τ be a topology such that* $\operatorname{adh}^{\natural}_{\tau} \mathbb{D}(\tau) \subset \mathbb{D}(\tau)$. *Let $R : (X, \xi) \rightrightarrows (Y, \tau)$ be a relation. The following are equivalent:*

1. *R is $\mathbb{D}$-compactoid;*

2. *R is $\mathbb{F}_1$-compactoid and Rx is $\mathbb{D}$-compact in τ for every $x \in X$;*

3. *R is $\mathbb{F}_1$-compactoid and Rx is $\left(\int \mathbb{D}\right)$-compact in τ for every $x \in X$;*

4. *R is $\left(\int \mathbb{D}\right)$-compactoid.*

Proof. ($1 \iff 2$) and ($3 \iff 4$) follow from Corollary 10 and ($1 \iff 4$) follows from [9, Theorem 8.1]. ■

The observation that perfect, countably perfect and closed maps can be characterized as $\mathbb{D}$-compactoid relations is due to S. Dolecki [7, section 10]. Recall that a surjection $f : X \to Y$ between two topological spaces is *closed* if the image of a closed set is closed and *perfect* (resp. *countably perfect*, resp. *inversely Lindelöf*) if it is closed with compact (resp. countably compact, resp. Lindelöf) fibers. Once the concept of closed maps is extended to convergence spaces, all the other notions extend as well in the obvious way. As observed in [7, section 10], preservation of closed sets by a map $f : (X, \xi) \to (Y, \tau)$ is equivalent to $\mathbb{F}_1$-compactoidness of the inverse map f^- when (X, ξ) is topological, but not if ξ is a general convergence. More precisely, calling a map $f : (X, \xi) \to (Y, \tau)$ *adherent* [7] if

$$y \in \operatorname{adh}_{\tau} f(H) \Longrightarrow \operatorname{adh}_{\xi} H \cap f^- y \neq \emptyset,$$

we have:

Lemma 12 1. *A map $f : (X, \xi) \to (Y, \tau)$ is adherent if and only if $f^- : (Y, \tau) \rightrightarrows (X, \xi)$ is an $\mathbb{F}_1$-compactoid relation;*

2. *If $f : (X, \xi) \to (Y, \tau)$ is adherent, then it is closed;*

3. *If $f : (X, \xi) \to (Y, \tau)$ is closed and if adherence of sets are closed in ξ (in particular if ξ is a topology), then f is adherent.*

Proof. (1) follows from the definition and is observed in [7, section 10].

(2) If $f(H)$ is not τ-closed, then there exists $y \in \operatorname{adh}_{\tau} f(H) \setminus f(H)$. Since f is adherent, there exists $x \in \operatorname{adh}_{\xi} H \cap f^- y$. But $x \notin H$ because $f(x) = y \notin f(H)$.Therefore H is not ξ-closed.

(3) is proved in [7, Proposition 10.2] even if this proposition is stated with a stronger assumption. ■

Hence, a map $f : (X, \xi) \to (Y, \tau)$ with a domain in which adherence of subsets are closed (in particular, a map with a topological domain) is adherent if and only if it is closed if and only if $f^- : (Y, \tau) \rightrightarrows (X, \xi)$ is an $\mathbb{F}_1$-compactoid relation. If the domain and range of a map are topological spaces, it is well known that closedness of the map amounts to upper semicontinuity of the inverse relation. It was observed (for instance in [6]) that a (multivalued) map is upper semicontinuous (u.s.c.) if and only if it is an $\mathbb{F}_1$-compactoid relation.

A surjection $f: X \to Y$ is $\mathbb{D}$-*perfect* if it is adherent with $\mathbb{D}$-compact fibers. In view of Corollary 11, compact valued u.s.c. maps between topological spaces, known as *usco maps*, are compactoid relations. Another direct consequence of Lemma 1 and of Corollary 11 is:

Theorem 13 *Let $f: (X,\xi) \to (Y,\tau)$ be a surjection, let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters, and let ξ be a topology such that $\mathrm{adh}^{\natural}_{\xi}\,\mathbb{D} \subset \mathbb{D}$. The following are equivalent:*

1. *f is $\mathbb{D}$-perfect;*
2. *$f^-: Y \rightrightarrows X$ is $\mathbb{D}$-compactoid;*
3. *$f^-: Y \rightrightarrows X$ is $\left(\int \mathbb{D}\right)$-compactoid;*
4. *f is $\left(\int \mathbb{D}\right)$-perfect.*

The equivalence between the first two points was first observed in [7, Proposition 10.2] but erroneously stated for general convergences as domain and range. Indeed, if $f: (X,\xi) \to (Y,\tau)$ is a surjective map between two convergence spaces and if $f^-: (Y,\tau) \rightrightarrows (X,\xi)$ is $\mathbb{D}$-compactoid then f is adherent and has $\mathbb{D}$-compact fibers; if on the other hand f is adherent and has $(\mathbb{D}/\mathbb{F}_1)$-compact fibers then $f^-: (Y,\tau) \rightrightarrows (X,\xi)$ is $\mathbb{D}$-compactoid. Hence, the two concepts are equivalent only when $\mathbb{D}$-compact sets are $(\mathbb{D}/\mathbb{F}_1)$-compact in ξ, for instance if $\mathrm{adh}^{\natural}_{\xi}(\mathbb{D}) \subset \mathbb{D}$ and ξ is a P-diagonal convergence (in particular if ξ is a topology).

S. Dolecki offered in [5] a unified treatment of various classes of quotient maps and preservation theorems under such maps in the general context of convergences. He extended the usual notions of quotient maps to convergence spaces in the following way: a surjection $f: (X,\xi) \to (Y,\tau)$ is $\mathbb{D}$-*quotient* if

$$y \in \mathrm{adh}_{\tau}\,\mathcal{H} \Longrightarrow f^-(y) \cap \mathrm{adh}_{\xi}\, f^-\mathcal{H} \neq \emptyset, \tag{6}$$

for every $\mathcal{H} \in \mathbb{D}(Y)$. When $\mathbb{D}$ is the class of all (resp. countably based, principal, principal of closed sets) filters, then continuous $\mathbb{D}$-quotient maps betweeen topological spaces are exactly biquotient (resp. countably biquotient, hereditarily quotient, quotient) maps. Now, I present a new characterization of $\mathbb{D}$-quotient maps as $\mathbb{D}$-compactoid relations, in this general context of convergence spaces. As mentioned before, the category of convergence spaces and continuous maps is topological, hence if $f: (X,\xi) \to Y$, there exists the finest convergence —called *final convergence* and denoted $f\xi$ — on Y making f continuous. Analogously, if $f: X \to (Y,\tau)$, there exists the coarsest convergence — called *initial convergence* and denoted $f^-\tau$ — on X making f continuous. If τ is topological, so is $f^-\tau$. In contrast, $f\xi$ can be non topological even when ξ is topological.

Theorem 14 *Let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters. Let $f: (X,\xi) \to (Y,\tau)$ be a surjection. The following are equivalent:*

1. $f:(X,\xi)\to(Y,\tau)$ *is* $\mathbb{D}$*-quotient;*
2. $\tau\geq \mathrm{Adh}_{\mathbb{D}}\, f\xi$;
3. $f:(X,f^{-}\tau)\to(Y,f\xi)$ *is a* $\mathbb{D}$*-compactoid relation.*

Proof. The equivalence $(1\iff 2)$ is [5, Theorem 1.2].

$(1\iff 3)$. Assume f is $\mathbb{D}$-quotient and let $x\in\lim_{f^{-}\tau}\mathcal{F}$. Then $f(x)\in\lim_{\tau} f(\mathcal{F})$, so that $f(x)\in\mathrm{adh}_{\tau}\mathcal{D}$ whenever $\mathcal{D}\in\mathbb{D}(Y)$ and $\mathcal{D}\#f(\mathcal{F})$. By (6), $f^{-}(f(x))\cap\mathrm{adh}_{\xi} f^{-}\mathcal{D}\neq\emptyset$ so that $f(x)\in f\left(\mathrm{adh}_{\xi} f^{-}\mathcal{D}\right)$. In view of [7, Lemma 2.1], $f(x)\in\mathrm{adh}_{f\xi}\mathcal{D}$.

Conversely, assume that $f:(X,f^{-}\tau)\to(Y,f\xi)$ is $\mathbb{D}$-compactoid and let $y\in\mathrm{adh}_{\tau}\mathcal{D}$. There exists $\mathcal{G}\#\mathcal{D}$ such that $y\in\lim_{\tau}\mathcal{G}$. By definition of $f^{-}\tau$, the filter $f^{-}\mathcal{G}$ is converging to every point of $f^{-}y$ for $f^{-}\tau$. In view of Proposition 7, $ff^{-}\mathcal{G}$ is $\mathbb{D}$-compactoid at $\{y\}$ for $f\xi$. Since f is surjective, $ff^{-}\mathcal{G}\approx\mathcal{G}$ and $\mathcal{G}\#\mathcal{D}$ so that $y\in\mathrm{adh}_{f\xi}\mathcal{D}=f\left(\mathrm{adh}_{\xi} f^{-}\mathcal{D}\right)$, by [7, Lemma 2.1]. Therefore, $f^{-}(y)\cap\mathrm{adh}_{\xi} f^{-}\mathcal{D}\neq\emptyset$. ■

Notice that even if the map has topological range and domain, you need to extend the notions to convergence spaces to obtain such a characterization.

4 Characterization of $\mathbb{D}$-compactoid filters in terms of products

The aim of this section is to show that the classical Kuratowski Theorem characterizing compactness of X in terms of closedness of the projections $p_Y: X\times Y\to Y$ for every topological space Y and its variants for other type of compactness (e.g., countable compactness, Lindelöfness), as well as product characterizations of various types of maps are all instances of a simple result on $\mathbb{D}$-compactoid filters. Namely,

Theorem 15 *Let* (X,ξ) *be a convergence space,* $A\subset X$*, and let* $\mathcal{F}$ *be a filter on* X*. Let* $\mathbb{D}$ *be a composable class of filters. The following are equivalent:*

1. $\mathcal{F}$ *is* $\mathbb{D}$*-compactoid at* A;
2. *For every convergence space* Y *and every compactoid* $\mathbb{D}$*-filter* $\mathcal{G}$ *at* $B\subset Y$*, the filter* $\mathcal{F}\times\mathcal{G}$ *is* $\mathbb{D}$*-compactoid at* $A\times B$;
3. *For every* $\mathbb{D}$*-based atomic* ([11]) *topological space* Y*, every* $y\in Y$ *and every* $\mathcal{G}$ *such that* $y\in\lim_Y\mathcal{G}$*, the filter* $\mathcal{F}\times\mathcal{G}$ *is* $\mathbb{F}_1$*-compactoid at* $A\times\{y\}$.

Proof. $(1\Longrightarrow 2)$.

Let $\mathcal{D}\in\mathbb{D}(X\times Y)$ such that $\mathcal{D}\#\left(\mathcal{F}\times\mathcal{G}\right)$. The filter $\mathcal{D}^{-}(\mathcal{G})\in\mathbb{D}(X)$ because $\mathcal{G}\in\mathbb{D}(Y)$ and $\mathbb{D}$ is composable. Moreover $\mathcal{D}^{-}(\mathcal{G})\#\mathcal{F}$ so that $\mathrm{adh}_X\mathcal{D}^{-}(\mathcal{G})\cap A\neq\emptyset$. Consequently, there exists a filter $\mathcal{W}$ with $x\in\lim_X\mathcal{W}\cap A$ such that

[11] A topological space with at at most one non-isolated point is called *atomic.* Such spaces have been also called *point-spaces* and *prime topological spaces.*

$\mathcal{W}\#\mathcal{D}^{-}(\mathcal{G})$. Therefore $\mathcal{D}(\mathcal{W})\#\mathcal{G}$ and $\mathrm{adh}_Y\, \mathcal{D}(\mathcal{W}) \cap B \neq \emptyset$ by compactoidness of $\mathcal{G}$. In other words, there is a filter $\mathcal{U}$ with $y \in \lim_Y \mathcal{U} \cap B$ such that $\mathcal{U}\#\mathcal{D}(\mathcal{W})$. Consequently, $(x, y) \in \mathrm{adh}_{X\times Y}\, \mathcal{D}$ because $(\mathcal{W} \times \mathcal{U})\,\#\mathcal{D}$.

$(2 \Longrightarrow 3)$ is obvious.

$(3 \Longrightarrow 1)$.

Assume that $\mathcal{F}$ is not $\mathbb{D}$-compactoid at A. Then, there exists a $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{F}$ such that $\mathrm{adh}_\xi\, \mathcal{D} \cap A = \emptyset$. Chose any point x_0 in X and let Y be a copy of X endowed with the atomic topology τ defined by $\mathcal{N}_\tau(x_0) = \mathcal{D} \wedge (x_0)$. Then $\mathcal{F} \times \mathcal{N}_\tau(x_0)$ is *not* $\mathbb{F}_1$-compactoid at $A \times \{x_0\}$: $\{(x, x) : x \neq x_0\}\#\mathcal{F} \times \mathcal{N}_\tau(x_0)$ because $\mathcal{D}\#\mathcal{F}$, but $\mathrm{adh}_{\xi\times\tau}\{(x, x) : x \neq x_0\} \cap (A \times \{x_0\}) = \emptyset$. Indeed, a filter on $\{(x, x) : x \neq x_0\}$ is of the form $\mathcal{G} \times \mathcal{G}$. If $x_0 \in \lim_\tau \mathcal{G}$ then $\mathcal{G} \geq \mathcal{D}$ and $\lim_\xi \mathcal{G} \cap A = \emptyset$. ■

Notice that this is a generalization of [12, Lemma 1]. Applied for $\mathcal{F} = \{X\} = \{A\}$, Theorem 15 rephrases as:

Corollary 16 *Let $\mathbb{D}$ be a composable class of filters and let X be a convergence space. The following are equivalent:*

1. *X is $\mathbb{D}$-compact;*
2. *for every $\mathbb{D}$-based convergence space Y, the projection $p_Y : X \times Y \to Y$ is $\mathbb{D}$-perfect;*
3. *for every $\mathbb{D}$-based atomic topological space Y, the projection $p_Y : X \times Y \to Y$ is adherent.*

Proof. $(1 \Longrightarrow 2)$ because the fact that $\{X\} \times \mathcal{G}$ is $\mathbb{D}$-compactoid at $X \times \{y\}$ for every $\mathbb{D}$-filter $\mathcal{G}$ such that $y \in \lim_Y \mathcal{G}$ amounts to $\mathbb{D}$-compactoidness of $p_Y^- : Y \rightrightarrows X \times Y$, which implies $\mathbb{D}$-perfectness of $p_Y : X \times Y \to Y$.

$(2 \Longrightarrow 3)$ by definition, and $(3 \Longrightarrow 1)$ because if $p_Y : X \times Y \to Y$ is adherent for every $\mathbb{D}$-based atomic topological space Y, then for every topological space Y, every $y \in Y$ and every $\mathbb{D}$-filter $\mathcal{G}$ that converges to y, the filter $\{X\} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $X \times \{y\}$. In view of Theorem 15, $\{X\}$ is compactoid, that is, X is compact. ■

In particular, for a topological space X, $\mathbb{D}$-compactness amounts to $\left(\int \mathbb{D}\right)$-compactness so that, in view of Lemma 6, we get:

Corollary 17 *Let $\mathbb{D}$ be a composable class of filters. Let X be a* topological *space. The following are equivalent:*

1. *X is $\mathbb{D}$-compact;*
2. *X is $\left(\int \mathbb{D}\right)$-compact;*
3. *for every $\left(\int \mathbb{D}\right)$-based convergence space Y, the projection $p_Y : X \times Y \to Y$ is $\left(\int \mathbb{D}\right)$-perfect;*
4. *for every $\mathbb{D}$-based atomic topological space Y, the projection $p_Y : X \times Y \to Y$ is closed.*

A similar result [27, Theorem 1] has been obtained by J. Vaughan for topological spaces. He used nets instead of filters. To a class Ω of directed sets, we can associate a class $\mathbb{D}_\Omega$ of filters by

$$\mathcal{F} \in \mathbb{D}_\Omega \Longleftrightarrow \exists D \in \not\leq, \exists f : D \to \mathcal{F} : d \leq d' \Longrightarrow f(d') \subset f(d).$$

The Ω-*net spaces* of [27] are topological spaces (X, ξ) such that $\xi = T\,\mathrm{Base}_{\mathbb{D}_\Omega}\,\xi$; Ω-*Fréchet spaces* are topological spaces (X, ξ) such that $\xi = P\,\mathrm{Base}_{\mathbb{D}_\Omega}\,\xi$ and Ω-*neighborhood spaces* are topological spaces (X, ξ) such that $\xi = \mathrm{Base}_{\mathbb{D}_\Omega}\,\xi$. It is easy to show that a subspace of a topological space (X, ξ) satisfying $\xi = T\,\mathrm{Base}_{\mathbb{D}}\,\xi$ is $\left(\int \mathbb{D}\right)$-based (see for instance [10]). Therefore [27, Theorem 1] follows from Corollary 17. In particular, when $\mathbb{D}$ ranges over $\mathbb{F}$, $\mathbb{F}_\omega$ and $\mathbb{F}_{\wedge\omega}$, Corollary 17 leads to

Corollary 18 1. *(Kuratowski [11, Theorem 3.1.16]) The following are equivalent for a topological space X:*

   (a) X is compact;

   (b) $p_Y : X \times Y \to Y$ is perfect for every topological space Y;

   (c) $p_Y : X \times Y \to Y$ is closed for every topological space Y.

2. *(Noble [25, Corollary 2.4]) The following are equivalent for a topological space X:*

   (a) X is countably compact;

   (b) $p_Y : X \times Y \to Y$ is countably perfect for every subsequential ([12]) topological space Y;

   (c) $p_Y : X \times Y \to Y$ is closed for every first-countable topological space Y.

3. *(Noble [25, Corollary 2.3]) The following are equivalent for a topological space X:*

   (a) X is Lindelöf;

   (b) $p_Y : X \times Y \to Y$ is inversely Lindelöf for every topological P-space ([13]) Y;

   (c) $p_Y : X \times Y \to Y$ is closed for every topological P-space Y.

On the other hand, applied to the case where A is a singleton, Theorem 15 rephrases in convergence theoretic terms as follows:

Theorem 19 *Let $\mathbb{D}$ be a composable class of filters and let ξ and θ be two convergences on X. The following are equivalent:*

[12] A topological space is *sequential* if every sequentially closed subset is closed and *subsequential* if it is homeomorphic to subspace of a sequential space.

[13] A topological space is *a P-space* if every countable intersection of open subsets is open; equivalently if it is $\mathbb{F}_{\wedge\omega}$-based.

1. $\theta \geq \mathrm{Adh}_{\mathbb{D}}\,\xi$;

2. $\theta \times \mathrm{Base}_{\mathbb{D}}\,\tau \geq \mathrm{Adh}_{\mathbb{D}}\,(\xi \times \tau)$ *for every convergence* τ;

3. $\theta \times \tau \geq P\,(\xi \times \tau)$ *for every* $\mathbb{D}$*-based atomic topology* τ.

Proof. ($1 \Longrightarrow 2$). Let $x \in \lim_\theta \mathcal{F}$ and let $y \in \lim_\tau \mathcal{G}$ with $\mathcal{G} \in \mathbb{D}$. By assumption, $x \in \lim_{\mathrm{Adh}_{\mathbb{D}}\,\xi} \mathcal{F}$; in other words, $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\{x\}$ and $\mathcal{G} \in \mathbb{D}$ is compactoid at $\{y\}$. By Theorem 15, $\mathcal{F} \times \mathcal{G}$ is $\mathbb{D}$ -compactoid at $\{(x,y)\}$, that is, $(x,y) \in \lim_{\mathrm{Adh}_{\mathbb{D}}(\xi\times\tau)}(\mathcal{F} \times \mathcal{G})$.

($2 \Longrightarrow 3$) is obvious and ($3 \Longrightarrow 1$) follows from ($3 \Longrightarrow 1$) in Theorem 15. Indeed, if $x \in \lim_\theta \mathcal{F}$,then for every atomic topological space (Y,τ) and every $\mathbb{D}$-filter $\mathcal{G}$ that converges to y in Y, $(x,y) \in \lim_{P(\xi\times\tau)}(\mathcal{F} \times \mathcal{G})$, that is, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $\{(x,y)\}$.so that $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\{x\}$. Hence $x \in \lim_{\mathrm{Adh}_{\mathbb{D}}\,\xi} \mathcal{F}$. ■

The result above was essentially proved in [21, Theorem 7.1] but was not stated explicitely in [21]. In view of Theorem 1, we obtain:

Corollary 20 *Let* $\mathbb{J} \subset \mathbb{D}$ *be two classes of filters containing principal filters. Assume that a product of two* $\mathbb{D}$*-filters is a* $\mathbb{D}$*-filter. The following are equivalent:*

1. ξ *is* $(\mathbb{J}/\mathbb{D})$*-accessible;*

2. $\xi \times \tau$ *is* $(\mathbb{J}/\mathbb{D})$*-accessible for every* $\mathbb{J}$*-based convergence space* (Y,τ);

3. $\xi \times \tau$ *is* $(\mathbb{F}_1/\mathbb{D})$*-accessible for every atomic* $\mathbb{J}$*-based topological space* (Y,τ).

Proof. Notice that $\mathrm{Base}_{\mathbb{J}} \geq \mathrm{Base}_{\mathbb{D}}$ because $\mathbb{J} \subset \mathbb{D}$.

($1 \Longrightarrow 2$) . If $\xi \geq \mathrm{Adh}_{\mathbb{J}}\,\mathrm{Base}_{\mathbb{D}}\,\xi$ and $\tau = \mathrm{Base}_{\mathbb{J}}\,\tau$, then

$$\xi \times \tau \geq \mathrm{Adh}_{\mathbb{J}}\,\mathrm{Base}_{\mathbb{D}}\,\xi \times \tau \geq \mathrm{Adh}_{\mathbb{J}}\,\mathrm{Base}_{\mathbb{D}}\,\xi \times \mathrm{Base}_{\mathbb{D}}\,\tau = \mathrm{Adh}_{\mathbb{J}}\,\mathrm{Base}_{\mathbb{D}}\,(\xi \times \tau)\,,$$

so that $\xi \times \tau$ is $(\mathbb{J}/\mathbb{D})$-accessible.

($2 \Longrightarrow 3$) is clear because $\mathbb{F}_1 \subset \mathbb{J}$.

($3 \Longrightarrow 1$) The convergence ξ satisfies

$$\xi \times \tau \geq P\,\mathrm{Base}_{\mathbb{D}}\,(\xi \times \tau) = P\,(\mathrm{Base}_{\mathbb{D}}\,\xi \times \tau)$$

for every $\mathbb{J}$-based atomic topology τ. By Theorem 19, $\xi \geq \mathrm{Adh}_{\mathbb{J}}\,\mathrm{Base}_{\mathbb{D}}\,\xi$. ■

In particular, when $\mathbb{J} = \mathbb{D} = \mathbb{F}_\omega$, it shows the following generalization to convergence spaces of [19, Propostions 4.D.4 and 4.D.5]:

Corollary 21 *A convergence space is strongly Fréchet if and only if its product with every first-countable convergence (equivalently, every atomic first-countable topological space) is strongly Fréchet (equivalently Fréchet).*

An $\mathbb{F}_1$-based convergence is called *finitely generated.* Finitely generated topologies are often called *Alexandroff topologies.* When $\mathbb{J} = \mathbb{F}_1$ and $\mathbb{D} = \mathbb{F}_\omega$, Corollary 20 particularizes to

Corollary 22 *[21] A topological (or convergence) space is Fréchet if and only if its product with every finitely generated convergence space (equivalently, Alexandroff topology) is Fréchet .*

On the other hand, applying Theorem 15 for the image of a general filter under a relation, I obtain the following corollary for (possibly multi-valued) maps.

Corollary 23 *Let $\mathbb{D}$ be a composable class of filters and let $R : X \rightrightarrows Z$. The following are equivalent:*

1. *R is a $\mathbb{D}$-compactoid relation;*
2. *$R \times Id_Y : X \times Y \rightrightarrows Z \times Y$ is a $\mathbb{D}$-compactoid relation for every $\mathbb{D}$-based convergence space Y;*
3. *$R \times Id_Y : X \times Y \rightrightarrows Z \times Y$ is an $\mathbb{F}_1$-compactoid relation for every atomic $\mathbb{D}$-based topological space Y.*

In view of Theorem 13, the last result leads to:

Corollary 24 *Let $\mathbb{D}$ be a composable class of filters, let X be a topological space, and let $f : X \to Y$ be a surjective map. The following are equivalent:*

1. *f is $\mathbb{D}$-perfect;*
2. *$f \times Id_Y$ is $\mathbb{D}$-perfect for every $\mathbb{D}$-based convergence space Y;*
3. *$f \times Id_Y$ is $\left(\int \mathbb{D}\right)$-perfect for every $\left(\int \mathbb{D}\right)$-based topological space Y;*
4. *$f \times Id_Y$ is closed for every $\mathbb{D}$-based topological space Y.*

In particular,

Corollary 25 *[26, Corollary 3.5 (iii), (iv), (v) and (vi)] Let X be a topological space, and let $f : X \to Y$ be a surjective map.*

1. *The following are equivalent:*
   (a) *f is perfect;*
   (b) *$f \times Id_W$ is perfect for every topological space W;*
   (c) *$f \times Id_W$ is closed for every topological space W.*
2. *The following are equivalent:*
   (a) *f is countably perfect;*
   (b) *$f \times Id_W$ is countably perfect for every subsequential topological space W;*
   (c) *$f \times Id_W$ is closed for every first-countable topological space W.*

3. *The following are equivalent:*
   (a) f *is inversely Lindelöf;*
   (b) $f \times Id_W$ *is inversely Lindelöf for every topological* P*-space* W;
   (c) $f \times Id_W$ *is closed for every topological* P*-space* W.

Similarily, in view of of Theorem 14, we obtain:

Corollary 26 *Let* $\mathbb{D}$ *be a composable class of filters and let* $f : X \to Y$ *be a surjective map. The following are equivalent:*

1. f *is* $\mathbb{D}$*-quotient;*
2. $f \times Id_Y$ *is* $\mathbb{D}$*-quotient for every* $\mathbb{D}$*-based convergence space* Y;
3. $f \times Id_Y$ *is hereditarily quotient for every* $\mathbb{D}$*-based topological space* Y.

In particular, calling $\mathbb{F}_{\wedge\omega}$-quotient maps *weakly biquotient* [17], we obtain:

Corollary 27 1. *(Michael [18]) The following are equivalent for a surjective map* $f : X \to Y$ *:*
   (a) f *is biquotient;*
   (b) $f \times Id_W$ *is biquotient for every convergence space* W;
   (c) $f \times Id_W$ *is hereditarily quotient for every topological space* W.

2. *(Michael [19, Propositions 4.3 and 4.4]) The following are equivalent for a surjective map* $f : X \to Y$ *:*
   (a) f *is countably biquotient;*
   (b) $f \times Id_W$ *is countably biquotient for every first-countable convergence space* W;
   (c) $f \times Id_W$ *is hereditarily quotient for every first-countable topological space* W.

3. *The following are equivalent for a surjective map* $f : X \to Y$ *:*
   (a) f *is weakly biquotient;*
   (b) $f \times Id_W$ *is weakly biquotient for every* $\mathbb{F}_{\wedge\omega}$*-based convergence space* W;
   (c) $f \times Id_W$ *is hereditarily quotient for every topological* P*-space* W.

Notice that even if X and Y are topological, the final convergence $f\xi$ may not be. Therefore, $\mathbb{D}$-quotientness and $\left(\int \mathbb{D}\right)$-quotientness are not equivalent.

5 Products of $\mathbb{D}$-compactoid filters

In Section 4, $\mathbb{D}$-compactoid filters are characterized as those filters whose product with every compactoid $\mathbb{D}$-filters is $\mathbb{D}$-compactoid. In this section, we consider the following related question: What are the filters whose product with every $\mathbb{D}$-compactoid filter (of a given class $\mathbb{J}$) is $\mathbb{D}$-compactoid?

This question was answered in [12], where the following notion was introduced :

A filter $\mathcal{F}$ is $\mathbb{M}$-*compactoidly* $\mathbb{J}$ *to* $\mathbb{D}$ *meshable* at A, or $\mathcal{F}$ is an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_{\#}$-filter at A, if

$$\mathcal{J} \in \mathbb{J}, \mathcal{J}\#\mathcal{F} \Longrightarrow \exists \mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{J} \text{ and } \mathcal{D} \text{ is } \mathbb{M}\text{-compactoid at } A.$$

Before proceeding with applications, I show that the notion of an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_{\#}$-filter, while seemingly complicated and unatural, is instrumental not only in answering the question above but also in characterizing a large number of classical concepts. The following examples hopefully will convince the reader that the discomfort of three different parameters is compensated by the number of notions handled simultaneously.

The notion of total countable compactness was first introduced by Z. Frolík [13] for a study of product of countably compact and pseudocompact spaces and rediscovered under various names by several authors (see [28, p. 212]). A topological space X is *totally countably compact* if every countably based filter has a finer (equivalently, meshes a) compactoid countably based filter. The name comes from *total* nets of Pettis. Obviously, a topological space is totally countably compact if and only if $\{X\}$ is compactoidly $\mathbb{F}_\omega$ to $\mathbb{F}_\omega$ meshable. In [28], J. Vaughan studied more generally under which condition a product of $\mathbb{D}$-compact spaces is $\mathbb{D}$-compact, under mild conditions on the class of filters $\mathbb{D}$. He used in particular the concept of a totally $\mathbb{D}$-compact space X, which amounts to $\{X\}$ being a compactoidly $(\mathbb{D}/\mathbb{D})_{\#}$-filter.

On the other hand, as observed in [12] in the case of topological spaces, Theorem 1 can be completed by the following immediate rephrasing of the notion of $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_{\#}$-filters relative to a singleton in convergence theoretic terms.

Proposition 28 *Let $\mathbb{D}$, $\mathbb{J}$ and $\mathbb{M}$ be three classes of filters, and let ξ and θ be two convergences on X. The following are equivalent:*

1. $\theta \geq \mathrm{Adh}_{\mathbb{J}} \mathrm{Base}_{\mathbb{D}} \mathrm{Adh}_{\mathbb{M}} \xi$;
2. $\mathcal{F}$ *is an* $\mathbb{M}$-*compactoidly* $(\mathbb{J}/\mathbb{D})_{\#}$ *filter at* $\{x\}$ *in* ξ *whenever* $x \in \lim_\theta \mathcal{F}$.

   In particular, $\xi = \mathrm{Adh}_{\mathbb{M}} \xi$ *is* $(\mathbb{J}/\mathbb{D})$-*accessible if and only if* $\mathcal{F}$ *is an* $\mathbb{M}$-*compactoidly* $(\mathbb{J}/\mathbb{D})_{\#}$-*filter at* $\{x\}$ *whenever* $x \in \lim \mathcal{F}$.

In view of Table 1, this applies to a variety of classical local topological properties.

A *relation* $R:(X,\xi)\rightrightarrows(Y,\tau)$ *is* $\mathbb{M}$*-compactoidly* $(\mathbb{J}/\mathbb{D})$*-meshable* if

$$\mathcal{F}\underset{\xi}{\rightarrow}x\Longrightarrow R(\mathcal{F})\text{ is }\mathbb{M}\text{-compactoidly }(\mathbb{J}/\mathbb{D})\text{-meshable at }Rx\text{ in }\tau.$$

Theorem 29 *Let* $\mathbb{M}\subset\mathbb{J}$*, let* $\tau=\mathrm{Adh}_{\mathbb{M}}\tau$ *and let* $f:(X,\xi)\to(Y,\tau)$ *be a continuous surjection. The map* f *is* $\mathbb{M}$*-quotient with* $(\mathbb{J}/\mathbb{D})$*-accessible range if and only if* $f:(X,f^{-}\tau)\to(Y,f\xi)$ *is an* $\mathbb{M}$*-compactoidly* $(\mathbb{J}/\mathbb{D})$*-meshable relation.*

Proof. Assume that f is $\mathbb{M}$-quotient with $(\mathbb{J}/\mathbb{D})$-accessible range and let $x\in\lim_{f^{-}\tau}\mathcal{F}$. Then $y=f(x)\in\lim_{\tau}f(\mathcal{F})$. Let $\mathcal{J}$ be a $\mathbb{J}$-filter such that $\mathcal{J}\#f(\mathcal{F})$. Since $y\in\mathrm{adh}_{\tau}\mathcal{J}$ and τ is $(\mathbb{J}/\mathbb{D})$-accessible, there exists a $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{J}$ such that $y\in\lim_{\tau}\mathcal{D}$. To show that $f(\mathcal{F})$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at y in $f\xi$, it remains to show that $\mathcal{D}$ is $\mathbb{M}$-compactoid at $\{y\}$ for $f\xi$, that is, that $y\in\lim_{\mathrm{Adh}_{\mathbb{M}}f\xi}\mathcal{D}$, which follows from the $\mathbb{M}$-quotientness of f.

Conversely, assume that $f:(X,f^{-}\tau)\to(Y,f\xi)$ is an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable relation, and let $y\in\lim_{\tau}\mathcal{G}$. Then $f^{-}(\mathcal{G})$ converges to any point $x\in f^{-}y$ for $f^{-}\tau$. Therefore, $f(f^{-}\mathcal{G})$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at $\{y\}$ in $f\xi$. Because f is a surjection, $f(f^{-}\mathcal{G})=\mathcal{G}$. Let $\mathcal{M}\in\mathbb{M}\subset\mathbb{J}$ such that $\mathcal{M}\#\mathcal{G}$. There exists a $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{M}$ which is $\mathbb{M}$-compactoid at $\{y\}$ in $f\xi$. Hence, $y\in\mathrm{adh}_{f\xi}\mathcal{M}$, so that $y\in\lim_{\mathrm{Adh}_{\mathbb{M}}f\xi}\mathcal{G}$. Therefore, f is $\mathbb{M}$-quotient. Moreover, if $y\in\mathrm{adh}_{\tau}\mathcal{J}$ for a $\mathbb{J}$-filter $\mathcal{J}$, then there exists $\mathcal{G}\#\mathcal{J}$ such that $y\in\lim_{\tau}\mathcal{G}$. By the previous argument, $\mathcal{G}$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at $\{y\}$ in $f\xi$. In particular, there exists a $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{J}$ which is $\mathbb{M}$-compactoid at $\{y\}$ in $f\xi$. In other words, $y\in\lim_{\mathrm{Adh}_{\mathbb{M}}f\xi}\mathcal{D}$. Since $f:(X,\xi)\to(Y,\tau)$ is continuous, $\tau\leq f\xi$ so that $\mathrm{Adh}_{\mathbb{M}}\tau=\tau\leq\mathrm{Adh}_{\mathbb{M}}f\xi$. Hence $y\in\lim_{\tau}\mathcal{D}$ and τ is $(\mathbb{J}/\mathbb{D})$-accessible. ∎

$\mathbb{M}$	$\mathbb{J}$	$\mathbb{D}$	map f as in Theorem 29
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_1$	hereditarily quotient with finitely generated range
$\mathbb{F}_1$	$\mathbb{F}_1$	$\mathbb{F}_\omega$	hereditarily quotient with Fréchet range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	hereditarily quotient with strongly Fréchet range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_\omega$	hereditarily quotient with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}$	hereditarily quotient
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_1$	countably biquotient with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably biquotient with strongly Fréchet range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}_\omega$	countably biquotient with bisequential range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}$	countably biquotient
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_1$	biquotient with finitely generated range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_\omega$	biquotient with bisequential range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	biquotient

Theorem 30 *Let* $\mathbb{M}\subset\mathbb{J}$ *and* $\mathbb{D}$ *be three classes of filters, where* $\mathbb{J}$ *and* $\mathbb{D}$ *are* $\mathbb{F}_1$*-composable. Let* $\tau=\mathrm{Adh}_{\mathbb{M}}\tau$ *and let* ξ *be a* P*-diagonal convergence such that* $\mathrm{adh}_{\xi}^{\natural}(\mathbb{M})\subset\mathbb{M}$ *. Let* $f:(X,\xi)\to(Y,\tau)$ *be a continuous surjection. The map* f *is* $\mathbb{M}$*-perfect with* $(\mathbb{J}/\mathbb{D})$*-accessible range if and only if* $f^{-}:(Y,\tau)\rightrightarrows(X,\xi)$ *is an* $\mathbb{M}$*-compactoidly* $(\mathbb{J}/\mathbb{D})$*-meshable relation.*

Proof. Assume that f is $\mathbb{M}$-perfect with $(\mathbb{J}/\mathbb{D})$-accessible range and let $y \in \lim_\tau \mathcal{G}$. Consider a $\mathbb{J}$-filter $\mathcal{J}\#f^-\mathcal{G}$. By $\mathbb{F}_1$-composability, $f(\mathcal{J})$ is a $\mathbb{J}$-filter. Moreover $f(\mathcal{J})\#\mathcal{G}$ so that $y \in \mathrm{adh}_\tau f(\mathcal{J})$. Since τ is $(\mathbb{J}/\mathbb{D})$-accessible, there exists a $\mathbb{D}$-filter $\mathcal{D}\#f(\mathcal{J})$ such that $y \in \lim_\tau \mathcal{D}$. In view of Corollary 11, $f^- : (Y,\tau) \rightrightarrows (X,\xi)$ is $\mathbb{M}$-compactoid because f is $\mathbb{M}$-perfect. Therefore, $f^-\mathcal{D}$ is $\mathbb{M}$-compactoid at f^-y in ξ. Moreover, $f^-\mathcal{D} \in \mathbb{D}$ because $\mathbb{D}$ is $\mathbb{F}_1$-composable and $f^-\mathcal{D}\#\mathcal{J}$. Hence, $f^-\mathcal{G}$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at f^-y in ξ.

Conversely, assume that $f^- : (Y,\tau) \rightrightarrows (X,\xi)$ is an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable relation. It is in particular an $\mathbb{M}$-compactoid relation because $\mathbb{M} \subset \mathbb{J}$. In view of Corollary 11, f is $\mathbb{M}$-perfect. Now assume that $y \,\mathrm{adh}_\tau\, \mathcal{J}$ where $\mathcal{J} \in \mathbb{J}$. There exists $\mathcal{G}\#\mathcal{J}$ such that $y \in \lim_\tau \mathcal{G}$. Therefore, $f^-\mathcal{G}$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at f^-y in ξ. The filter $f^-\mathcal{J}$ meshes with $f^-\mathcal{G}$ because f is surjective, and is a $\mathbb{J}$-filter because $\mathbb{J}$ is $\mathbb{F}_1$-composable. Hence, there exists a $\mathbb{D}$-filter $\mathcal{D}\#f^-\mathcal{J}$ which is $\mathbb{M}$-compactoid at f^-y in ξ. By continuity of f, the filter $f(\mathcal{D})$ is $\mathbb{M}$-compactoid at $\{y\}$ in τ (Corollary 8). In view of Proposition 2, $y \in \lim_{\mathrm{Adh}_\mathbb{M} \tau} f(\mathcal{D})$. Moreover, the filter $f(\mathcal{D})$ meshes with $\mathcal{J}$ and is a $\mathbb{D}$-filter, by $\mathbb{F}_1$-composability of $\mathbb{D}$. Since, $\tau = \mathrm{Adh}_\mathbb{M} \tau$, $y \in \lim_\tau f(\mathcal{D})$ and τ is $(\mathbb{J}/\mathbb{D})$-accessible. ■

$\mathbb{M}$	$\mathbb{J}$	$\mathbb{D}$	map f as in Theorem 30
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_1$	closed with finitely generated range
$\mathbb{F}_1$	$\mathbb{F}_1$	$\mathbb{F}_\omega$	closed with Fréchet range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	closed with strongly Fréchet range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}_\omega$	closed with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	$\mathbb{F}$	closed
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_1$	countably perfect with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably perfect with strongly Fréchet range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}_\omega$	countably perfect with bisequential range
$\mathbb{F}_\omega$	$\mathbb{F}$	$\mathbb{F}$	countably perfect
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_1$	perfect with finitely generated range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}_\omega$	perfect with bisequential range
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	perfect

The purpose of the remaining part of the paper is now to present applications of the main result of the companion paper [12], a version (extended to convergence spaces) of which is given below. The proof for convergence spaces rather than topological spaces is unchanged, except for (4) $\Longrightarrow$ (1).

Theorem 31 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters and let $\mathbb{J}$ and $\mathbb{K}$ be two $\mathbb{D}$-composable classes of filters. Let $\mathcal{F} \in \mathbb{K}(X)$ and $A \subset X$. The following are equivalent:*

1. *$\mathcal{F}$ is a $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_\#$- filter at $A \subset X$;*
2. *for every Y, every $B \subset Y$ and every $(\mathbb{K}/\mathbb{J})_{\#\geq}$-filter $\mathcal{G}$ which is a compactoidly $(\mathbb{D}/\mathbb{M})_\#$-filter at B, the filter $\mathcal{F}\times\mathcal{G}$ is an $\mathbb{M}$-compactoidly $(\mathbb{D}/\mathbb{D}\times\mathbb{M})_\#$-filter at $A \times B$;*

3. *for every $(\mathbb{D}/\mathbb{M})$-accessible space Y, every $B \subset Y$ and every $\mathbb{J}$-filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at B, the filter $\mathcal{F} \times \mathcal{G}$ is $(\mathbb{D} \cap \mathbb{M})$-compactoid at $A \times B$;*

4. *for every $\mathbb{M}$-based convergence space Y and $y \in Y$, and for every $\mathbb{J}$-filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at $\{y\}$, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $A \times \{y\}$;*

5. *for every $\mathbb{M}$-based (possibly non-Hausdorff) topological space Y and $B \subset Y$, and for every $\mathbb{J}$-filter $\mathcal{G}$ which is $\mathbb{D}$-compactoid at B, the filter $\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $A \times B$.*

Proof. $(4 \Longrightarrow 1)$. If $\mathcal{F}$ is not $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_{\#}$ at A, then there exists a $\mathbb{J}$-filter $\mathcal{J}\#\mathcal{F}$ such that for every $\mathbb{D}$-filter $\mathcal{D}\#\mathcal{J}$, there exists a $\mathbb{M}$-filter $\mathcal{M}_{\mathcal{D}}\#\mathcal{D}$ such that $\operatorname{adh} \mathcal{M}_{\mathcal{D}} \cap A = \emptyset$. Pick $y_0 \in A$ and denote by Y a copy of X endowed with the atomic $\mathbb{M}$-based convergence structure defined by $y_0 \in \lim \mathcal{G}$ iff there exists $\mathcal{D}\#\mathcal{J}$ such that $\mathcal{G} \geq \mathcal{M}_{\mathcal{D}} \wedge \{y_0\}$. Then $\mathcal{J}$ is $\mathbb{D}$-compactoid at $\{y_0\}$ in Y, but $\mathcal{F} \times \mathcal{J}$ is not $\mathbb{F}_1$-compactoid at $A \times \{y_0\}$. Indeed, $\Delta = \{(x,x) : x \in X, x \neq y_0\} \subset X \times Y$ is in $\mathbb{F}_1$ and $\Delta\#\mathcal{F} \times \mathcal{J}$ because $\mathcal{F}\#\mathcal{J}$. But $\operatorname{adh} \Delta \cap A \times \{y_0\} = \varnothing$. Indeed, a filter on Δ can be assumed to be of the form $\mathcal{H} \times \mathcal{H}$. Now if $\mathcal{H}$ converges to $\{y_0\}$ in Y, then $\mathcal{H} \geq \mathcal{M}_{\mathcal{D}}$ so that $\mathcal{H}$ cannot converge to $y_0 \in A$ in X, since $\operatorname{adh} \mathcal{M}_{\mathcal{D}} \cap A = \emptyset$. ∎

From the viewpoint of convergence, there is no reason to distinguish between a sequence and the filter generated by the family of its tails. Therefore, in this paper, sequences are identified to their associated filter and I will freely treat sequences as filters. For instance, given a filter $\mathcal{M}$, I consider the set $\mathcal{E}(\mathcal{M}) = \{(x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \geq \mathcal{M}\}$ of sequences finer than $\mathcal{M}$ by applying this convention.

Proposition 32 *[12, Proposition 5]Let $\mathbb{M}$ be a class of filters such that $\mathcal{E}(\mathcal{M}) \neq \emptyset$ whenever $\mathcal{M} \in \mathbb{M}$.*

Assume that for every $(\mathbb{F}_1/\mathbb{M})$-accessible atomic topological space Y and every $\mathbb{J}$-filter $\mathcal{J}$, which is compactoidly $\mathbb{D}$ to $\mathbb{M}$ meshable relative to the non-isolated point $\{\infty\}$ of Y, the filter $\mathcal{F} \times \mathcal{J}$ is an $\mathbb{F}_1$-compactoidly $(\mathbb{F}_1/\mathbb{M})_{\#}$-filter at $A \times \{\infty\}$. Then $\mathcal{F}$ is an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})_{\#}$- filter at A.

6 Applications

6.1 Global properties

As observed in [12], the part $(1 \Longrightarrow 3)$ of Theorem 31 applied to principal filters $\mathcal{F} = \{X\}$ and $\mathcal{G} = \{Y\}$, for various instances of $\mathbb{D} = \mathbb{J}$ and of $\mathbb{M}$ allows to recover results of J. Vaughan [28], and also to provide new variants. For instance:

Theorem 33 *[12]*

1. *The product of a countably compact space and a compactoidly $(\mathbb{F}_\omega/\mathbb{F}_\omega)$-meshable space is countably compact.*

2. *The product of a strongly Fréchet countably compact space and a $\mathbb{F}_\omega$-compactoidly $(\mathbb{F}_\omega/\mathbb{F}_\omega)$-meshable space is countably compact.*

For instance, compact, sequentially compact, countably compact k-spaces are all examples of compactoidly $(\mathbb{F}_\omega/\mathbb{F}_\omega)$-meshable space and every countably compact space is a $\mathbb{F}_\omega$-compactoidly $(\mathbb{F}_\omega/\mathbb{F}_\omega)$-meshable space.

Denote by $\mathcal{O}_\xi(\mathcal{A})$ the family $\{O$ open: $\exists A \in \mathcal{A},\ A \subset O\}$ whenever $\mathcal{A}$ is a family of subsets of a convergence space (X,ξ). Accordingly, $\mathcal{O}_\xi(\mathbb{D})$ will denote the class of $\mathbb{D}$-filters $\mathcal{D}$ such that $\mathcal{D} = (\mathcal{O}_\xi(\mathcal{D}))^\uparrow$. Notice that a topological space X is feebly compact (that is, pseudocompact if X is additionally Tychonoff) if and only if $\{X\}$ is $\mathcal{O}(\mathbb{F}_\omega)$-compactoid.

Theorem 34 *[12]*

1. *The product of a feebly compact space and a compactoidly $(\mathcal{O}(\mathbb{F}_\omega)/\mathcal{O}(\mathbb{F}_\omega))$-meshable space is feebly compact.*

2. *The product of a $(\mathcal{O}(\mathbb{F}_\omega)/\mathbb{F}_\omega)$-accessible (in particular strongly Fréchet) feebly compact space and a $\mathbb{F}_\omega$-compactoidly $(\mathcal{O}(\mathbb{F}_\omega)/\mathcal{O}(\mathbb{F}_\omega))$-meshable space is feebly compact.*

Theorem 35 *[12]*

1. *The product of a Lindelöf space and a compactoidly $(\mathbb{F}_{\wedge\omega}/\mathbb{F}_{\wedge\omega})$-meshable space is Lindelöf.*

2. *The product of a weakly bisequential Lindelöf space and a $\mathbb{F}_\omega$-compactoidly $(\mathbb{F}_{\wedge\omega}/\mathbb{F}_{\wedge\omega})$-meshable space is Lindelöf.*

6.2 Local properties

As observed in [12], Theorem 31 and Proposition 32 applied in the case of compactoidness relative to a singleton leads to the following.

Theorem 36 *[12, Corollary 4] Let $\mathbb{D} \subset \mathbb{M}$ be two composable classes of filters containing principal filters and assume that there exists a sequence $(x_n)_{n\in\mathbb{N}} \geq \mathcal{M}$ whenever $\mathcal{M} \in \mathbb{M}$. The following are equivalent for a topological space X :*

1. *X is $((\mathbb{D}/\mathbb{M})_{\#\geq}/\mathbb{D})$-accessible;*

2. *$X \times Y$ is $(\mathbb{D}/\mathbb{M})$-accessible for every $(\mathbb{D}/\mathbb{M})$-accessible topological space Y;*

3. *$X \times Y$ is $(\mathbb{F}_1/\mathbb{M})$-accessible for every $(\mathbb{D}/\mathbb{M})$-accessible atomic topological space Y.*

For $\mathbb{D} = \mathbb{F}_1$ and $\mathbb{M} = \mathbb{F}_\omega$, we obtain:

Corollary 37 *[21] A topological space X is finitely generated if and only if its product with every Fréchet topological space is Fréchet.*

If $\mathbb{D} = \mathbb{M} = \mathbb{F}_\omega$, we obtain:

Corollary 38 *[15] A topological space X is productively Fréchet if and only if its product with every strongly Fréchet topological space is (strongly) Fréchet.*

6.3 Products of Maps

In view of Theorems 13, 14, 30 and 29, Theorem 31 has important consequences in terms of product of maps. More specifically:

Theorem 39 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters and let $\mathbb{J}$ be a $\mathbb{D}$-composable class of filters. The following are equivalent for a relation $R : X \rightrightarrows Y$:*

1. *R is an $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable relation*;
2. *for every compactoidly $(\mathbb{D}/\mathbb{M})$-meshable relation $G : Z \rightrightarrows W$ where the convergence space Z is $\mathbb{J}$-based, the relation $R \times G : X \times Z \rightrightarrows Y \times W$ is an $\mathbb{M}$-compactoidly $(\mathbb{D}/\mathbb{D} \times \mathbb{M})$-meshable relation;*
3. *for every $\mathbb{D}$-compactoid relation $G : Z \rightrightarrows W$ where the convergence spaces W and Z are respectively $(\mathbb{D}/\mathbb{M})$-accessible and $\mathbb{J}$-based, the relation $R \times G : X \times Z \rightrightarrows Y \times W$ is $(\mathbb{D} \cap \mathbb{M})$-compactoid;*
4. *for every $\mathbb{M}$-based convergence space W, the relation $R \times Id : X \times \mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} W \rightrightarrows Y \times W$ is $\mathbb{F}_1$-compactoid;*
5. *for every map $g : W \to Z$,where W is an $\mathbb{M}$-based topological space and Z is a $\mathbb{J}$-based atomic topological space, whose inverse relation $g^- : Z \rightrightarrows W$ is $\mathbb{D}$-compactoid, the relation $R \times g^- : X \times Z \rightrightarrows Y \times W$ is $\mathbb{F}_1$-compactoid.*

Proof. $(1 \Longrightarrow 2)$ Let $x \in \lim_X \mathcal{F}$ and $z \in \lim_Z \mathcal{G}$. We can assume $\mathcal{G}$ and hence $G(\mathcal{G})$ to be $\mathbb{J}$-filters. By assumption, $G(\mathcal{G})$ is a $\mathbb{J}$-filter that is compactoidly $(\mathbb{D}/\mathbb{M})$-meshable at Gy, and $R(\mathcal{F})$ is $\mathbb{M}$-compactoidly $(\mathbb{J}/\mathbb{D})$-meshable at Rx. By Theorem 31 $(1 \Longrightarrow 2)$, $R(\mathcal{F}) \times G(\mathcal{G})$ is $\mathbb{M}$-compactoidly $(\mathbb{D}/\mathbb{D} \times \mathbb{M})$-meshable at $Rx \times Gy$ in $Y \times W$.

$(2 \Longrightarrow 3)$, $(3 \Longrightarrow 4)$ and $(3 \Longrightarrow 5)$ are obvious.

$(4 \Longrightarrow 1)$. In view of Theorem 31, it is sufficient to show that $x \in \lim \mathcal{F}$ implies that $R(\mathcal{F}) \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $Rx \times \{w\}$ in $Y \times W$ whenever $\mathcal{G}$ is a $\mathbb{D}$-compactoid at $\{w\}$ $\mathbb{J}$-filter, where W is an $\mathbb{M}$-based convergence space. Notice that $w \in \lim_{\mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} W} \mathcal{G}$. Therefore $(R \times Id)(\mathcal{F} \times \mathcal{G}) = R(\mathcal{F}) \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $Rx \times \{w\}$ in $Y \times W$ and the conclusion follows.

$(5 \Longrightarrow 1)$. In view of Theorem 31, it is sufficient to show that $x \in \lim \mathcal{F}$ implies that $R(\mathcal{F}) \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $Rx \times B$ whenever $\mathcal{G}$ is $\mathbb{D}$-compactoid at $B \subset W$, where W is an $\mathbb{M}$-based topological space. For each such $\mathcal{G}$, consider the relation $G_{\mathcal{G}} : Z \rightrightarrows W$, where Z is the $\mathbb{J}$-based atomic topological space $W \oplus \{\mathcal{G}\}$, defined by $G_{\mathcal{G}}(w) = \{w\}$ for every $w \in W$ and $G_{\mathcal{G}}(\{\mathcal{G}\}) = B$. The filter $G_{\mathcal{G}}(\mathcal{G}) = \mathcal{G}$ is $\mathbb{D}$-compactoid at $B \subset W$ by construction, so that, by hypothesis, $R\mathcal{F} \times \mathcal{G}$ is $\mathbb{F}_1$-compactoid at $Rx \times B$ and the conclusion follows. Notice that the inverse relation is a map $g_{\mathcal{G}}$. ■

Theorem 39 (restricted to $\mathbb{J} = \mathbb{F}$) can be combined with Theorem 30 to the effect that:

Corollary 40 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters. Let $f : X \to Y$ be a continuous surjection between two topological spaces. The following are equivalent:*

1. *f is $\mathbb{M}$-perfect with $(\mathbb{F}/\mathbb{D})$-accessible range;*
2. *$f \times g$ is $(\mathbb{D} \cap \mathbb{M})$-perfect, for every $\mathbb{D}$-perfect map g with $(\mathbb{D}/\mathbb{M})$-accessible domain*;
3. *$f \times g$ is closed, for every $\mathbb{D}$-perfect map g with $\mathbb{M}$-based domain.*

Notice that the statement corresponding to Theorem 39 (2) is omitted in Corollary 40. The reason is that the hypothesis $\mathbb{M} \subset \mathbb{J}$ of Theorem 30 is in general not fullfilled so that this statement cannot be interpreted in terms of $\mathbb{D}$-perfect maps via Theorem 30. However, when $\mathbb{D} = \mathbb{F}$, Theorem 39 (2) and (4) can be interpreted properly, leading to the following generalization of Corollary 25.

Corollary 41 *Let $\mathbb{M}$ be a composable classes of filters containing principal filters. Let $f : X \to Y$ be a continuous surjection between two topological spaces. The following are equivalent:*

1. *f is $\mathbb{M}$-perfect;*
2. *$f \times g$ is $\mathbb{M}$-perfect, for every perfect map g with $(\mathbb{F}/\mathbb{M})$-accessible domain*;
3. *$f \times g$ is closed, for every perfect map $\mathbb{M}$-based domain;*
4. *$f \times Id_Y$ is closed for every $\mathbb{M}$-based topological space Y.*

The following table gathers the corresponding results. Conditions in parenthesis are equivalent to the condition given in the same cell.

$\mathbb{D}$	$\mathbb{M}$	$f \times g$ is	for every g	iff f is
$\mathbb{F}_1$	$\mathbb{F}_1$	closed	closed with finitely generated range	closed with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_1$	closed	countably perfect with finitely generated range	closed with bisequential range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	closed	closed with Fréchet range (first-countable domain)	countably perfect with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably perfect (closed)	countably perfect with strongly Fréchet range (first-countable domain)	countably perfect with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	closed	closed	perfect with finitely generated range
$\mathbb{F}$	$\mathbb{F}_1$	closed	perfect with finitely generated range (identity of finitely generated)	closed
$\mathbb{F}_\omega$	$\mathbb{F}$	countably perfect (closed)	countably perfect	perfect with bisequential range
$\mathbb{F}$	$\mathbb{F}_\omega$	countably perfect (closed)	perfect with bisequential range (identity of first-countable)	countably perfect
$\mathbb{F}$	$\mathbb{F}$	perfect (closed)	perfect (identity map)	perfect

Similarily, Theorem 39 (restricted to $\mathbb{J} = \mathbb{F}$) can also be combined with Theorem 29 to the effect that (taking again into account the restrictions applying to Theorem 29):

Corollary 42 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters. Let $f : X \to Y$ be a continuous surjection between two topological spaces. The following are equivalent:*

1. *f is $\mathbb{M}$-quotient with $(\mathbb{F}/\mathbb{D})$-accessible range;*
2. *$f \times g$ is $(\mathbb{D} \cap \mathbb{M})$-quotient, for every $\mathbb{D}$-perfect map g with $(\mathbb{D}/\mathbb{M})$-accessible domain;*
3. *$f \times g$ is hereditarily quotient, for every $\mathbb{D}$-quotient map g with $\mathbb{M}$-based domain.*

Corollary 43 *Let $\mathbb{M}$ be a composable classes of filters containing principal filters. Let $f : X \to Y$ be a continuous surjection between two topological spaces. The following are equivalent:*

1. *f is $\mathbb{M}$-quotient;*
2. *$f \times g$ is $\mathbb{M}$-quotient, for every biquotient map g with $(\mathbb{F}/\mathbb{M})$-accessible domain;*

3. $f \times g$ is hereditarily quotient, for every biquotient map $\mathbb{M}$-based domain;

4. $f \times Id_Y$ is hereditarily quotient for every $\mathbb{M}$-based topological space Y.

$\mathbb{D}$	$\mathbb{M}$	$f \times g$ is	for every g	iff f is
$\mathbb{F}_1$	$\mathbb{F}_1$	hereditarily quotient	hereditarily quotient with finitely generated range	hereditarily quotient with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_1$	hereditarily quotient	countably biquotient with finitely generated range	hereditarily quotient with bisequential range
$\mathbb{F}_1$	$\mathbb{F}_\omega$	hereditarily quotient	hereditarily quotient with Fréchet range (first-countable domain)	countably biquotient with finitely generated range
$\mathbb{F}_\omega$	$\mathbb{F}_\omega$	countably biquotient (hereditarily quotient)	countably biquotient with strongly Fréchet range (first-countable domain)	countably biquotient with bisequential range
$\mathbb{F}_1$	$\mathbb{F}$	hereditarily quotient	hereditarily quotient	biquotient with finitely generated range
$\mathbb{F}$	$\mathbb{F}_1$	hereditarily quotient	biquotient with finitely generated range (identity of finitely generated)	hereditarily quotient
$\mathbb{F}_\omega$	$\mathbb{F}$	countably biquotient (hereditarily quotient)	countably biquotient	biquotient with bisequential range
$\mathbb{F}$	$\mathbb{F}_\omega$	countably biquotient (hereditarily quotient)	biquotient with bisequential range (identity of first-countable)	countably biquotient
$\mathbb{F}$	$\mathbb{F}$	biquotient (hereditarily quotient)	biquotient (identity map)	biquotient

6.4 Coreflectively modified duality

In a series of papers [10], [21], [23], [24] the author developed a categorical method to deal with topological product theorems, which relates product problems with properties of function spaces and commutation of functors with products. Applications of this method range from a unified treatment of a wide number of classical results [21], [23] to solutions of an old topological problem [20] on one hand, and of a problem of convergence theory (pertaining to Lindelöf and countably compact convergence spaces) [22] on the other hand. The key to concretely apply the abstract results of [21], [23], [24] is to internally characterize couples of convergences (ξ, θ) (on the same underlying set) satisfying

$$\theta \times F\tau \geq G(\xi \times \tau),$$

for every $\tau \geq H\tau$ for specific instances of concrete endofunctors F, G and H of the category of convergence spaces and continuous maps.

In view of Proposition 28, Theorem 31 rephrases as follows when A is a singleton.

Theorem 44 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters and let $\mathbb{J}$ be a $\mathbb{D}$-composable class of filters. The following are equivalent:*

1. $\theta \geq \mathrm{Adh}_{\mathbb{J}} \mathrm{Base}_{\mathbb{D}} \mathrm{Adh}_{\mathbb{M}} \xi$;
2. $\theta \times \mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} \mathrm{Base}_{\mathbb{M}} S\tau \geq \mathrm{Adh}_{\mathbb{D}} \mathrm{Base}_{\mathbb{M}} \mathrm{Adh}_{\mathbb{M}} (\xi \times \tau)$;
3. *for every* $\tau \geq \mathrm{Adh}_{\mathbb{D}} \mathrm{Base}_{\mathbb{M}} \tau$,
$$\theta \times \mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} \tau \geq \mathrm{Adh}_{\mathbb{D}\cap\mathbb{M}} (\xi \times \tau);$$
4. *for every* $\tau = \mathrm{Base}_{\mathbb{M}} \tau$,
$$\theta \times \mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} \tau \geq P(\xi \times \tau).$$

This generalizes [21, Corollay 7.2 and Proposition 7.3] (corresponding to the case $\mathbb{J} = \mathbb{F}$ and $\mathbb{D} \subset \mathbb{M}$) whose important consequences are exposed in [21] and [23]. In particular, relationships between a topological (or convergence) space and the function spaces over it endowed with the continuous convergence can be deduced from Theorem 31. Beattie and Butzmann [2] call a pseudotopological space a *Choquet space* and call a space *countably Choquet* if a countably based filter converges to a point whenever all of its ultrafilter do. In other words, a convergence ξ is countably Choquet, or in our terminology *countably pseudotopological*, if $\xi \leq \mathrm{First}\, S\xi$. More generally, I call $\mathbb{J}$-*pseudotological* a convergence satisfying $\xi \leq \mathrm{Base}_{\mathbb{J}} S\xi$ and $\mathbb{J}$-*paratopological* a convergence satisfying $\xi \leq \mathrm{Base}_{\mathbb{J}} P_\omega \xi$.

Combining Theorem 44 and [21, Theorem 3.1], we get (for $\theta = \xi$) the following new characterizations of bisequentiality, strong and productive Fréchetness in terms of function spaces:

Corollary 45 *Let $\mathbb{D}$ and $\mathbb{M}$ be two composable classes of filters containing principal filters and let $\mathbb{J}$ be a $\mathbb{D}$-composable class of filters. A convergence $\xi = \mathrm{Adh}_{\mathbb{M}} \xi$ is $(\mathbb{J}/\mathbb{D})$-accessible if and only if $\mathrm{Base}_{\mathbb{J}} \mathrm{Adh}_{\mathbb{D}} \mathrm{Base}_{\mathbb{M}}[\xi, \sigma] \geq [\xi, \sigma]$ for every $\sigma = \mathrm{Adh}_{\mathbb{D}} \sigma$ (equivalently for every pretopology σ).*
In particular, when $\mathbb{D} = \mathbb{F}_\omega$ and $\mathbb{M} = \mathbb{F}$:

1. *A pseudotopology ξ is bisequential if and only if the continuous convergence $[\xi, \sigma]$ is a paratopology for every paratopology (equivalently every pretopology) σ;*
2. *A pseudotopology ξ is productively Fréchet if and only if $[\xi, \sigma]$ is $(\mathbb{F}_\omega/\mathbb{F}_\omega)_{\#\geq}$-paratopological for every paratopology (equivalently every pretopology) σ;*
3. *A pseudotopology ξ is strongly Fréchet if and only if $[\xi, \sigma]$ is countably paratopological for every paratopology (equivalently every pretopology) σ.*

This is a sample example. Many others can be found in [21] and [23].

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