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by

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Abstract

The aim of this paper is to find optimal alternatives bivariate ranked set sample for one sample location model bivariate sign test. Our numerical and theoretical results indicated that the optimal designs for bivariate sign test are the alternative designs with quantifying order statistics with labels $\left\{\left(\frac{r+1}{2},\frac{r+1}{2}\right)\right\}$, when the set size r is odd and $\left\{\left(\frac{r}{2}+1,\ \frac{r}{2}\right),\left(\frac{r}{2},\ \frac{r}{2}+1\right)\right\}$ when the set size r is even. The asymptotic distribution and Pitman efficiencies of those designs are derived. Simulation study is conducted to investigate the power of the proposed optimal designs. Illustration using real data with Bootstrap algorithm for P-value estimation is used.

Key Words: Bivariate Ranked Set Sample; Location Model; Median Ranked Set Sample; Pitman efficiencies; Ranked set Sample; Simple Random Sample; and Sign Test.
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1. Introduction

Since McIntyre (1952) introduced the idea of rank set sampling (RSS) for estimating the population mean, many authors investigated this method and its variation. RSS has been applied in agricultural, environmental, ecological, engineering and epidemiological studies, because of its cost savings nature. Also, in many cases RSS produces unbiased estimators with smaller variance than the counterpart estimators produced by simple random sampling (SRS).

The RSS procedure can be described as follows: identify a group of r^2 sampling units from the target population. Then, partition the identified units randomly into r disjoint subsets each having a pre-assigned size r. In most practical situations, the size r is 2, 3 or 4. Each subset is then ranked by a suitable ranking method (e.g., prior information, visual inspection or by the subject-matter experimenter himself, … etc.) The *i-th* ordered statistic from the *i-th* subset, $X_{i(i)}$, $i = 1, \ldots, r$, is quantified (actual measurement) and $X_{1(1)}, X_{2(2)}, \ldots, X_{r(r)}$ constitute the RSS. This represents one cycle. The whole procedure can be repeated m times as needed to get a RSS of size $n=mr$. Takahasi and Wakimoto (1968) established mathematical proves of RSS for estimating the population mean. Stokes and Sager (1988) suggested an estimator of the distribution functions $F(.)$ based on RSS using empirical distribution function (edf). They indicated that the estimator of the distribution function $\hat{F}_{RSS}(.)$ based on RSS is an unbiased estimator of $F(.)$ and is more efficient than the empirical distribution function based on SRS. Also, they showed that $\hat{F}_{RSS}(.)$ has asymptotic normal distribution. For more references on RSS up to 1999 see, Kaur et al. (1995) and Patil et al. (1999).

Recently, Chen (2000) and Samawi (2001) discussed quantiles estimation using RSS. Chen (2001) suggested an optimal ranked set sample scheme (ORSS) for inference on population quantiles.

Bohn and Wolfe (1992, 1994), generalized the Mann-Whitney two-sample procedure using perfect ranked set sample method. Kvam and Samaniego (1994), suggested a non-parametric maximum likelihood estimation as an alternative estimator of the cumulative distribution function (c.d.f) using RSS.

Hettmansperger (1995) considered sign test based on RSS. Koti and Babu (1996) studied the exact distribution of the RSS sign test. They showed that the test is more efficient than the counterpart SRS sign test. Barabesi (1998) provided a simpler and faster method for computing the exact distribution of the RSS sign test.

Several researchers in the literature introduced the optimality of RSS sign test via Pitman asymptotic efficiency. Öztürk (1999) and Öztürk and Wolfe (2000) showed that the median ranked set sample (MRSS) is the optimal sampling schemes for sign test via Pitman asymptotic efficiency. However, Samawi and Abu-Dayyeh (2003) gave the exact power and distribution function of the MRSS sign test in case of finite sample size.

Multiple characteristics estimation using RSS was discussed by Patil et al (1994), and Norris et al (1995). Al-Saleh and Zheng (2002) suggested a bivariate ranked set sampling procedure (BVRSS) for multiple characteristics estimation.

Samawi et al (2006) investigated sign test for one-sample bivariate location model using BVRSS. They showed that this test provides a more powerful test than the sign test based on a bivariate simple random sample (BVSRS).

In this paper, we propose to find optimal sign tests for one-sample bivariate location model using alternatives bivariate ranked set sampling.

1.1 Bivariate ranked set sampling (BVRSS)

Al-Saleh and Zheng (2002), bivariate ranked set sample procedure is as follows: Suppose (X, Y) is a bivariate random vector with joint probability density function (p.d.f.) $f(x, y)$.

1. Select randomly r^4 sampling units with respect to the variable of interest from the target population.
2. Randomly allocated the units into r^2 pools each of size r^2 so that each pool is a square matrix with r rows and r columns.
3. In the first pool, choose the minimum value by a suitable ranking method with respect to the first characteristic X, for each row in this pool.
4. The actual quantification is done on the pair that corresponds to the minimum value of the second characteristic, Y identified by a suitable ranking method. This pair represents the first element of the BVRSS, which resembles the label (1, 1).

5. Repeat Steps 2 and 3 for the second pool, but in Step 3, the pair that corresponds to the second minimum value with respect to the second characteristic Y, is chosen for actual quantification. This pair resembles the label (1, 2).

6. The process continues until the label $(r,\ r)$ is resembled from the r^2-th (last) pool.

7. The above procedure represents a BVRSS of size r^2 and we can repeat the above procedure m times to get a BVRSS of size $n=mr^2$.

1.1.1 Some characteristics of a BVRSS

Let $[(X_{ijk},\ Y_{ijk}),\ i=1,\ 2,\ ...,\ r\ ,\ j=1,\ 2,\ ...,\ r,\ k=1,\ 2,\ ...,\ m]$ be mr^2 i.i.d ordered pairs from a bivariate probability density function, say $f\,(x\,,y\,)$; $(x\,,\ y\,)\in R^2$. The definition of a BVRSS is introduced by Al-Saleh and Zheng (2002) as follows: $\left[\ (X_{[i](j)k},\ \mathrm{Y}_{(i)[j]k}),\ i=1,\ 2,\ ...,\ r;\ \ j=1,\ 2,\ ...,\ r\ \right]$ and $k=1,\ 2,\ ....,\ m$ denotes such a sample from $f\,(x\,,y\,)$. Let $f_{X_{[i](j)},\,Y_{(i)[j]}}\left(x\,,y\,\right)$ be the joint p.d.f. of $(X_{[i](j)k},\ Y_{(i)[j]k})$, $k=1,\ 2,\ ....,\ m$. Then as in Al-Saleh and Zheng (2002),

$$f_{X_{[i](j)},\,Y_{(i)[j]}}\left(x\,,y\,\right)\ =f_{Y_{(i)[j]}}(y\,)\frac{f_{X_{(j)}}(x\,)f_{Y\,|X}\,(y\,|\,x\,)}{f_{Y_{[j]}}(y\,)} \tag{1.1}$$

where $f_{X_{(j)}}$ is the density of the $j-th$ order statistic for a SRS of size r from the marginal density of $f_X\,(x\,)$, $f_{Y_{[j]}}(y\,)$, is the density of the corresponding variable Y and it is given by

$$f_{Y_{[j]}}(y\,)=\int_{-\infty}^{\infty}f_{X_{(j)}}(x\,)f_{Y\,|X}\,(y\,|\,x\,)\ dx \tag{1.2}$$

and $f_{Y_{(i)[j]}}(y)$ is the density of the $i-th$ order statistic of an i.i.d. sample from $f_{Y_{[j]}}(y)$, i.e. $f_{Y_{(i)[j]}}(y)=\ d(F_{Y_{[j]}}(y))^{i-1}(1-F_{Y_{[j]}}(y))^{r-i}\,f_{Y_{[j]}}(y)$ $\quad$ (1.3) where $F_{Y_{[j]}}(y) = \int_{-\infty}^{y} \left(\int_{-\infty}^{\infty} f_{X_{(j)}}(x) f_{Y|X}(y \mid x)\, dx \right) dw$

and $d = \dfrac{r!}{(i-1)!\,(r-i)!}$. Therefore,

$$f_{X_{[i](j)},\, Y_{(i)[j]}}(x, y) = d_1 (F_{Y_{[j]}}(y))^{i-1} (1 - F_{Y_{[j]}}(y))^{r-i} (F_X(x))^{j-1} (1 - F_X(x))^{r-j} f(x, y) \tag{1.4}$$

where $d_1 = \dfrac{r!}{(i-1)!\,(r-i)!} \dfrac{r!}{(j-1)!\,(r-j)!}$.

Also, Al-Saleh and Zheng (2002) showed the following

(1) $f(x, y) = \dfrac{1}{r^2} \sum_{j=1}^{r} \sum_{i=1}^{r} f_{X_{[i](j)},\, Y_{(i)[j]}}(x, y)$, (1.5)

(2) $f_X(x) = \dfrac{1}{r^2} \sum_{j=1}^{r} \sum_{i=1}^{r} f_{X_{[i](j)}}(x)$, (1.6)

(3) $f_Y(y) = \dfrac{1}{r^2} \sum_{j=1}^{r} \sum_{i=1}^{r} f_{Y_{(i)[j]}}(y)$ (1.7)

For further description and proofs see Al-Saleh and Zheng (2002).

1.2 Sign Test for One-Sample Bivariate Location Model

1.2.1 General setting

As described by Hettmansperger (1984), assume that the bivariate random variable (X, Y) has an absolutely continuous cumulative distribution function (cdf) $F(x - \theta_1, y - \theta_2)$ with absolutely continuous marginal (cdfs) $F_X(x - \theta_1)$ and $F_Y(y - \theta_2)$ where the location parameters θ_1 and θ_2 are the marginal medians and $F_X, F_Y \in \Omega_0 = \{F : F$ is absolutely continuous and $F(0) = \frac{1}{2}$, uniquely$\}$. We are interested in testing $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ versus $H_A : \underset{\sim}{\theta} \neq \underset{\sim}{0}$, where $\underset{\sim}{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ and $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The hypothesis $H_0 : \underset{\sim}{\theta} = \underset{\sim}{\theta}_0$ can be transformed into $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ by subtracting the given $\underset{\sim}{\theta}_0$ from the observation vectors.

1.2.2 Sign test based on a BVSRS

Hettmansperger (1984), showed that since no further structural assumptions on $F(x, y)$ was made, the vector of sign statistic is appropriate for the construction of a test for the hypotheses given above. The form of the sign statistic with zero expectation under the null hypothesis is as follows:

Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be a BVSRS from $F(x, y)$. Then the BVSRS sign test statistic is given by

$$\underset{\sim}{S}' = \left(S_x = \sum_{i=1}^{n} \text{sgn} X_i = 2\sum_{i=1}^{n} I(X_i > 0) - n \ a.e., \ S_y = \sum_{i=1}^{n} \text{sgn} Y_i = 2\sum_{i=1}^{n} I(Y_i > 0) - n \ a.e. \right)$$ where

$I(.)$ is the indicator function which is defined by:

$$I(t > 0) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\text{sgn}(t) = -1$ if $t < 0$, 0 if $t = 0$, and $+1$ if $t > 0$. The asymptotic distribution function of $\underset{\sim}{S}'$ is obtained as follows: Under the null hypothesis,

$$\frac{1}{\sqrt{n}} \underset{\sim}{S}' \xrightarrow{D} N_2(\underset{\sim}{0}, V), \text{ where } V = \begin{pmatrix} 1 & v_{xy} \\ v_{xy} & 1 \end{pmatrix}, \text{ and}$$

$v_{xy} = P(X \le 0, Y \le 0) + P(X > 0, Y > 0) - P(X \le 0, Y > 0) - P(X > 0, Y \le 0)$. It is easy to verify $v_{xy} = 4P(X \le 0, Y \le 0) - 1$. From the above notation a consistent estimator of v_{xy} is $\hat{v}_{xy} = \frac{1}{n}\sum_{i=1}^{n} \text{sgn} X_i \ \text{sgn} Y_i$. Then, under the null hypothesis, a consistent estimator of V is $\hat{V} = \begin{pmatrix} 1 & \hat{v}_{xy} \\ \hat{v}_{xy} & 1 \end{pmatrix}$. Therefore, the test statistics is given by

$S_{BVSRS} = \underset{\sim}{S}'(n\hat{V})^{-1}\underset{\sim}{S}$ is asymptotically distributed as $\chi^2(2)$.

A practical and simple form of S_{BVSRS} is given by

$$S_{BVSRS} = \frac{(N_{00} - N_{11})^2}{N_{00} + N_{11}} + \frac{(N_{01} - N_{10})^2}{N_{01} + N_{10}}, \tag{1.8}$$

where $N_{00} = \#(X_i \le 0,\ Y_i \le 0),\ N_{11} = \#(X_i > 0,\ Y_i > 0),\ N_{01} = \#(X_i \le 0,\ Y_i > 0)$ and $N_{10} = \#(X_i > 0,\ Y_i \le 0),\ i = 1,\ 2,\ ...,\ n$. (See Hettmansperger, 1984.)

Furthermore, Hettmansperger (1984) derived Pitman efficiency of the BVSRS sign test along a sequence of alternatives, $\underset{\sim}{\theta}_n = \frac{\underset{\sim}{\theta}}{\sqrt{n}}$, converging to $\underset{\sim}{0}$ as follows: The limiting distribution of the statistic S_{BVSRS} along a sequence of alternatives, $\underset{\sim}{\theta}_n = \frac{\underset{\sim}{\theta}}{\sqrt{n}}$, converging to $\underset{\sim}{0}$, will be a noncentral Chi-square with 2 degrees of freedom, and the noncentrality parameter is $\underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu}$, where $\underset{\sim}{\mu} = 2(\theta_1 f_X(0),\ \theta_2 f_Y(0))'$ and $V = \begin{pmatrix} 1 & v_{xy} \\ v_{xy} & 1 \end{pmatrix}$ as defined above. Since the power of the test is an increasing function of the noncentrality parameter, then Hettmansperger (1984) defined $\underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu}$ as Pitman efficiency of the BVSRS sign test.

2. Alternative BVRSS for Sign Test

This section introduces a plan to find an alternative BVRSS scheme which gives the optimal design for the sign test, with zero expectation, for one-sample bivariate location model.

2.1 Alternative BVRSS design

An alternative bivariate ranked set sampling (ABVRSS) is a sampling protocol that quantifies the same order statistics in each pool using similar BVRSS protocol as described by Al-Saleh and Zheng (2002). Define $\ell(A)$ to be the cardinality of a set *A*, then $\ell(A)$ = the number of elements in a set *A*. Let J_{ABVRSS} = {set of all possible alternative BVRSS designs} = $\{J_1, J_2, ..., J_{\ell(J_{ABVRSS})}\}$, for example, when *r=2*, then J_{ABVRSS} = {{(1, 1)}, {(1, 2)}, {(2, 1)}, {(2, 2)}, {(1, 1), (1, 2)}, {(1, 1), (2, 1)}, {(1, 1), (2, 2)}, {(1, 2), (2, 1)}, {(1, 2),(2, 2)}, {(2, 1),(2, 2)}, {(1, 1), (1, 2), (2, 1)}, {(1, 1), (1, 2), (2, 2)}, {(1, 1), (2, 1), (2, 2)}, {(1, 2), (2, 1), (2, 2)}, {(1, 1), (1, 2), (2, 1), (2, 2)}}. Then for *r=2*, $\ell(J_{ABVRSS}) = \sum_{i=1}^{4}\binom{4}{i} = 15$. In general, for a set of size *r*,

$\ell(J_{ABVRSS}) = \sum_{i=1}^{r^2}\binom{r^2}{i} = 2^{r^2} - 1$. Now, for an integer s, $s \in \{1, 2, ..., 2^{r^2} - 1\}$, let $J_s \in J_{ABVRSS}$, be the set of judgment ranks of ordered pairs labels for the observations to be quantified.

Our sampling protocol involves selecting $m\,\ell(J_s)r^2$ units from an infinite population. These units are partitioned into $m\,\ell(J_s)$ pools each having r^2 units.

From each pool, by using the same procedure discussed for a BVRSS protocol by Al-Saleh and Zheng (2002), we quantify only one of the ordered pairs labels in J_s, therefore they are mutually independent.

2.2 Sign test using an alternative BVRSS

Let (X, Y) be a bivariate random variable have an absolutely continuous cumulative distribution function (cdf) $F(x - \theta_1, y - \theta_2)$ with absolutely continuous marginal. Furthermore, assume that $f(x, y) = f(-x, -y)$. To test $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ versus $H_A : \underset{\sim}{\theta} \neq \underset{\sim}{0}$, for $J_s \in J_{ABVRSS}$, $s = 1, 2, ..., 2^{r^2} - 1$, let

$J_s = \{(c_1, d_1), ..., (c_{\ell(J_s)}, d_{\ell(J_s)})\}$, then

$\left\{\left(X_{[c_1](d_1)k}, Y_{(c_1)[d_1]k}\right), ..., \left(X_{[c_{\ell(J_s)}](d_{\ell(J_s)})k}, Y_{(c_{\ell(J_s)})[d_{\ell(J_s)}]k}\right)\right\}$, where

$k = 1, 2, ..., m$, be an alternative BVRSS from $F(x, y)$ with $n = m\,\ell(J_s)$. Thus, the sign test statistic based on ABVRSS is $\underset{\sim}{S}'_T = \left(S_{XT}, S_{YT}\right)$, where

$S_{XT} = \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} \mathrm{sgn} X_{[c_u](d_u)k} = 2\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} > 0) - n$ a.s and

$$S_{YT} = \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} \text{sgn} Y_{(c_u)[d_u]k} = 2 \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} I(Y_{(c_u)[d_u]k} > 0) - n \text{ a.s.}$$

The mean and the variance of $\underset{\sim}{S}'_T$ under H_0 are as follows:

$$E(S_{XT}) = \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} E(\text{sgn} X_{[c_u](d_u)k}) = 2 \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} E(I(X_{[c_u](d_u)k} > 0)) - n$$
$$= n\left[1 - \frac{2}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} F_{X_{[c_u](d_u)}}(0)\right],$$

where $n = m\ell(J_s)$. Similarly, $E(S_{YT}) = n\left[1 - \frac{2}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} F_{Y_{(c_u)[d_u]}}(0)\right]$.

Also,

$$V(S_{XT}) = \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} V(\text{sgn} X_{[c_u](d_u)k}) = 4 \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} V(I(X_{[c_u](d_u)k} > 0))$$
$$= n\left[\frac{4}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} \left(1 - F_{X_{[c_u](d_u)}}(0)\right) F_{X_{[c_u](d_u)}}(0)\right].$$

Similarly, $V(S_{YT}) = n\left[\frac{4}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} \left(1 - F_{Y_{(c_u)[d_u]}}(0)\right) F_{Y_{(c_u)[d_u]}}(0)\right]$.

and

$$COV(S_{XT}, S_{YT}) = COV\left(\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} \text{sgn} X_{[c_u](d_u)k}, \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} \text{sgn} Y_{(c_u)[d_u]k}\right)$$

$$= n\left[\frac{4}{\ell(J_S)} \sum_{u=1}^{\ell(J_S)} [1 - F_{X_{[c_u](d_u)}, Y_{(c_u)[d_u]}}(0,0)] - [(1 - F_{X_{[c_u](d_u)}}(0))(1 - F_{Y_{(c_u)[d_u]}}(0)]\right].$$

Note that, Samawi et. al. (2006) BVRSS sign test is just a special case when $J_s = J_{\ell(J_{ABVRSS})}$. In this paper we consider only the cases when $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$. Next, we prove only two cases for which $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$.

<u>Theorem 2.1</u> Under the assumptions of ABVRSS protocol and if

1. $\left\{(c_1, d_1) : c_1 = \frac{r+1}{2},\ d_1 = \frac{r+1}{2}\right\}$, when r is odd.

2. $\left\{(c_1, d_1),\ (c_2, d_2) : c_1 = \frac{r}{2},\ d_1 = \frac{r}{2} + 1,\ c_2 = \frac{r}{2} + 1,\ d_2 = \frac{r}{2}\right\}$, when r is even.

Then, $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$.

Proof:

Case 1: r is odd. To show that $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$, it is suffices to show that

$$F_{Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(0) = \int_{-\infty}^{0} c(F_{Y_{[\frac{r+1}{2}]}}(y))^{\frac{r-1}{2}}(1-F_{Y_{[\frac{r+1}{2}]}}(y))^{\frac{r-1}{2}} f_{Y_{[\frac{r+1}{2}]}}(y)dy = 0.5 \quad (2.1)$$

and

$$F_{X_{[\frac{r+1}{2}](\frac{r+1}{2})}}(0) = \int_{-\infty}^{0} c(F_{X_{[\frac{r+1}{2}]}}(x))^{\frac{r-1}{2}}(1-F_{X_{[\frac{r+1}{2}]}}(x))^{\frac{r-1}{2}} f_{X_{[\frac{r+1}{2}]}}(x)dx = 0.5 \quad (2.2)$$

where $F_{Y_{[\frac{r+1}{2}]}}(y) = \int_{-\infty}^{y} f_{Y_{[\frac{r+1}{2}]}}(w)dw$, $F_{X_{[\frac{r+1}{2}]}}(x) = \int_{-\infty}^{x} f_{X_{[\frac{r+1}{2}]}}(v)dv$

and $c = \dfrac{r!}{(\frac{r-1}{2})!\,(\frac{r-1}{2})!}$.

To show (2.1), let $u = F_{Y_{[\frac{r+1}{2}]}}(y)$ then

$$F_{Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(0) = \int_{0}^{F_{Y_{[\frac{r+1}{2}]}}(0)} c(u)^{\frac{r-1}{2}}(1-u)^{\frac{r-1}{2}} du .$$

Note that, this integral is an incomplete beta, which is symmetric about 0.5. Therefore, to show

$$F_{Y_{(\frac{r+1}{2})[\frac{r+1}{2}]}}(0) = \int_{0}^{F_{Y_{[\frac{r+1}{2}]}}(0)} c(u)^{\frac{r-1}{2}}(1-u)^{\frac{r-1}{2}} du = 0.5$$, it is suffices to show that

$F_{Y_{[\frac{r+1}{2}]}}(0) = 0.5$ and hence we need to show that $f_{Y_{[\frac{r+1}{2}]}}(y)$ is symmetric about 0. Since

$$f_{Y_{[\frac{r+1}{2}]}}(y) = \int_{-\infty}^{\infty} c(F_X(x))^{\frac{r-1}{2}}(1-F_X(x))^{\frac{r-1}{2}} f(x,y)dx ,$$ then

$$f_{Y_{[\frac{r+1}{2}]}}(-y) = \int_{-\infty}^{\infty} c(F_X(x))^{\frac{r-1}{2}}(1-F_X(x))^{\frac{r-1}{2}} f(x,-y)dx .$$ Now, let $x=-t$, then

$f_{Y_{[\frac{r+1}{2}]}}(-y) = \int_{-\infty}^{\infty} c(F_X(-t))^{\frac{r-1}{2}}(1-F_X(-t))^{\frac{r-1}{2}} f(-t,-y)dt$. Since $f_X(x)$ is symmetric about 0 and $f(-t,-y) = f(t,y)$, then

$f_{Y_{[\frac{r+1}{2}]}}(-y) = \int_{-\infty}^{\infty} c(1-F_X(t))^{\frac{r-1}{2}}(F_X(t))^{\frac{r-1}{2}} f(t,y)dt = f_{Y_{[\frac{r+1}{2}]}}(y)$. Similarly, we can show (2.2).

Case 2: *r* is even. To show that $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$, it is suffices to show that

$$F_{Y_{(\frac{r}{2})[\frac{r}{2}+1]}}(0) + F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = 1 \text{ or } F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = 1 - F_{Y_{(\frac{r}{2})[\frac{r}{2}+1]}}(0) \tag{2.3}$$

and $$F_{X_{[\frac{r}{2}](\frac{r}{2}+1)}}(0) + F_{X_{[\frac{r}{2}+1](\frac{r}{2})}}(0) = 1 \text{ or } F_{X_{[\frac{r}{2}+1](\frac{r}{2})}}(0) = 1 - F_{X_{[\frac{r}{2}](\frac{r}{2}+1)}}(0) \tag{2.4}$$

To show (2.3) we use similar arguments as in Case 1, thus we have

$$F_{Y_{(\frac{r}{2})[\frac{r}{2}+1]}}(0) = \int_{0}^{F_{Y_{[\frac{r}{2}+1]}}(0)} c_1(u)^{\frac{r}{2}}(1-u)^{\frac{r}{2}-1} du$$

and

$$F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = \int_{0}^{F_{Y_{[\frac{r}{2}]}}(0)} c_1(u)^{\frac{r}{2}-1}(1-u)^{\frac{r}{2}} du,$$

where $c_1 = \dfrac{r!}{(\frac{r}{2}-1)!\ (\frac{r}{2})!}$. Now let *v=1-u*, then

$$F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = \int_{1-F_{Y_{[\frac{r}{2}]}}(0)}^{1} c_1(1-v)^{\frac{r}{2}-1}(v)^{\frac{r}{2}} dv.$$

Now,

$$F_{Y_{(\frac{r}{2})[\frac{r}{2}+1]}}(0) + F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = \int_0^{F_{Y_{[\frac{r}{2}+1]}}(0)} c_1 (u)^{\frac{r}{2}} (1-u)^{\frac{r}{2}-1} du$$

$$+ \int_{1-F_{Y_{[\frac{r}{2}]}}(0)}^{1} c_1 (1-v)^{\frac{r}{2}-1} (v)^{\frac{r}{2}} dv$$

$$= 1 - \int_{F_{Y_{[\frac{r}{2}+1]}}(0)}^{1} c_1 (u)^{\frac{r}{2}} (1-u)^{\frac{r}{2}-1} du$$

$$+ \int_{1-F_{Y_{[\frac{r}{2}]}}(0)}^{1} c_1 (1-v)^{\frac{r}{2}-1} (v)^{\frac{r}{2}} dv .$$

Thus, $F_{Y_{(\frac{r}{2})[\frac{r}{2}+1]}}(0) + F_{Y_{(\frac{r}{2}+1)[\frac{r}{2}]}}(0) = 1$, if and only if $F_{Y_{[\frac{r}{2}+1]}}(0) = 1 - F_{Y_{[\frac{r}{2}]}}(0)$.

Now, since

$f_{Y_{[\frac{r}{2}+1]}}(y) = \int_{-\infty}^{\infty} c_1 (F_X(x))^{\frac{r}{2}} (1-F_X(x))^{\frac{r}{2}-1} f(x,y)dx,$ then by using similar arguments as

in Case 1, $f_{Y_{[\frac{r}{2}+1]}}(-y) = \int_{-\infty}^{\infty} c_1 (1-F_X(x))^{\frac{r}{2}} (F_X(x))^{\frac{r}{2}-1} f(x,y)dx = f_{Y_{[\frac{r}{2}]}}(y), \forall y.$

Thus, $\int_{-\infty}^{t} f_{Y_{[\frac{r}{2}+1]}}(-y)dy = \int_{-\infty}^{t} f_{Y_{[\frac{r}{2}]}}(y)dy.$ Now let $y = -u$, then

$\int_{-t}^{\infty} f_{Y_{[\frac{r}{2}+1]}}(u)du = \int_{-\infty}^{t} f_{Y_{[\frac{r}{2}]}}(y)dy,$ i.e., $1 - F_{Y_{[\frac{r}{2}+1]}}(-t) = F_{Y_{[\frac{r}{2}]}}(t).$

Therefore, only at t=0, $1 - F_{Y_{[\frac{r}{2}+1]}}(0) = F_{Y_{[\frac{r}{2}]}}(0)$ and hence

$F_{Y_{[\frac{r}{2}+1]}}(0) = 1 - F_{Y_{[\frac{r}{2}]}}(0).$ Similarly, we can show (2.4)

Note that, there are other cases in Table 2.1 and Table 2.2 with $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$ can be proved in similar manner. However, we proved only the above two cases because they have the highest relative Pitman efficiency comparing with other cases.

Theorem 2.2 Under the null hypothesis $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$, for a fixed value of $\ell(J_s)$ and a large value of m, and hence for a large n, where $n = m\ell(J_s)$. Assume that $E(\underset{\sim}{S}'_T) = \underset{\sim}{0}$, then $\frac{1}{\sqrt{n}}\underset{\sim}{S}'_T \xrightarrow{D} N_2(\underset{\sim}{0}, V_T)$, where,

$$V_T = \begin{pmatrix} V_{xxT} & V_{xyT} \\ V_{xyT} & V_{yyT} \end{pmatrix}, V_{xxT} = \left[\frac{4}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} \left(1 - F_{X_{[c_u](d_u)}}(0)\right) F_{X_{[c_u](d_u)}}(0)\right],$$

$V_{xyT} =$

$$\frac{4}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} [1 - F_{X_{[c_u](d_u)}, Y_{(c_u)[d_u]}}(0,0)] - [(1 - F_{X_{[c_u](d_u)}}(0))(1 - F_{Y_{(c_u)[d_u]}}(0)],$$

and $V_{yyT} = \frac{4}{\ell(J_s)} \sum_{u=1}^{\ell(J_s)} \left(1 - F_{Y_{(c_u)[d_u]}}(0)\right) F_{Y_{(c_u)[d_u]}}(0)$.

Proof: First, S_{XT} can be defined as the sum of independent identical distribution variates, $S_{XT} = 2\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} > 0) - n = \sum_{k=1}^{m} S_{XTk}$ where

$S_{XTk} = 2\sum_{u=1}^{\ell(J_s)} I(X_{[c_u](d_u)k} > 0) - \ell(J_s)$, with

$E(S_{XTk}) = 0$ and $V(S_{XTk}) = \left[4\sum_{u=1}^{\ell(J_s)} \left(1 - F_{X_{[c_u](d_u)}}(0)\right) F_{X_{[c_u](d_u)}}(0)\right] < \infty$

Similarly, $S_{YT} = 2\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} I(Y_{(c_u)[d_u]k} > 0) - n = \sum_{k=1}^{m} S_{YTk}$ where

$S_{YTk} = 2\sum_{u=1}^{\ell(J_s)} I(Y_{(c_u)[d_u]k} > 0) - \ell(J_s)$, with

$E(S_{YTk}) = 0$ and $V(S_{YTk}) = \left[4\sum_{u=1}^{\ell(J_s)} \left(1 - F_{Y_{(c_u)[d_u]}}(0)\right) F_{Y_{(c_u)[d_u]}}(0)\right] < \infty$,

$k = 1, 2, \ldots, m$, for a fixed value of $\ell(J_s)$. Therefore, Theorem A12 page 303 in Hettmansperger (1984) implies that the limiting distribution is

$$\frac{1}{\sqrt{n}} \underset{\sim}{S}_T' \xrightarrow{D} N_2(\underset{\sim}{0}, V_T).$$

2.3 Asymptotic relative Pitman efficiency

Similar to Hettmansperger (1984) and Samawi et. al. (2006) we derive the limiting distribution of $\frac{1}{\sqrt{n}} \underset{\sim}{S}_T'$ along a sequence of alternatives, $\underset{\sim}{\theta}_n = \frac{\underset{\sim}{\theta}}{\sqrt{n}}$, converging to $\underset{\sim}{0}$ when using ABVRSS. Note that, $E_{\underset{\sim}{\theta}_n}\left(\frac{1}{\sqrt{n}} \underset{\sim}{S}_T\right)$ converges to $\underset{\sim}{\mu}_T = (\mu_{TX}, \mu_{TY})'$, where $\theta_1 \frac{\partial}{\partial \theta_1}\left(E_{\theta_{1n}}\left(\frac{1}{\sqrt{n}} S_{XT}\right)\right) = \theta_1 \frac{\partial}{\partial \theta_1}\left(E_{\theta_{1n}}\left(\frac{1}{\sqrt{n}} 2 \sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} I(X_{[c_u](d_u)k} > 0) - n\right)\right)$

$$= \frac{2\theta_1}{\sqrt{n}} \frac{\partial}{\partial \theta_1}\left(\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} [1 - F_{X_{[c_u](d_u)k}}(-\theta_{1n})] - n\right), \text{where } \theta_{1n} = \frac{\theta_1}{\sqrt{n}}$$

$$= \frac{2\theta_1}{n}\left(\sum_{u=1}^{\ell(J_s)} \sum_{k=1}^{m} f_{X_{[c_u](d_u)k}}(-\theta_{1n})\right) \text{ converges to } \frac{2\theta_1}{\ell(J_s)}\left(\sum_{u=1}^{\ell(J_s)} f_{X_{[c_u](d_u)}}(0)\right).$$

Similarly, $E_{\theta_{2n}}\left(\frac{1}{\sqrt{n}} S_{YT}\right)$ converges to $\mu_{TY} = \frac{2\theta_2}{\ell(J_s)}\left(\sum_{u=1}^{\ell(J_S)} f_{Y_{(c_u)[d_u]}}(0)\right)$ along a sequence of alternatives θ_{2n}. Thus, along a sequence of alternatives $\underset{\sim}{\theta}_n$, $\frac{1}{\sqrt{n}} \underset{\sim}{S}_T'$ has an asymptotic distribution $N_2(\underset{\sim}{\mu}_T, V_T)$, where $V_T = \begin{pmatrix} V_{xxT} & V_{xyT} \\ V_{xyT} & V_{yyT} \end{pmatrix}$ as defined in Theorem 2.2. Therefore, the limiting distribution of the statistic S_{ABVRSS}, along $\underset{\sim}{\theta}_n$, will be a noncentral chi-square with 2 degrees of freedom, and a noncentrality parameter $\underset{\sim}{\mu}_T' V_T^{-1} \underset{\sim}{\mu}_T$.

Again the asymptotic relative efficiency of ABVRSS sign test relative BVSRS sign test $\left(AEff\,(S_{ABVRSS}, S_{BVSRS})\right)$ can be obtained as the ratio of their noncentrality parameters. Hence, $AEff\,(S_{ABVRSS}, S_{BVSRS}) = \left[\dfrac{\underset{\sim}{\mu}'_T V_T^{-1} \underset{\sim}{\mu}_T}{\underset{\sim}{\mu}' V^{-1} \underset{\sim}{\mu}}\right]$, where $\underset{\sim}{\mu}$ and V were defined above.

2.4 Numerical analysis

The performance of the ABVRSS sign test will be compared with the performance of the BVSRS sign test via their asymptotic relative Pitman efficiency for testing $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ versus $H_A : \underset{\sim}{\theta} \neq \underset{\sim}{0}$, in the case of bivariate normal distribution. We will be consider for the following cases: (r =2) with correlation coefficient (ρ = 0.5), (r = 3) with correlation coefficient (ρ = 0.9, 0.5, 0.1) and (r=4) with correlation coefficient (ρ = 0.5). Note that we report only the cases when $E\,(\underset{\sim}{S}'_T) = \underset{\sim}{0}$ and the cases when $AEff\,(S_{ABVRSS}, S_{BVSRS})$ is more than the ordinary BVRSS.

Table 2.1 shows that the alternative design with labels {(1, 2), (2, 1)} in case of r=2 has the highest $AEff\,(S_{ABVRSS}, S_{BVSRS})$ among the other alternative designs with $E\,(\underset{\sim}{S}'_T) = \underset{\sim}{0}$. Note that this optimal design sign test has asymptotic relative efficiency higher than the ordinary BVRSS sign test. Although, the alternative design with label {(2, 1)} has the highest asymptotic relative efficiency among all alternative designs. However, the expected value of the sign test under this design does not equal zero. Since the non-zero expectation of the sign test of this design depends on the unknown parameters, this design can not be used in practice. Hence, it will not be consider in this paper.

Table 2.1. Asymptotic relative efficiency of ABVRSS sign test relative to BVSRS sign test.

$R = 2, \rho = 0.5$			
J_s	$AEff\,(S_{ABVRSS}, S_{BVSRS})$	$E\,(S_{XT})$	$E\,(S_{YT})$
(2,1)	1.8353	-0.2435	-0.2435
(1,1)(2,2)	1.2507	0	0
(1,2)(2,1)	1.6719	0	0
Ordinary BVRSS	1.4504	0	0

In the next tables we present only important cases to make this paper shorter. Table 2.2, Table 2.3 and Table 2.4, show that the alternative design with label (2, 2), in case of *r=3* has the highest $AEff\ (S_{ABVRSS}, S_{BVSRS})$ among all other alternative designs with $E\ (\underset{\sim}{S}'_T) = \underset{\sim}{0}$.

Table 2.2. Asymptotic relative efficiency of ABVRSS sign test relative to BVSRS sign test .

r =3*, ρ* = 0.5			
J_s	$AEff\ (S_{ABVRSS}, S_{BVSRS})$	$E\ (S_{XT})$	$E\ (S_{YT})$
(2,2)	2.6961	0	0
(1,2)(3,2)	2.4621	0	0
(1,2)(2,2)(3,2)	2.4622	0	0
(1,2)(2,1)(2,3)(3,2)	2.1417	0	0
(1,2)(1,3)(2,2)(3,1)(3,2)	2.0364	0	0
(1,2)(2,1)(2,2)(2,3)(3,2)	2.2324	0	0
Ordinary BVRSS	1.7197	0	0

Table 2.3. Asymptotic relative efficiency of ABVRSS sign test relative to BVSRS sign test.

r =3, ρ = 0.9			
J_s	$AEff\ (S_{ABVRSS}, S_{BVSRS})$	$E\ (S_{XT})$	$E\ (S_{YT})$
(2,2)	4.2800	0	0
Ordinary BVRSS	2.477	0	0

Table 2.4. Asymptotic relative efficiency of ABVRSS sign test relative to BVSRS sign test.

r =3, ρ = 0.1			
J_s	$AEff\ (S_{ABVRSS}, S_{BVSRS})$	$E\ (S_{XT})$	$E\ (S_{YT})$
(2,2)	2.3722	0	0
Ordinary BVRSS	1.2685	0	0

Moreover, Table 2.5, shows that the alternative design with labels {(2, 3), (3, 2)} in case of *r=4* has the highest $AEff\ (S_{ABVRSS}, S_{BVSRS})$ among other alternative designs with $E\ (\underset{\sim}{S}'_T) = \underset{\sim}{0}$.

Table 2.5. Asymptotic relative efficiency of ABVRSS sign test relative to BVSRS sign test.

$r = 4,\ \rho = 0.5$			
J_s	$AEff\,(S_{ABVRSS}, S_{BVSRS})$	$E\,(S_{XT})$	$E\,(S_{YT})$
(3,2)	4.0910	-0.1712	0.1712
(2,3)(3,2)	3.5565	0	0
Ordinary BVRSS	2.4561	0	0

In conclusion, it seems that, the median ranked set samples, are the optimal designs for bivariate sign tests. In general, when r is odd, the optimal design for bivariate sign test is with labels $J_s = \left\{ (\frac{r+1}{2}, \frac{r+1}{2}) \right\}$. Also, when r is even, the optimal design for bivariate sign test is with labels $J_s = \left\{ \left(\frac{r}{2},\ \frac{r}{2}+1\right),\ \left(\frac{r}{2}+1,\ \frac{r}{2}\right) \right\}$.

3. Optimal Designs Bivariate Sign Tests.

In this section we introduce the optimal bivariate ranked set sampling protocols (OBVRSS) sign test. Some theoretical results of the sign test for the one-sample bivariate location model using those (OBVRSS) designs are derived. Also, we investigate the power of the tests for those designs.

Suppose that, the hypothesis $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ versus $H_A : \underset{\sim}{\theta} \neq \underset{\sim}{0}$, where $\underset{\sim}{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ and $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is to be tested using sign test.

Case 1: Set size r is odd.

Let $\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k},\ Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} \right)$, $k = 1, 2, \ldots, n$ be OBVRSS$_O$ from $F(x,\ y)$.

Then the sign test statistic based on OBVRSS$_O$ is $\underset{\sim}{S}'_O = \left(S_{XO}, S_{YO} \right)$ where

$$S_{XO} = \sum_{k=1}^{n} \text{sgn} X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k} = 2 \sum_{k=1}^{n} I\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k} > 0\right) - n \text{ a.s and}$$

$$S_{YO} = \sum_{k=1}^{n} \operatorname{sgn} Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} = 2\sum_{k=1}^{n} I\left(Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} > 0\right) - n \text{ a.s}$$

Thus the mean and variance of $\underset{\sim}{S}'_O$ under H_0 are: $E(S_{YO}) = 0$ and $E(S_{XO}) = 0$ by Theorem 2.1. Also by Theorem 2.1, $V(S_{XO}) = n$ and $V(S_{YO}) = n$.

Moreover, $COV(S_{XO}, S_{YO}) = n\left[4P\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)} > 0,\ Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]} > 0\right) - 1\right]$.

Now, by Theorem 2.2, $\frac{1}{\sqrt{n}}\underset{\sim}{S}'_O \xrightarrow{D} N_2(\underset{\sim}{0},\ V_O)$, where,

$V_O = \begin{pmatrix} 1 & V_{xyO} \\ V_{xyO} & 1 \end{pmatrix}$, and $V_{xyO} = 4P\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k} > 0, Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} > 0\right) - 1$.

Now let $N_{00} = \# X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)} \le 0$ and $Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]} \le 0$,

$N_{01} = \# X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)} \le 0$ and $Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]} > 0$,

$N_{10} = \# X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)} > 0$ and $Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]} \le 0$,

$N_{11} = \# X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)} > 0$ and $Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]} > 0$, and $n = N = N_{00} + N_{01} + N_{10} + N_{11}$.

Using the above notation, a consistent estimator of

$P\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k} > 0,\ Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} > 0\right)$ is

$\hat{P}\left(X_{\left[\frac{r+1}{2}\right]\left(\frac{r+1}{2}\right)k} > 0,\ Y_{\left(\frac{r+1}{2}\right)\left[\frac{r+1}{2}\right]k} > 0\right) = \frac{N_{11}}{N}$. Therefore,

$$\hat{V}_O = \begin{pmatrix} 1 & \frac{4N_{11}}{N}-1 \\ \frac{4N_{11}}{N}-1 & 1 \end{pmatrix}$$ and hence $S_{OBVRSS_O} = \underset{\sim}{S}'_O (n\hat{V}_O)^{-1} \underset{\sim}{S}'_O$ is

asymptotically $\chi^2(2)$. Using simple algebra, it can be shown that S_{OBVRSS_O} can be written in simpler way as

$$S_{OBVRSS_O} = \frac{(N_{00}-N_{11})^2}{N_{00}+N_{11}} + \frac{(N_{01}-N_{10})^2}{N_{01}+N_{10}}.$$

Case 2: Set size r is even.

Let $\left\{ \left(X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)k}, Y_{\left(\frac{r}{2}+1\right)\left[\frac{r}{2}\right]k} \right), \left(X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)k}, Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]k} \right) \right\}$, $k=1, 2, ..., m$, be

OBVRSS$_E$ from $F(x, y)$. Then the sign test statistic based on OBVRSS$_E$ is

$\underset{\sim}{S}'_E = (S_{XE}, S_{YE})$ where $S_{XE} = \sum_{k=1}^{m} \text{sgn} X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)k} + \sum_{k=1}^{m} \text{sgn} X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)k}$

$$= 2\sum_{k=1}^{m} I(X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)k} > 0) - m + 2\sum_{k=1}^{m} I(X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)k} > 0) - m \quad \text{a.s}$$

$= 2\sum_{k=1}^{m} \left[I(X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)k} > 0) + I(X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)k} > 0) \right] - n$, where $n = 2m$. Also,

$$S_{YE} = \sum_{k=1}^{m} \text{sgn} Y_{\left(\frac{r}{2}+1\right)\left[\frac{r}{2}\right]k} + \sum_{k=1}^{m} \text{sgn} Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]k} \quad \text{a.s}$$

$$= 2\sum_{k=1}^{m} \left[I(Y_{\left[\left(\frac{r}{2}+1\right)\right]\left(\frac{r}{2}\right)k} > 0) + I(Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]k} > 0) \right] - n$$

Thus the mean and variance of $\underset{\sim}{S}'_E$ under H_0 are:

$E(S_{YO}) = 0$ and $E(S_{XE}) = 0$. by Theorem 2.1.

Also, $V(S_{XE}) = \sum_{k=1}^{m} V \ \text{sgn} X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)k} + \sum_{k=1}^{m} V \ \text{sgn} X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)k}$

$= 4n[1 - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0)]\ F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0)$ by Theorem 2.1. Similarly,

$V(S_{YE}) = 4n[1 - F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0)]\ F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0)$.

Moreover, using similar argument,

$COV(S_{XE}, S_{YE}) =$

$$= 2n\begin{bmatrix} P(X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)} > 0, Y_{\left(\frac{r}{2}+1\right)\left[\frac{r}{2}\right]} > 0) - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0) F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0) \\ +P(X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)} > 0, Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]} > 0) - [1 - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0)][1 - F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0)] \end{bmatrix}$$

By the Theorem 2.2, $\frac{1}{\sqrt{n}} \underset{\sim}{S}'_E \xrightarrow{D} N_2(\underset{\sim}{0}, V_E)$, where,

$$V_E = \begin{pmatrix} V_{xxE} & V_{xyE} \\ V_{xyE} & V_{yyE} \end{pmatrix},$$

$V_{xxE} = 4[1 - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0)]\ F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0),$

$V_{yyE} = 4[1 - F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0)]\ F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0),$ and

$$V_{xyE} = 2\begin{bmatrix} P(X_{\left[\frac{r}{2}+1\right]\left(\frac{r}{2}\right)} > 0, Y_{\left(\frac{r}{2}+1\right)\left[\frac{r}{2}\right]} > 0) - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0) F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0) \\ +P(X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)} > 0, Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]} > 0) - [1 - F_{X_{\left[\frac{r}{2}\right]\left(\frac{r}{2}+1\right)}}(0)][1 - F_{Y_{\left(\frac{r}{2}\right)\left[\frac{r}{2}+1\right]}}(0)] \end{bmatrix}.$$

Now define the two-way tables of counts as follows:

	$Y_{(\frac{r}{2})[\frac{r}{2}+1]}$		
$X_{[\frac{r}{2}](\frac{r}{2}+1)}$	≤ 0	*>0*	*Totals*
≤ 0	N_{100}	N_{101}	N_{10+}
>0	N_{110}	N_{111}	N_{11+}
Totals	N_{1+0}	N_{1+1}	N_1

	$Y_{(\frac{r}{2}+1)[\frac{r}{2}]}$		
$X_{[\frac{r}{2}+1](\frac{r}{2})}$	≤ 0	*>0*	*Totals*
≤ 0	N_{200}	N_{201}	N_{20+}
>0	N_{210}	N_{211}	N_{21+}
Totals	N_{2+0}	N_{2+1}	N_{2}

Also, define the following: $N_{00} = N_{100} + N_{200}$, $N_{01} = N_{101} + N_{201}$, $N_{10} = N_{110} + N_{210}$, $N_{11} = N_{111} + N_{211}$ and $n = 2m = N = N_{00} + N_{01} + N_{10} + N_{11}$, and $N = N_1 + N_2$ $(N_1 = N_2 = \frac{N}{2})$. Using the above notations a consistent estimator of V_E is $\hat{V}_E = \begin{pmatrix} \hat{V}_{xxE} & \hat{V}_{xyE} \\ \hat{V}_{xyE} & \hat{V}_{yyE} \end{pmatrix}$, where $\hat{V}_{xxE} = 16\frac{N_{11+}}{N}\frac{N_{10+}}{N}$, $\hat{V}_{yyE} = 16\frac{N_{1+1}}{N}\frac{N_{1+0}}{N}$, and $\hat{V}_{xyE} = 4\left[\frac{NN_{11} - 2N_{10+}N_{1+0} - 2N_{11+}N_{1+1}}{N^2}\right]$.

Therefore, $S_{\mathrm{OBVRSS_E}} = \underset{\sim}{S}'_E (n\hat{V}_E)^{-1} S_E$ is asymptotically $\chi^2(2)$. It is straight forward to show that $S_{\mathrm{OBVRSS_E}} = \frac{(N_{00} - N_{11})^2}{N_{00} + N_{11}} + \frac{(N_{01} - N_{10})^2}{N_{01} + N_{10}}$.

3.1 Simulation study

Bivariate normal distribution is used to investigate the power performance of OBVRSS sign test comparing with BVSRS sign test. Different values of ρ and θ (shifted parameter) were used. In particular, θ=0.0, 0.10, 0.15, 0.20, 0.25, 0.30, 0.35, r=3, 4, 5, m=36, 45, 100, 125 and $\rho = 0$, ± 0.2, ± 0.5, ± 0.9, were considered. Moreover, Plackett's class of bivariate distribution with fixed marginal distribution functions $F(x)$

and $G(y)$ are used to investigate the power performance of OBVRSS sign test comparing with BVSRS sign test. The Plackett's joint cdf is given by

$$H(x, y) = \begin{cases} \dfrac{S(x, y) - \left[S^2(x, y) - 4\psi(\psi - 1)F(x)G(y)\right]^{1/2}}{2(\psi - 1)} & \text{if } \psi \neq 1 \\ F(x)G(y) & \text{if } \psi = 1 \end{cases},$$

where $S(x, y) = 1 + (\psi - 1)[F(x) + G(y)]$ and the parameter ψ governs the dependence between *X* and *Y*. The reason for choosing this class of bivariate distributions is that it covers the full range of dependency. For more detailed description of Plackett's distribution, see Johnson (1987). Please note that we presenting some of our simulation in the paper to make it shorter. The other tables of simulation gave similar results.

Table 3.1 Power estimation of bivariate sign test for Normal distribution using (OBVRSS) and BVSRS (*m*=36, *r*=3)

ρ \ θ	0.0	0.10	0.15	0.20	0.25	0.30	0.35
0.0	0.0494 (0.0475)	0.0488 (0.0521)	0.0711 (0.1013)	0.1001 (0.1975)	0.1584 (0.3400)	0.2388 (0.5126)	0.3403 (0.6936)
0.2	0.0431 (0.0472)	0.053 (0.0608)	0.061 (0.1021)	0.0927 (0.1910)	0.1525 (0.3363)	0.2215 (0.5079)	0.3084 (0.6653)
-0.2	0.0434 (0.0467)	0.0454 (0.0558)	0.0641 (0.1089)	0.1105 (0.2032)	0.172 (0.3511)	0.2628 (0.5451)	0.3827 (0.7269)
0.5	0.0459 (0.0491)	0.0460 (0.0541)	0.0589 (0.1041)	0.0869 (0.1879)	0.1374 (0.3393)	0.1906 (0.5016)	0.2681 (0.6853)
-0.5	0.0433 (0.0462)	0.0472 (0.0600)	0.0699 (0.1235)	0.1188 (0.2447)	0.2120 (0.4412)	0.3183 (0.6369)	0.4661 (0.8139)
0.9	0.0389 (0.0477)	0.0405 (0.0578)	0.0490 (0.1247)	0.0725 (0.2416)	0.1097 (0.4094)	0.1615 (0.6087)	0.2218 (0.7834)
-0.9	0.0402 (0.0435)	0.0425 (0.0755)	0.0824 (0.2228)	0.1685 (0.5255)	0.3304 (0.7963)	0.5257 (0.9479)	0.7051 (0.9938)

Our simulation study confirmed the results obtained in Section 2. In particular, Table 3.1 to Table 3.3 indicate that using OBVRSS for bivariate sign tests is strictly more powerful than using BVSRS for all selected set size *r*, cycle *m* and all values of ρ and θ, when the underlying distribution is assumed to be Normal. Similar conclusion can be drawn when the underlying distribution is Plackett's distribution with uniform and exponential marginal for all values of the dependency parameter ψ . Also, from Table 3.1 to Table 3.3, it is clear that the power of the sign test increases as $\rho,\ \theta,\ m$ and r increase. However, in some cases the increasing pattern of the power is not clear, that may be due to simulation fluctuation, because we used different simulated data sets at each value of the given parameters.

Table 3.2 Power estimation of bivariate sign test for Normal distribution using (OBVRSS) and BVSRS (m=100, r=4)

ρ \ θ	0.0	0.10	0.15	0.20	0.25	0.30	0.35
0.0	0.0508 (0.0305)	0.1498 (0.3211)	0.2993 (0.6520)	0.4957 (0.9043)	0.7078 (0.9858)	0.8569 (0.9987)	0.9405 (0.9998)
0.2	0.0503 (0.0357)	0.1360 (0.3198)	0.2687 (0.6631)	0.4518 (0.8976)	0.6522 (0.9860)	0.8031 (0.9988)	0.9154 (1.0000)
-0.2	0.0478 (0.0315)	0.1707 (0.3235)	0.3436 (0.6701)	0.5542 (0.9054)	0.7614 (0.9855)	0.8982 (0.9990)	0.9672 (1.0000)
0.5	0.0464 (0.0376)	0.1249 (0.3454)	0. 2296 (0.6845)	0.3882 (0.9126)	0.5706 (0.9897)	0.7354 (0.9993)	0.8573 (1.0000)
-0.5	0.0502 (0.0242)	0.0873 (0.0962)	0.2047 (0.3541)	0.4326 (0.7103)	0.6719 (0.9345)	0.8656 (0.9925)	0.9914 (1.0000)
0.9	0.0462 (0.0420)	0.1021 (0.4444)	0.1856 (0.8031)	0.3070 (0.9711)	0.4568 (0.9985)	0.6207 (1.0000)	0.7620 (1.0000)
-0.9	0.0470 (0.0173)	0.4216 (0.5412)	0.7700 (0.9176)	0.9552 (0.9963)	0.9967 (0.9999)	0.9997 (1.0000)	1.0000 (1.0000)

Table 3.3 Power estimation of bivariate sign test for Normal distribution using (OBVRSS) and BVSRS (m=100, r=5)

ρ \ θ	0.0	0.10	0.15	0.20	0.25	0.30	0.35
0.0	0.0499 (0.0489)	0.0684 (0.1243)	0.1360 (0.4182)	0.2751 (0.7866)	0.4727 (0.9617)	0.6796 (0.9972)	0.8372 (0.9999)
0.2	0.0475 (0.0507)	0.0624 (0.1199)	0.1279 (0.4121)	0.2521 (0.7680)	0.4176 (0.9619)	0.6069 (0.9974)	0.7889 (0.9999)
-0.2	0.0537 (0.0454)	0.0685 (0.1237)	0.1503 (0.4307)	0.3048 (0.8016)	0.5199 (0.9693)	0.7354 (0.9981)	0.8878 (1.0000)
0.5	0.0467 (0.0513)	0.065 (0.1317)	0.1221 (0.4252)	0. 2125 (0.8033)	0.3635 (0.9716)	0.5416 (0.9982)	0.7165 (1.0000)
-0.5	0.0469 (0.0487)	0.7480 (0.1446)	0.1885 (0.5156)	0.3892 (0.8785)	0.6366 (0.9896)	0.8411 (1.0000)	0.9523 (1.0000)
0.9	0.04673 (0.0485)	0.05060 (0.1785)	0.0987 (0.6088)	0.1731 (0.9333)	0.2925 (0.9972)	0.4395 (0.9999)	0.5974 (1.0000)
-0.9	0.0438 (0.0486)	0.1043 (0.3021)	0.3519 (0.8933)	0.7233 (0.9986)	0.9385 (1.0000)	0.9923 (1.0000)	0.9996 (1.0000)

Plackett's distribution with dependency parameter ψ=0.5, 1, 2 are considered. Also, different types of marginal such as uniform (0, 1) and exponential with mean 2 are used for different value of *m* and *r.*

Table 3.4 Power estimation of bivariate sign test for Plackett's distribution with exponential marginal using (OBVRSS) and BVSRS(*m*=36*, r*=3)

θ \ ψ	0.5	1.0	2.0
0.00	0.0432 (0.0452)	0.0400 (0.0416)	0.0436 (0.0394)
0.05	0.0628 (0.0762)	0.0604 (0.0750)	0.0552 (0.0740)
0.10	0.1144 (0.2036)	0.1128 (0.1876)	0.0874 (0.1778)
0.15	0.2238 (0.4660)	0.1874 (0.3998)	0.1596 (0.3768)
0.20	0.4026 (0.7312)	0.3308 (0.6758)	0.2948 (0.6486)
0.25	0.6270 (0.9300)	0.5456 (0.8888)	0.4710 (0.8748)
0.30	0.8246 (0.9896)	0.7452 (0.9810)	0.6758 (0.9714)
0.35	0.9404 (0.9996)	0.8910 (0.9980)	0.8476 (0.9986)

Table 3.5 Power estimation of bivariate sign test for Plackett's distribution with uniform marginal using (OBVRSS) and BVSRS(*m*=36*, r*=3)

θ \ ψ	0.5	1.0	2.0
0.00	0.0452 (0.0446)	0.0412 (0.0418)	0.0448 (0.0382)
0.05	0.1028 (0.1842)	0.0966 (0.1632)	0.0874 (0.1624)
0.10	0.3246 (0.6410)	0.2774 (0.5940)	0.2398 (0.5438)
0.15	0.6708 (0.9518)	0.6012 (0.9216)	0.5096 (0.9068)
0.20	0.9216 (0.9980)	0.8572 (0.9970)	0.8030 (0.9942)
0.25	0.9902 (1.0000)	0.9800 (1.0000)	0.9578 (1.0000)
0.30	0.9994 (1.0000)	0.9941 (1.0000)	0.9946 (1.0000)
0.35	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)

Table 3.6 Power estimation of bivariate sign test for Plackett's distribution with uniform marginal using (OBVRSS) and BVSRS(*m*=100*, r*=4)

θ \ ψ	0.5	1.0	2.0
0.00	0.0485 (0.0297)	0.0468 (0.0333)	0.0503 (0.0344)
0.05	0.2614 (0.5092)	0.2154 (0.4918)	0.1890 (0.4897)
0.10	0.7851 (0.9920)	0.7142 (0.9850)	0.6414 (0.9879)
0.15	0.9920 (1.0000)	0.9768 (1.0000)	0.9544 (1.0000)
0.20	1.0000 (1.0000)	0.9997 (1.0000)	0.9988 (1.0000)
0.25	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.30	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)
0.35	1.0000 (1.0000)	1.0000 (1.0000)	1.0000 (1.0000)

Table 3.7 Power estimation of bivariate sign test for Plackett's distribution with exponential and uniform marginal using (OBVRSS) and BVSRS(m=36, r=3)

θ \ ψ	0.5	1.0	2.0
0.00	0.0452 (0.0392)	0.0412 (0.0414)	0.0450 (0.0414)
0.05	0.0740 (0.1550)	0.0764 (0.1178)	0.0826 (0.1328)
0.10	0.1568 (0.3812)	0.1870 (0.3802)	0.2150 (0.4308)
0.15	0.3632 (0.7378)	0.3950 (0.7494)	0.4552 (0.8174)
0.20	0.6162 (0.9460)	0.6704 (0.9592)	0.7322 (0.9798)
0.25	0.8366 (0.9970)	0.8790 (0.9970)	0.9296 (0.9986)
0.30	0.9612 (1.0000)	0.9754 (1.0000)	0.9984 (1.0000)
0.35	0.9962 (1.0000)	0.9986 (1.0000)	0.9994 (1.0000)

Moreover, when the underlying distribution is the Plackett's distribution, the power of OBVRSS sign test increases as m and r increase. However, there is no clear pattern that the power of the bivariate sign test changes as ψ changes. Also, our simulation indicates that the power of OBVRSS sign test, when the underlying distribution is Plackett's distribution is higher than when the underlying distribution is Normal. Furthermore, the bivariate sign test is the highest when the underlying distribution is bivariate uniform.

Furthermore, using computer simulation we found that when n =60, the critical values of our S_{OBVRSS} are very close to asymptotic critical value. Thus we suggest to use least n=60 to make sure that our asymptotic results are valid.

4. Illustration Using Real Data and Bootstrap Algorithms

In this chapter real data set is used from the Rural Health Study to illustrate the proposed optimal alternative BVRSS designs for sign test. Bootstrap approach is used to estimate the P-value of the bivariate sign tests. Final comments and suggestions are provided.

4.1 Illustration

4.1.1 Data description

The Rural Health study is a prospective longitudinal cohort study, of 8 years follow-up from 1981 to 1989, of 3673 individual (1420 men and 2253 women) aged 65 or older living in Washington and Iowa countries of the state of Iowa. This study is one of four study supported by the National institute on aging and collectively referred to as EPESE, Established Populations for Epidemiological Studies of the Elderly, see Rubstein and Lemke (1993) and Brock et al. (1986). Due to missing observations the cohort (population) size was reduced to 3062, with 1141 males and 1921 females.

Table 4.1 The selected samples

OBVRSS Sample of (r=3 and m=30)					BVSRS of size n=30			
Female		Male			Female		Male	
Age	Survival Time	Age	Survival Time		Age	Survival Time	Age	Survival Time
70	90	75	90		87	40	76	87
78	89	76	88		78	90	70	90
68	89	57	89		66	14	70	89
74	89	73	90		69	89	69	89
81	89	74	89		81	89	78	89
79	89	78	31		74	89	83	40
73	89	78	82		73	89	77	89
69	88	79	36		71	90	70	89
74	87	78	88		79	64	80	74
69	83	69	85		75	45	68	88
74	89	70	90		74	89	68	85
73	90	79	89		67	90	71	42
81	89	77	68		84	39	67	89
72	88	72	88		70	89	70	90
72	89	69	89		76	88	68	84
72	89	75	81		74	89	70	64
80	80	68	63		87	63	78	51
76	90	77	90		72	66	73	14
75	89	74	23		80	92	78	31
74	89	75	90		73	89	80	90
71	89	73	89		67	90	76	22
71	89	70	90		68	89	70	90
72	98	70	89		76	89	67	91
71	89	69	50		71	89	69	60
74	89	73	89		66	89	73	89
67	89	68	90		92	71	72	32
75	89	71	67		69	90	67	34
67	84	76	74		90	70	71	60
71	90	78	90		79	90	73	90
74	88	69	88		66	88	77	89

In this illustration OBVRSS with m=30 and r=3 and BVSRS with n=30, are selected randomly from each population (males and females). For OBVRSS, ranking was conducted on the age variable at baseline. At the first stage median RSS, based on age, is selected. Then at the end of the study, the ranking was conducted on the number of months that the first stage selected sample members were survived. At the second stage, we quantified only the median RSS, based on survival time, in order to obtain the OBVRSS of size n=30 (m=30, r=3). Table 4.1 contains the selected OBVRSS and BVSRS samples.

4.1.2 Bootstrap algorithm for estimating the P-value of the sign tests

When the sample size is small, the distribution of the test bivariate sign test statistic, say S_A, is complicated and then can not be found explicitly. Thus, the exact P-value calculation for sample size n<60 is not feasible without knowing the underlying distribution. In this section we introduce a simple bootstrap method for calculating the P-value of the sign test for any given bivariate data. For general description of the bootstrap method of estimation see Efron and Tibshirani (1993).

Suppose that a bivariate random sample is drawn from a population using the OBVRSS or BVSRS sampling method. The bootstrap algorithm for approximating the bootstrap P-value of the test for testing the hypothesis

$H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ versus $H_A : \underset{\sim}{\theta} \neq \underset{\sim}{0}$, where $\underset{\sim}{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ and $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is as follows:

1) Calculate the sample test statistic (say S_A), based on the original sample.
2) Estimate $\underset{\sim}{\theta}$ from the data; say $\hat{\underset{\sim}{\theta}}$. Shift (X_i, Y_i) to $(X_i - \hat{\theta}_1, Y_i - \hat{\theta}_2)$, *i=1, 2, ..., n.*
3) Define $\hat{F}(x, y)$ by placing a mass probability $p_i = \dfrac{1}{n}$ on $(X_i,\ Y_i),\ i = 1,\ 2,\ ...,\ n$.
4) Generate a resample $(X_i^*,\ Y_i^*),\ i = 1,\ 2,\ ...,\ n$ from $\hat{F}(x, y)$.
5) calculate $S_{Ab}^*,\ \ b = 1, 2,\ ..., B$ based on the sample of step 4.

Repeat steps 3, 4 and 5 B times.

Then the bootstrap P-value, $P^* = P(S_A^* \geq S_A \mid \hat{F}(x, y))$, can be approximated by

$P^* = \frac{1}{B}\sum_{b=1}^{B} I(S_{Ab}^* \geq S_A)$. Note that the above algorithm where used when r is odd.

Similar procedure can be used to get an estimate of the P-value using Bootstrap algorithm when r is even, as follow:

Since $OBVRSS_E$ sample has two different i.i.d sample each of size m, therefore, in step 3 we define $\hat{F}_1(x, y)$ using the mass probability $p_{1i} = \frac{1}{m}$ for the first sample and $\hat{F}_2(x, y)$ using the mass probability $p_{2i} = \frac{1}{m}$ for the second sample. Also, in step 4 we generate independently $(X_{1i}^*, Y_{1i}^*),\ i = 1, 2, ..., \mathrm{m}$ and $(X_{2i}^*, Y_{2i}^*),\ i = 1, 2, ..., \mathrm{m}$ from $\hat{F}_1(x, y)$ and $\hat{F}_2(x, y)$ respectively.

4.1.3 Results of the illustration

Table 4.2 summarizes the results of bivariate sign tests based on the selected samples from Table 4.1. Our objective is test the hypotheses $H_0: \theta_{Age} = 73, \theta_{Survival} = 74$ for the males samples and $H_0: \theta_{Age} = 74, \theta_{Survival} = 89$ for the females samples. Bootstrap method is used to estimate the P-value of the bivariate sign tests based on B=5000 iterations. Also, the Bootstrap mean square error is calculated based on 5000 iterations.

Table 4.2 Summary of the results for the bivariate sign tests

Sample		Test Statistics	Bootstrap P-value	Bootstrap MSE
BVSRS	Males	04.140	0.407	0.000233
	Females	10.269	0.141	0.000130
OBVRSS	Males	19.180	0.0013	0.000013
	Females	16.886	0.0047	0.000046

Clearly, Table 4.2 shows that the above hypotheses are failed to be rejected for the BVSRS. However, those hypotheses are to be rejected for the OBVRSS. That is may be due to the fact that OBVRSS sign test is strictly more powerful than BVSRS sign test.

4.2 Comments and suggestions for future studies

Based on our theoretical and numerical results, we recommend using our proposed OBVRSS sampling design for the bivariate sign tests for one-sample location model. Our proposed designs are more powerful than BVSRS and BVRSS designs for the bivariate sign test. Also, our proposed optimal designs are easy to calculate using the same simple form as in BVSRS.

Moreover, since our calculation indicated that there are other optimal designs with higher Pitman relative efficiency than our proposed designs; however, their expectations are not equal to zero. We suggest further investigation to find similar sign test with zero expectation for those designs.

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