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Abstract

A set of vectors is k-independent if all its subsets with no more than k elements are linearly independent. We obtain a result concerning the maximal possible cardinality $Ind_q(n,k)$ of a k-independent set of vectors in the n-dimensional vector space F_q^n over the finite field F_q of order q. Namely, we give a necessary and sufficient condition for $Ind_q(n,k) = n+1$.

1 Introduction

For q a prime power, let F_q denote the finite field of order q, and let F_q^n denote the n-dimensional vector space of all n-tuples over F_q. For an integer k, with $1 \leq k \leq n$, we say that a set of vectors $A \subseteq F_q^n$ is **k-independent** if all its subsets with at most k elements are linearly independent. We are interested in the maximal possible cardinality, $Ind_q(n,k)$, of a k-independent subset of F_q^n. It is not hard to see that we have

$$q^n - 1 = Ind_q(n,1) \geq Ind_q(n,2) \geq \cdots \geq Ind_q(n,n) \geq n+1. \qquad (1.1)$$

Indeed, any set of nonzero vectors is 1-independent; $(k+1)$-independence implies k-independence; and finally, the $(n+1)$-element set consisting of the standard basis plus the "all-ones" vector is clearly n-independent.

The first inequality in (1.1) becomes an equality when $q = 2$, for over F_2, 2-independence is equivalent to 1-independence. The general formula

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for $Ind_q(n,2)$ given in the observation below follows from the fact that two (nonzero) vectors are linearly independent if and only if neither is a scalar multiple of the other.

Observation 1. *Let q be a prime power, and $n \geq 1$ an integer. Then*

$$Ind_q(n,2) = \frac{q^n - 1}{q - 1}. \tag{1.2}$$

In [1], the authors investigated formulae for $Ind_2(n,k)$ in two extreme cases: the well known cases when $k \leq 3$ and the cases when $k \geq 2n/3$. The results from [1] are stated in the theorem below, where m and n are positive integers.

Theorem 1. *The following formulae hold:*

(a)

$$Ind_2(n,3) = 2^{n-1}, \;\; \textit{for} \;\; n \geq 3. \tag{1.3}$$

(b)

$$Ind_2(n, n-m) = n+1, \;\; \textit{for} \;\; n \geq 3m+2, \; m \geq 0. \tag{1.4}$$

(c)

$$Ind_2(n, n-m) = n+2, \;\; \textit{for} \;\; n = 3m+i, \; i = 0,1, \; m \geq 2. \tag{1.5}$$

In this paper we generalize the result stated in part (b) of Theorem 1. We present a simple condition on q, n and k which is both necessary and sufficient for $Ind_q(n,k) = n+1$ to hold. Here is our main result.

Theorem 2. *Let q be a prime power, and let k and n be integers with $2 \leq k \leq n$. Then $Ind_q(n,k) = n+1$ if and only if*

$$\frac{q}{q+1}(n+1) \leq k.$$

In particular, in the case $q = 2$, Theorem 2 says that the inequality in Theorem 1 (b) is not only sufficient, but also necessary.

Note also that with q and n fixed, our current result in particular evaluates $Ind_q(n,k)$ for the top $\lfloor (n-1)/(q+1) \rfloor$ values of k, where $\lfloor \; \rfloor$ denotes the floor, or the largest-integer, function. In particular, when $q = 2$, our result evaluates $Ind_q(n,k)$ for all values of k in the range $(2n+2)/3 \leq k \leq n$.

2 k-Extensions and k-Completions

Clearly, when calculating $Ind_q(n,k)$, we can restrict our attention to **maximal** k-independent sets; i.e. those k-independent sets that don't have proper supersets that are still k-independent.

Observation 2. *Every maximal k-independent set contains a basis of F_q^n over F_q.*

Proof. For $X \subseteq F_q^n$, we use $span\,(X)$ to denote the linear subspace generated by X. If $A \subseteq F_q^n$ is maximal k-independent then every element of F_q^n is a linear combination of (less than k) elements of A; i.e. $span\,(A) = F_q^n$. Consider a maximal linearly independent $B \subseteq A$. If follows (by maximality of B) that $A \subseteq span\,(B)$, and therefore $F_q^n = span\,(A) \subseteq span\,(B)$; i.e. B is a basis of F_q^n. □

Since k-independence is preserved by automorphisms of F_q^n, in the light of Observation 2, while studying $Ind_q(n,k)$ we can restrict our attention even further, namely to the supersets of the standard basis, which we denote by $\mathcal{B}$. We shall say that a set $W \subseteq F_q^n$ is a k**-extension** (of $\mathcal{B}$) if W is disjoint from $\mathcal{B}$, and $W \cup \mathcal{B}$ is k-independent; if $W \cup \mathcal{B}$ is *maximal* k-independent then W will be called a k**-completion** (of $\mathcal{B}$). Let $Cpl_q(n,k)$ denote the maximal possible cardinality of a k-completion in F_q^n. The above remarks imply that

$$Ind_q(n,k) = n + Cpl_q(n,k). \tag{2.1}$$

Theorem 2 determines exactly for which q, n and k, $Cpl_q(n,k) = 1$; i.e. for which q, n and k, singletons are the only possible nonempty k-extensions of the standard basis $\mathcal{B}$, and therefore, the only possible k-completions in F_q^n.

3 The Proof of The Main Result

Throughout this section q is a prime power, and n and k are integers with $2 \le k \le n$. We begin by introducing more notation.

The cardinality of a set X will be denoted by $|X|$. For $\mathbf{a} \in F_q^n$, we define the *support* of $\mathbf{a}$, written supp($\mathbf{a}$), by

$$\mathrm{supp}(\mathbf{a}) = \{i : a_i \neq 0, i = 1, \dots, n\},$$

where $\mathbf{a} = (a_1, \dots, a_n)$. We will write $\|\mathbf{a}\|$ for $|\mathrm{supp}(\mathbf{a})|$. (Note that $\| \; \| : F_q^n \to R^+$ satisfies the usual norm conditions, where the absolute value is replaced by the trivial valuation on F_q. In particular, $\|\alpha\mathbf{a}\| = \|\mathbf{a}\|$, for every $\alpha \in F_q^*$.)

Our first lemma gives a characterization of k-extensions in terms of $\| \; \|$.

Lemma 1. *Suppose that $W \neq \emptyset$ is disjoint from the standard basis $\mathcal{B}$. Then W is a k-extension if and only if for every nonempty $U \subseteq W$ and $\{\alpha_{\mathbf{u}} : \mathbf{u} \in U\} \subseteq F_q^*$, we have*

$$\left\| \sum_{\mathbf{u} \in U} \alpha_{\mathbf{u}} \mathbf{u} \right\| > k - |U|.$$

Proof. Suppose first that W is a k-extension, and let

$$\mathbf{w} = \sum_{\mathbf{u} \in U} \alpha_{\mathbf{u}} \mathbf{u}$$

be as above. By expanding $\mathbf{w}$ in the standard basis we get

$$\mathbf{w} = \sum_{\mathbf{u} \in U} \alpha_{\mathbf{u}} \mathbf{u} = \sum_{\mathbf{v} \in C} \beta_{\mathbf{v}} \mathbf{v}$$

for some $C \subseteq \mathcal{B}$, with $|C| = \|\mathbf{w}\|$, and $\beta_{\mathbf{v}} \in F_q^*$, for $\mathbf{v} \in C$. It follows that $U \cup C$ is a linearly dependent subset of the k-independent set $W \cup \mathcal{B}$, and therefore its cardinality $|U \cup C| = |U| + \|\mathbf{w}\|$ must be greater than k; i.e. $||\mathbf{w}|| > k - |U|$, as required.

Next, suppose that W is not a k-extension, (i.e. $W \cup \mathcal{B}$ is not k-independent). Then for some $U \subseteq W, C \subseteq \mathcal{B}$ with $|U| + |C| \leq k$, and some $\alpha_{\mathbf{u}}, \beta_{\mathbf{v}} \in F_q^*$, for $\mathbf{u} \in U$ and $\mathbf{v} \in C$, we have

$$\sum_{\mathbf{u} \in U} \alpha_{\mathbf{u}} \mathbf{u} + \sum_{\mathbf{v} \in C} \beta_{\mathbf{v}} \mathbf{v} = \mathbf{0}.$$

In particular,

$$\left\|\sum_{\mathbf{u}\in U}\alpha_{\mathbf{u}}\mathbf{u}\right\| = \left\|-\sum_{\mathbf{v}\in C}\beta_{\mathbf{v}}\mathbf{v}\right\| = |C| \le k - |U|. \qquad \square$$

Lemma 1 will be used in the proof of Theorem 2 through the following corollary.

Corollary 1.

(a) *If W is a k-extension then $\|\mathbf{a}\| \ge k$, for every $\mathbf{a} \in W$.*

(b) *A singleton $\{\mathbf{a}\} \subseteq F_q^n - \mathcal{B}$ is a k-extension if and only if $\|\mathbf{a}\| \ge k$.*

(c) *Suppose $\mathbf{a}, \mathbf{b} \in F_q^n - \mathcal{B}$ are distinct. Then $\{\mathbf{a}, \mathbf{b}\}$ is a k-extension if and only if $\|\mathbf{a}\|, \|\mathbf{b}\| \ge k$ and $\|\alpha\mathbf{a} + \beta\mathbf{b}\| \ge k-1$, for all $\alpha, \beta \in F_q^*$.*

(d) *Suppose $W \subseteq F_q^n - \mathcal{B}$ consists of vectors with pairwise disjoint supports. Then W is a k-extension if and only if $\|\mathbf{a}\| \ge k$, for every $\mathbf{a} \in W$.*

Proof. The proofs of parts (a), (b), and (c) are straightforward from Lemma 1. In proving part (d) we use the fact that if U consists of vectors with pairwise disjoint supports then for every $\{\alpha_{\mathbf{u}} : \mathbf{u} \in U\} \subseteq F_q^*$

$$\left\|\sum_{\mathbf{u}\in U}\alpha_{\mathbf{u}}\mathbf{u}\right\| = \sum_{\mathbf{u}\in U}\|\alpha_{\mathbf{u}}\mathbf{u}\| = \sum_{\mathbf{u}\in U}\|\mathbf{u}\|. \qquad \square$$

One consequence of Corollary 1, stated in the next observation, is a slight improvement on the lower bound on $Ind_q(n,k)$ given in the introduction ($Ind_q(n,k) \ge n+1$). Recall that $\lfloor\ \rfloor$ denotes the floor function.

Observation 3. $Ind_q(n,k) \ge n + \lfloor n/k \rfloor$.

Proof. Let $m = \lfloor n/k \rfloor$. Partition the set $\{1, \ldots, km\}$ into k-element subsets $A_1, \ldots, A_m$. For each $i = 1, \ldots, m$, let $\mathbf{a}_i$ be any vector with $\text{supp}(\mathbf{a}_i) = A_i$. The set $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is a k-extension by Corollary 1(d). $\square$

Next, in connection with Corollary 1(c), we are going to take a closer look at $\|\alpha\mathbf{a}+\beta\mathbf{b}\|$, for $\mathbf{a},\mathbf{b}\in F_q^n$ and $\alpha,\beta\in F_q^*$. Let A and B denote the support of $\mathbf{a}=(a_1,\ldots,a_n)$ and $\mathbf{b}=(b_1,\ldots,b_n)$. For any $\xi\in F_q^*$ we define $R_\xi(\mathbf{a},\mathbf{b}):=\{i\in A\cap B : a_i/b_i=\xi\}$. It is not hard to see that the support of $\alpha\mathbf{a}+\beta\mathbf{b}$ equals $A\cup B-R_{-\beta/\alpha}(\mathbf{a},\mathbf{b})$. In particular,

$$\|\alpha\mathbf{a}+\beta\mathbf{b}\| = |A\cup B| - \left|R_{-\beta/\alpha}(\mathbf{a},\mathbf{b})\right|.$$

So if $\mu(\mathbf{a},\mathbf{b})=\max_{\xi\in F_q^*}|R_\xi(\mathbf{a},\mathbf{b})|$ then we have

$$\min_{\alpha,\beta\in F_q^*}\|\alpha\mathbf{a}+\beta\mathbf{b}\| = |A\cup B|-\mu(\mathbf{a},\mathbf{b}). \tag{3.1}$$

Note also that

$$|A\cap B|\le (q-1)\mu(\mathbf{a},\mathbf{b}). \tag{3.2}$$

Indeed,with $R_\xi=R_\xi(\mathbf{a},\mathbf{b})$ we have

$$|A\cap B| = \left|\bigcup_{\xi\in F_q^*}R_\xi\right| = \sum_{\xi\in F_q^*}|R_\xi| \le (q-1)\max_{\xi\in F_q^*}|R_\xi|.$$

Lemma 2. *Suppose $\mathbf{a},\mathbf{b}\in F_q^n$ are distinct, and let A and B denote the support of $\mathbf{a}$ and $\mathbf{b}$, respectively.*

(a) *If $\|\mathbf{a}\|,\|\mathbf{b}\|\ge k$ then $\{\mathbf{a},\mathbf{b}\}$ is a k-extension iff*

$$\mu(\mathbf{a},\mathbf{b})\le |A\cup B|-k+1. \tag{3.3}$$

(b) *If $\{\mathbf{a},\mathbf{b}\}$ is a k-extension then*

$$2k-n\le |A\cap B|\le (q-1)(n-k+1).$$

Proof. Part (a) follows from Corollary 1(c), since by (3.1) the inequality (3.3) is equivalent to $\min_{\alpha,\beta\in F_q^*}\|\alpha\mathbf{a}+\beta\mathbf{b}\|\ge k-1$.

For part (b), by Corollary 1(a), we have $|A|,|B|\ge k$, and so the first inequality follows because

$$|A|+|B|-|A\cap B| = |A\cup B|\le n.$$

The second inequality follows from (3.2) and part (a) of this lemma:

$$|A\cap B|\le (q-1)\mu(\mathbf{a},\mathbf{b})\le (q-1)(|A\cup B|-k+1). \qquad \square$$

Corollary 2. *If $Ind_q(n,k) \geq n+2$, then $q-1 \geq \frac{2k-n}{n-k+1}$.*

Proof. Suppose that $Ind_q(n,k) \geq n+2$; i.e $Cpl_q(n,k) \geq 2$ (cf. 2.1). Then there are $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a}, \mathbf{b}\}$ is a two-element k-extension. But then by Lemma 2(b),

$$q-1 \geq \frac{|\text{supp}(\mathbf{a}) \cap \text{supp}(\mathbf{b})|}{n-k+1} \geq \frac{2k-n}{n-k+1}. \qquad \square$$

In the proof of our last lemma we shall need a basic combinatorial observation. Suppose X and Y are finite sets with $Y \neq \emptyset$. By a partition of X indexed by the elements of Y we mean any family $\{X_y : y \in Y\}$ of subsets of X such that the union of the family equals X, and its members are pairwise disjoint, with some of them possibly empty. Below, $\lceil\ \rceil$ denotes the ceiling function ($\lceil x \rceil$ is the smallest integer not smaller than x.)

Observation 4. *For any finite sets X, Y, with $Y \neq \emptyset$, there is a partition of X indexed by the elements of Y so that $max_{y \in Y} |X_y| \leq \lceil |X|/|Y| \rceil$.*

Lemma 3. *Suppose that r and s are positive integers with $r, s \leq n \leq r+s$. Then there exist distinct $\mathbf{a}, \mathbf{b} \in F_q^n$ such that*

(a) $\|a\| = r,\ \|b\| = s,$

(b) $\mu(\mathbf{a}, \mathbf{b}) \leq\ \lceil (r+s-n)/(q-1) \rceil,$

(c) $|\text{supp}(\mathbf{a}) \cup \text{supp}(\mathbf{b})| = n.$

Proof. Let $\mathbf{a} = (1, \ldots, 1, 0, \ldots, 0)$, with $\|\mathbf{a}\| = r$. Let X be the $(r+s-n)$-element set $\{n-s+1, \ldots, r\}$. Let $\left\{X_\beta : \beta \in F_q^*\right\}$ be a partition of X such that $max_{\beta \in F_q^*} |X_\beta| \leq \lceil (r+s-n)/(q-1) \rceil$ (cf. Observation 4). We define $\mathbf{b} = (b_1, \ldots, b_n)$ by

$$b_i = \begin{cases} 0 & \text{if } i \leq n-s \\ \beta & \text{if } i \in X_\beta \\ 1 & \text{if } i > r. \end{cases}$$

It is clear that $\|\mathbf{b}\| = s$. Also,

$$\mu(\mathbf{a}, \mathbf{b}) = max_{\beta \in F_q^*} |X_\beta| \leq \lceil (r + s - n)/(q - 1) \rceil .$$

To see that condition (c) holds note that $\text{supp}(\mathbf{a}) \cap \text{supp}(\mathbf{b}) = X$, and so $|\text{supp}(\mathbf{a}) \cup \text{supp}(\mathbf{b})| = \|\mathbf{a}\| + \|\mathbf{b}\| - |X| = r + s - (r + s - n) = n$. □

Proof of Theorem 2: Note that the condition $k \geq \frac{q}{q+1}(n+1)$ is equivalent to

$$q < \frac{2k - n}{n - k + 1} + 1. \tag{3.4}$$

If (3.4) holds then Corollary 2 implies that $Ind_q(n, k) \leq n+1$; i.e. $Ind_q(n, k) = n + 1$ (see the remark preceeding Observation 3).

Now suppose that (3.4) does not hold. Note that this implies

$$\left\lceil \frac{2k - n}{q - 1} \right\rceil \leq n - k + 1. \tag{3.5}$$

We will show that $Ind_q(n, k) \geq n+2$. By Observation 3 this is true if $2k \leq n$. Suppose then that $2k > n$. Using Lemma 3 with $r = s = k$, there exist distinct $\mathbf{a}, \mathbf{b} \in F_q^n$ such that $\|\mathbf{a}\| = \|\mathbf{b}\| = k$, $\mu(\mathbf{a}, \mathbf{b}) \leq \lceil (2k - n)/(q - 1) \rceil$, and $|\text{supp}(\mathbf{a}) \cup \text{supp}(\mathbf{b})| = n$. To complete the proof it is enough to show that $\{\mathbf{a}, \mathbf{b}\}$ is a k-extension. The latter follows from Lemma 2(a) since by (3.5) and the properties of $\mathbf{a}$ and $\mathbf{b}$ above we have

$$\mu(\mathbf{a}, \mathbf{b}) \leq n - k + 1 = |\text{supp}(\mathbf{a}) \cup \text{supp}(\mathbf{b})| - k + 1. \qquad \square$$

4 An application to sets of k-orthogonal hypercubes

In [1, Section 3], numerous applications of Theorem 1 were given related to the construction of hypercubes and orthogonal arrays, pseudo (t, m, s)-nets, and linear codes. We refer the reader to the paper [1] and the references cited therein for a comprehensive account of these applications.

We now present an application of our current results to the construction of sets of orthogonal hypercubes. By a *hypercube of dimension* n *and order* b is meant a $b \times \cdots \times b$ array consisting of b^n cells, based upon b distinct symbols arranged so that each of the b symbols appears the same number of times, namely $b^n/b = b^{n-1}$ times. For $2 \leq k \leq n$, a set of k such hypercubes is said to be k*-orthogonal* if upon superpositioning of the k hypercubes, each of the b^k distinct ordered k-tuples appears the same number of times, i.e. $b^n/b^k = b^{n-k}$ times. Finally a set of $r \geq k$ such hypercubes is said to be k*-orthogonal* if any subset of k hypercubes is k-orthogonal. When $n = k = 2$ these ideas reduce to the usual notion of mutually orthogonal latin squares of order b.

Given a set of k-independent vectors of length n over F_q, we can build sets of k-orthogonal hypercubes of order q and dimension n. Let $a_1x_1 + \cdots + a_nx_n$ denote a vector of length n over F_q in a k-orthogonal set. One can then construct a hypercube of order q and dimension n by placing the field element $a_1b_1 + \cdots + a_nb_n$ in the cell of the hypercube labeled by $(b_1, \ldots, b_n)$, where each $b_i \in F_q$. Since each coefficient vector $(a_1, \ldots, a_n)$ has at least one nonzero entry, it is clear that the array represented by the vector is indeed a hypercube of dimension n and order q.

Moreover, given k such vectors from a k-independent set, the corresponding set of k hypercubes will be k-orthogonal. This follows from the fact that the k vectors are k-independent over F_q, and hence the $k \times n$ matrix obtained from the coefficients of the k vectors will have rank k. Hence each element of F_q^k will be picked up exactly q^{n-k} times, so the k hypercubes are indeed k-orthogonal. This construction thus yields $Ind_q(n, k)$, k-orthogonal hypercubes of dimension n and order q.

We now raise a question regarding hypercubes of prime power orders. Let q, n, and k be such that they satisfy Theorem 2 so that $Ind_q(n, k) = n + 1$. Then as above, we can construct $n + 1$ hypercubes, each of dimension n and order q, which are k-orthogonal.

Question: If q is a prime power and the values of q, n and k satisfy Theorem 2 so that $Ind_q(n, k) = n + 1$, is it possible to have more than $n + 1$ hypercubes of order q and dimension n which are k-orthogonal?

We close by referring the reader to [2] and [3] for discussions of latin squares and hypercubes.

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