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Singularity and L^2-Dimension of Self-Similar Measures

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SINGULARITY AND L^2-DIMENSION OF SELF-SIMILAR MEASURES

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Dedicated to Professor Man-Keung Siu.

Abstract. We study the singularity of self-similar measures defined by nonuniformly contractive iterated functions systems of similitudes with overlaps. In the case the contraction ratios of the similitudes are exponentially commensurable, we describe an method to compute the L^2-dimension of the associated self-similar measures.

1. Introduction

Let $\{S_i\}_{i=1}^q$ be an *iterated function system* (IFS) of contractive similitudes on $\mathbb{R}^d$ defined by

$$S_i(x) = \rho_i R_i x + b_i, \qquad 1 \le i \le q, \tag{1.1}$$

where for each i, $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$, and R_i is a $d \times d$ orthogonal matrix. It is well known that there exists a unique nonempty compact set F, called a *self-similar set* (or *attractor*), that satisfies

$$F = \bigcup_{i=1}^q S_i(F).$$

Moreover, to any set of probability weights $\{p_i\}_{i=1}^q$ (i.e., $0 < p_i < 1$ and $\sum_{i=1}^q p_i = 1$), there corresponds a unique Borel probability measure satisfying the *self-similar identity*

$$\mu = \sum_{i=1}^q p_i \mu \circ S_i^{-1}. \tag{1.2}$$

(See Hutchinson [H], Falconer [F1]). μ is called a *self-similar measure.* If $\rho_i = \rho$ for all $i = 1, \dots, q$, we say that the iterated function system is *uniformly contractive*; otherwise we say that it is *nonuniformly contractive.*

It is a classical problem to determine whether a self-similar measure μ is absolutely continuous or singular. This problem has been studied extensively for the infinite Bernoulli convolutions and for uniformly contractive iterated function systems (see

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[E], [G], [L1], [L2], [So], [PS1], [LNR]). For nonuniformly contractive iterated function system, it is known that μ is singular if $\prod_{i=1}^q (p_i/\rho_i^d)^{p_i} > 1$ (see [NSB], [NW2]). If $\prod_{i=1}^q (p_i/\rho_i^d)^{p_i} = 1$, then μ is absolutely continuous if and only if $\{S_i\}_{i=1}^q$ satisfies the open set condition [NW2]. However, much less is known in the case

$$\prod_{i=1}^{q}\left(\frac{p_i}{\rho_i^d}\right)^{p_i} < 1, \qquad \text{i.}e., \qquad \frac{\sum_{i=1}^q p_i \log p_i}{\sum_{i=1}^q p_i \log \rho_i} > d. \tag{1.3}$$

In [NW2], it is shown that under certain conditions, inequality (1.3) guarantees that for Lebesgue a.e. $(\rho_1, \dots, \rho_q)$ the corresponding measure μ is absolutely continuous. These conditions are verified for a class of iterated function systems. It is therefore interesting to investigate the exceptional case in which μ is singular.

The purpose of this paper is to study the singularity of a self-similar measure μ, especially when (1.3) holds. In the uniformly contractive case, it is known that μ is singular if ρ^{-1} is a Pisot number (see [E], [LNR], [PSS]). The proof relies on showing that the Fourier transform $\hat{\mu}(\xi)$ does not tend to zero as ξ tends to infinity. In the nonuniformly contractive case that we consider in this paper, this technique fails because $\hat{\mu}(\xi)$ is not an infinite product. Nevertheless, by imposing the *weak separation property* (see Section 2), some interesting results can be obtained. It follows from a result in [LNR] that if $\{S_i\}_{i=1}^q$ has the weak separation property, and if β is the unique number satisfying

$$0 < \varlimsup_{h\to 0^+} \frac{1}{h^{d+\beta}} \int \mu\big(B_h(x)\big)^2\, dx < \infty, \tag{1.4}$$

then μ is absolutely continuous if and only if $\beta = d$. In other words, μ is absolutely continuous if and only if the L^2-*dimension* (see [S]) of μ equals d. This provides us with a criterion for deciding whether μ is absolutely continuous or singular by computing its L^2-dimension and showing that (1.4) holds.

The computation of the L^2-dimension of a measure is itself an interesting question because of its importance in the multifractal decomposition of the measure. In the absence of the open set condition, there are only partial results in this direction. In the uniformly contractive case, this problem has been studied in [L1], [L2], [LN2], [FLN], [LNR]. Most of the known calculations rely on the Perron-Frobenius theorem on nonnegative matrices. The situation is different in the nonuniformly contractive case and the Perron-Frobenius theorem cannot be adapted. Nevertheless, if the contraction ratios of the similitudes are exponentially commensurable, we will show in this paper that analogous results can be obtained. Let

$$S_i(x) = \rho^{n_i} R_i x + b_i, \qquad i = 1, \dots, q, \tag{1.5}$$

where each n_i is a positive integer. Under suitable conditions on ρ, R_i and b_i, we can show that the iterated function system has the weak separation property and we can formulate an algorithm to compute the L^2-dimension of any associated self-similar measure. The main idea is to replace the Perron-Frobenius theorem by a vector-valued renewal theorem proved by Lau *et al.* [LWC].

To compute the critical exponent β in (1.4) we begin by modifying a technique for uniformly contractive iterated function systems. Let G be the group generated by the orthogonal transformations $R_1, \dots, R_q$. Let $g \in G$, $a \in \mathbb{R}^d$, and fix an integer m satisfying $0 \le m \le n_q$. For $h > 0$ and $\alpha > 0$ define

$$\Phi^{(\alpha)}_{(m,g,a)}(h) = \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} \mu\big(B_{\rho^m h}(\rho^m g x + a)\big)\mu\big(B_h(x)\big)\, dx.$$

By applying the self-similar identity in (1.2) repeatedly, it can be shown (Proposition 3.1) that

$$\Phi^{(\alpha)}_{(m,g,a)}(h) = \sum_{k=n_1}^{n_q} c_k \Phi^{(\alpha)}_{(m_k,g_k,a_k)}\left(\frac{h}{\rho^k}\right), \tag{1.6}$$

where $g_k \in G$, $a_k \in \mathbb{R}^d$, c_k are constants independent of h, and m_k, ℓ_k are integers satisfying $0 \le m_k \le n_q$ and $n_1 \le \ell_k \le n_q$. We call (m, g, a) a *state* and denote the above equality symbolically as

$$T(m, g, a; h) = \sum_{k=n_1}^{n_q} c_k \left(m_k, g_k, a_k; \frac{h}{\rho^k}\right).$$

We can regard T as an operator on the set of states. We say that (m_k, g_k, a_k) is a state *generated by* (m, g, a) if the corresponding c_k is nonzero.

Starting from the state $\mathbf{0} = (0, I, 0)$, where I is the identity in G, we let $T(\{\mathbf{0}\})$ denote the set of all states generated from $\mathbf{0}$. Inductively, for each positive integer $n \ge 2$, we let $T^n(\{\mathbf{0}\})$ denote the set of all states generated from the states in $T^{n-1}(\{\mathbf{0}\})$. Let $\mathcal{S}$ denote the collection of all such states, i.e.,

$$\mathcal{S} = \bigcup_{n=0}^{\infty} T^n\big(\{\mathbf{0}\}\big),$$

where $T^0(\{\mathbf{0}\}) := \{\mathbf{0}\}$.

Define

$$\mathcal{S}_1 := \big\{(m, g, a) \in \mathcal{S} :\ (\rho^m g x + a, x) \in \operatorname{supp}(\mu) \times \operatorname{supp}(\mu) \text{ for some } x \in \mathbb{R}^d\big\}.$$

It follows from definition that a state (m, g, a) belongs to $\mathcal{S} \setminus \mathcal{S}_1$ if and only if $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ for all sufficiently small $h > 0$. If $\mathcal{S}_1$ is a finite set, then we can set up a finite matrix to compute the critical exponent β.

We now assume that $\mathcal{S}_1$ is finite and order the functions $\Phi^{(\alpha)}_{(m,g,a)}(h)$ for $(m,g,a)\in\mathcal{S}_1$ in vector form as

$$\big(\Phi^{(\alpha)}_1(h),\dots,\Phi^{(\alpha)}_\ell(h)\big) := \mathbf{\Phi}^{(\alpha)}(h).$$

Then it can be proved (see Propositions 3.1 and 3.2) that there exists $h_o>0$ such that

$$\mathbf{\Phi}^{(\alpha)}(h)=\sum_{k=n_1}^{n_q} A_k\mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho^k}\right)\quad \text{for } 0<h<h_o, \tag{1.7}$$

where

$$A_k=A_k(\alpha)=\begin{bmatrix} a_{11}^{(k)} & \dots & a_{1\ell}^{(k)}\\ \vdots & & \vdots\\ a_{\ell 1}^{(k)} & \dots & a_{\ell\ell}^{(k)}\end{bmatrix},$$

and $a_{ij}^{(k)}=a_{ij}^{(k)}(\alpha)\ge 0$ are independent of h.

Equation (1.7) can be expressed as a vector-valued renewal equation. Let μ_{ji} be the discrete measure supported on $\{-k\log\rho\}_{k=n_1}^{n_q}$ with mass $a_{ij}^{(k)}$ at $-k\log\rho$ and define

$$\mathbf{f}(x):=\big(f_1(x),\dots,f_\ell(x)\big)\quad\text{where}\quad f_i(x):=\Phi_i^{(\alpha)}(e^{-x}),\quad i=1,\dots,\ell.$$

We will show in Section 3 that (1.7) can be expressed as

$$\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha+\mathbf{z},\quad x\in\mathbb{R},$$

where $\mathbf{z}$ is some error function and

$$\mathbf{M}_\alpha=\begin{bmatrix} \mu_{11} & \dots & \mu_{1\ell}\\ \vdots & & \vdots\\ \mu_{\ell 1} & \dots & \mu_{\ell\ell}\end{bmatrix}. \tag{1.8}$$

Define

$$\mathbf{M}_\alpha(\infty)=\begin{bmatrix} \mu_{11}(\mathbb{R}) & \dots & \mu_{1\ell}(\mathbb{R})\\ \vdots & & \vdots\\ \mu_{\ell 1}(\mathbb{R}) & \dots & \mu_{\ell\ell}(\mathbb{R})\end{bmatrix}, \tag{1.9}$$

and let $\dim_2(\mu)$ denote the L^2-dimension of μ. Then by using the vector-valued renewal theorem in [LWC] we obtain our main theorem.

Theorem 1.1. *Let $\{S_i\}_{i=1}^q$ be defined by (1.5) and let μ be an associated self-similar measure. Assume that μ is not a point mass and that the set $\mathcal{S}_1$ is finite. Let $\mathbf{M}_\alpha(\infty)$ be the matrix defined in (1.9). Suppose $\alpha_o>0$ is the unique real number such that the spectral radius of $\mathbf{M}_{\alpha_o}(\infty)$ is equal to 1. Then $\dim_2(\mu)=\alpha_o$.*

Lastly, we apply our algorithm to the iterated function system defined by

$$S_1(x)=\rho x,\qquad S_2(x)=\rho^2x+(1-\rho^2),$$

where $1/2<\rho<1$, $\rho+\rho^2>1$, and ρ^{-1} is a Pisot number.

2. A condition for singularity

In this section we assume that $\{S_i\}_{i=1}^q$ is an IFS on $\mathbb{R}^d$ of contractive similitudes as defined in (1.1), F is the attractor, and μ is an associated self-similar measure defined by a set of probability weights $\{p_i\}_{i=1}^q$ as in (1.2).

We need some standard notation. Let $\rho := \min\{\rho_i : 1 \le i \le q\}$. Let $\Sigma^* := \bigcup_{n=0}^{\infty}\{1,\dots,q\}^n$ be the set of all finite sequences, with $\{1,\dots,q\}^0 := \{\emptyset\}$. For $\mathbf{j} = (j_1,\dots,j_n) \in \Sigma^*$, let $|\mathbf{j}| = n$ denote the length of $\mathbf{j}$ and

$$S_{\mathbf{j}} := S_{j_1}\circ\cdots\circ S_{j_n}, \quad \rho_{\mathbf{j}} := \rho_{j_1}\cdots\rho_{j_n}, \quad R_{\mathbf{j}} := R_{j_1}\cdots R_{j_n}, \quad p_{\mathbf{j}} := p_{j_1}\cdots p_{j_n}$$

(with $r_\emptyset := 1$, $p_\emptyset := 1$, and $S_\emptyset :=$ identity). For $0 < b < 1$, let

$$\mathcal{J}_b := \left\{\mathbf{j} = (j_1,\dots,j_n) \in \Sigma^* : \rho_{\mathbf{j}} \le b < \rho_{j_1,\dots,j_{n-1}}\right\},$$
$$\mathcal{A}_b := \{S_{\mathbf{j}} : \mathbf{j} \in \mathcal{J}_b\}.$$

For $S = S_{\mathbf{j}} \in \mathcal{A}_b$, let $\rho_S := \rho_{\mathbf{j}}$ denote the contraction ratio of S, and let

$$p_S := \sum\{p_{\mathbf{j}} : S_{\mathbf{j}} = S,\ \mathbf{j} \in \mathcal{J}_b\}.$$

Throughout this section we assume that $\{S_i\}_{i=1}^q$ has the *weak separation property* (WSP) (see [LN1] and [LW]): There exist $z_o \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$ such that for any index $\mathbf{i} \in \Sigma^*$ and $z = S_{\mathbf{i}}(z_o)$, any closed ρ^n-ball contains at most ℓ distinct $S_{\mathbf{j}}(z)$, $\mathbf{j} \in \Lambda_n$.

Let $\dim_{\mathrm{H}}(F)$ denote the Hausdorff dimension of the set F and let $\mathcal{H}^\alpha|_F$ be the restriction of the α-dimensional Hausdorff measure to F. The two theorems below are proved in [LW]. They generalize corresponding results in [LNR] by allowing the S_i's to have different contraction ratios.

Theorem 2.1. *Let $\{S_i\}_{i=1}^q$ be an IFS of contractive similitudes on $\mathbb{R}^d$ that has the weak separation property, let F be the attractor with $\dim_{\mathrm{H}}(F) = \alpha$, and let μ be a self-similar measure as defined in (1.2). Then μ is singular with respect to $\mathcal{H}^\alpha|_F$ if and only if there exists $b > 0$ and $S \in \mathcal{A}_b$ such that $p_S > \rho_S^\alpha$.*

Theorem 2.2. *Assume the same hypotheses of Theorem 2.1. If μ is absolutely continuous with respect to $H^\alpha|_F$, then the density of μ is bounded.*

As a simple corollary to Theorem 2.2, we have

Corollary 2.3. *Assume the same hypotheses of Theorem 2.1. Then μ is absolutely continuous with respect to Lebesgue measure if and only if the L^2-density of μ exists.*

Let μ be a finite positive Borel measure on $\mathbb{R}^d$ with compact support. The *lower* L^2*-dimension* of μ (see [S]) is defined as

$$\begin{aligned}\underline{\dim}_2(\mu) &= \varliminf_{h\to 0} \frac{\log \int \mu(B_h(x))^2\,dx}{\log h} - d\\ &= \sup\Big\{\alpha : \ \varlimsup_{h\to 0^+} \frac{1}{h^{d+\alpha}} \int \mu\big(B_h(x)\big)^2\,dx < \infty\Big\},\end{aligned}$$

where, as is throughout the rest of the paper, the integrals are over $\mathbb{R}^d$. Similarly we can define the *upper* L^2*-dimension* of μ. It follows from a more general result of Peres and Solomyak [PS2] that for self-similar measures μ, $\underline{\dim}_2(\mu) = \overline{\dim}_2(\mu) := \dim_2(\mu)$.

The L^2-dimension of a measure is closely related to its absolute continuity. It is well known (see [HL]) that

$$\varlimsup_{h\to 0^+} \frac{1}{h^{2d}} \int \mu\big(B_h(x)\big)^2\,dx < \infty$$

if and only if μ is absolutely continuous with density $d\mu/dx \in L^2(\mathbb{R}^d)$. By using this result and Corollary 2.3, we obtain the following theorem immediately.

Theorem 2.4. *Suppose $\{S_i\}_{i=1}^q$ has the WSP and let β be the unique number satisfying*

$$0 < \varlimsup_{h\to 0^+} \frac{1}{h^{d+\beta}} \int \mu\big(B_h(x)\big)^2\,dx < \infty.$$

Then μ is absolutely continuous if and only if $\beta = d$, i.e., $\dim_2(\mu) = d$.

If $\dim_2(\mu) < d$, its exact value is an indication of the "degree of singularity" of the measure. If the IFS satisfies the open set condition, then $\dim_2(\mu)$ is given by the unique solution α of the following equation (see e.g., [CM])

$$\sum_{i=1}^q p_i^2 \rho_i^{-\alpha} = 1. \tag{2.1}$$

In the absence of the open set condition, only some partial results are known. This problem has been studied in [L1], [L2], [LN2], [FLN], and [LNR]. In these investigations, the iterated function systems are assumed to be uniformly contractive. In this paper we are interested in the case the IFS maps have different, but exponentially commensurable, contraction ratios.

3. IFS with commensurable contraction ratios

For general nonuniformly contractive IFS's that have the WSP, it is not clear how $\dim_2(\mu)$ can be computed. Nevertheless, we can obtain some results if the contraction ratios of the S_i are exponentially commensurable. In this section we will formulate a set of conditions and derive an iteration formula for computing $\dim_2(\mu)$ for μ defined by an IFS on $\mathbb{R}^d$ of the form

$$S_i(x) = \rho^{n_i} R_i x + b_i, \quad i = 1, \dots, q, \tag{3.1}$$

where n_i are integers satisfying $1 \le n_1 \le n_2 \le \cdots \le n_q$, R_i are orthogonal transformations, and $b_i \in \mathbb{R}^d$. μ satisfies the self-similar identity

$$\mu = \sum_{i=1}^{q} p_i \mu \circ S_i^{-1}, \tag{3.2}$$

where $p_i > 0$ for $1 \le i \le q$ and $\sum_{i=1}^q p_i = 1$.

Let G be the group generated by the orthogonal transformations $R_1, \dots, R_q$. Fix an integer m satisfying $0 \le m \le n_q$ and let $g \in G$ and $a \in \mathbb{R}^d$. For $h > 0$ and $\alpha > 0$ define

$$\Phi^{(\alpha)}_{(m,g,a)}(h) = \frac{1}{h^{d+\alpha}} \int \mu\big(B_{\rho^m h}(\rho^m g x + a)\big)\mu\big(B_h(x)\big)\, dx. \tag{3.3}$$

Proposition 3.1. *Assume $\{S_i\}_{i=1}^q$ is as in (3.1). Let $0 \le m \le n_q$, $g \in G$, and $a \in \mathbb{R}^d$. Then each $\Phi^{(\alpha)}_{(m,g,a)}(h)$ can be expressed as a finite linear combination of $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(\rho^{-\ell_k} h)$, where $g_k \in G$, $a_k \in \mathbb{R}^d$, and m_k, ℓ_k are integers satisfying $0 \le m_k \le n_q$ and $n_1 \le \ell_k \le n_q$.*

Proof. Applying identity (3.2) to the definition of $\Phi^{(\alpha)}_{(m,g,a)}(h)$, we get

$$\begin{aligned}\Phi^{(\alpha)}_{(m,g,a)}(h) &= \sum_{i,j=1}^{q} \frac{p_i p_j}{h^{d+\alpha}} \int \mu\left(B_{\rho^{m-n_i} h}\big(S_i^{-1}(\rho^m g x + a)\big)\right) \mu\left(B_{\rho^{-n_j} h}\big(S_j^{-1}(x)\big)\right) dx \\ &:= \sum_{i,j=1}^{q} p_i p_j \Phi_{ij}.\end{aligned}$$

We will show that each Φ_{ij} satisfies the conclusion of the proposition. Consider the following three cases.

Case 1. $0 \le n_j + m - n_i \le n_q$. Using the change of variables $y = S_j^{-1}(x)$, we have

$$\begin{aligned}\Phi_{ij} &= \frac{\rho^{d n_j}}{h^{d+\alpha}} \int \mu\left(B_{\rho^{m-n_i} h}\big(S_i^{-1}(\rho^m g S_j(y) + a)\big)\right) \mu\big(B_{\rho^{-n_j} h}(y)\big)\, dy \\ &= \rho^{-n_j \alpha} \Phi^{(\alpha)}_{(m_{ij}, g_{ij}, a_{ij})}(\rho^{-n_j} h),\end{aligned}$$

where

$$m_{ij} = n_j + m - n_i, \quad g_{ij} = R_i^{-1} g R_j, \quad a_{ij} = \rho^{m-n_i} R_i^{-1} g b_j + \rho^{-n_i} R_i^{-1} a - \rho^{-n_i} R_i^{-1} b_i.$$

Case 2. $n_j+m-n_i<0$. In this case we have $0<n_i-m-n_j\le n_q$. By using the change of variables $y=S_i^{-1}(\rho^m gx+a)$, we get

$$\begin{aligned}\Phi_{ij} &= \frac{\rho^{d(n_i-m)}}{h^{d+\alpha}}\int \mu\big(B_{\rho^{m-n_i}h}(y)\big)\mu\left(B_{\rho^{-n_j}h}\big(S_j^{-1}(\rho^{-m}g^{-1}(S_i(y)-a))\big)\right)\,dy\\ &= \rho^{(m-n_i)\alpha}\Phi^{(\alpha)}_{(m_{ij},g_{ij},a_{ij})}\big(\rho^{-(n_i-m)}h\big),\end{aligned}$$

where

$$\begin{aligned} m_{ij} &= n_i-m-n_j, \qquad g_{ij}=R_j^{-1}g^{-1}R_i,\\ a_{ij} &= -\rho^{-n_j-m}R_j^{-1}g^{-1}a+\rho^{-n_j-m}R_j^{-1}g^{-1}b_i-\rho^{-n_j}R_j^{-1}b_j.\end{aligned}$$

Case 3. $n_j+m-n_i>n_q$. This is the remaining case. In this case Φ_{ij} cannot be expressed directly as a scalar multiple of some $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(\rho^{-n_k}h)$ by a simple change of variables. We need to apply the self-similar identity again to the first factor in the integrand of Φ_{ij}, i.e., the quantity $\mu(B_{\rho^{m-n_i}h}(S_i^{-1}(\rho^m gx+a)))$. This yields

$$\begin{aligned}\Phi_{ij} &= \sum_{k=1}^{q}\frac{p_k}{h^{d+\alpha}}\int \mu\left(B_{\rho^{m-n_i-n_k}h}\big(S_k^{-1}S_i^{-1}(\rho^m gx+a)\big)\right)\mu\left(B_{\rho^{-n_j}h}\big(S_j^{-1}(x)\big)\right)\,dx\\ &:= \sum_{k=1}^{q}p_k\Phi_{ijk}.\end{aligned}$$

Take any term Φ_{ijk}, $1\le k\le q$ and suppose that $n_j+m-n_i-n_k\le n_q$. Since $n_j+m-n_i>n_q$ we have

$$0\le n_q-n_k<n_j+m-n_i-n_k\le n_q.$$

This takes us back to Case 1 and therefore Φ_{ijk} satisfies the conclusion of the proposition. If $n_j+m-n_i-n_k>n_q$, then we are back to Case 3. We repeat the above process and break Φ_{ijk} down in turn into a linear combination of integrals as above. Note that $n_j+m-n_i-n_k$ is strictly less than n_j+m-n_i. Hence, the process must stop in a finite number of steps, and we can express Φ_{ij} as a linear combination of $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(\rho^{-\ell_k}h)$ in the desired form. This proves the proposition. □

According to Proposition 3.1 we have the following iteration formula

$$\Phi^{(\alpha)}_{(m,g,a)}(h)=\sum_{k=n_1}^{n_q}c_k\Phi^{(\alpha)}_{(m_k,g_k,a_k)}\left(\frac{h}{\rho^k}\right). \tag{3.4}$$

We call (m,g,a) a *state* and denote the above equality symbolically as

$$T(m,g,a;h)=\sum_{k=n_1}^{n_q}c_k\left(m_k,g_k,a_k;\frac{h}{\rho^k}\right).$$

We say that (m_k,g_k,a_k) is a state *generated by* (m,g,a) if the corresponding c_k is nonzero.

Starting from the state $\mathbf{0} = (0, I, 0)$, where I is the identity in G, we let $T(\{\mathbf{0}\})$ denote the set of all states generated from $\mathbf{0}$. Inductively, for each positive integer $n \geq 2$, we let $T^n(\{\mathbf{0}\})$ denote the set of all states generated from the states in $T^{n-1}(\{\mathbf{0}\})$. Let $\mathcal{S}$ denote the collection of all such states, i.e.,

$$\mathcal{S} = \bigcup_{n=0}^{\infty} T^n(\{\mathbf{0}\}),$$

where $T^0(\{\mathbf{0}\}) := \{\mathbf{0}\}$.

Note that $\Phi^{(\alpha)}_{(m,g,a)}(h) > 0$ for all $h > 0$ if and only if there exists some $x \in \mathbb{R}^d$ such that $(\rho^m gx + a, x) \in F \times F$, where $F = \operatorname{supp}(\mu)$, the attractor of $\{S_i\}_{i=1}^q$. We define

$$\mathcal{S}_1 := \big\{(m, g, a) \in \mathcal{S} : \ (\rho^m gx + a, x) \in F \times F \text{ for some } x \in \mathbb{R}^d\big\}.$$

Proposition 3.2. *Assume the same hypotheses of Proposition 3.1. Then*

(a) $(m, g, a) \in \mathcal{S} \setminus \mathcal{S}_1$ *if and only if* $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ *for all* $h > 0$ *sufficiently small.*
(b) T *is invariant on* $\mathcal{S} \setminus \mathcal{S}_1$.
(c) *If* $\mathcal{S}_1$ *is finite, then there exists* $h_o > 0$ *such that* $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ *for all* $0 < h < h_o$ *and for all* $(m, g, a) \in \mathcal{S} \setminus \mathcal{S}_1$.

Proof. Part (a) follows directly from definitions.

(b) Let $(m, g, a) \in \mathcal{S} \setminus \mathcal{S}_1$. Then $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ for all $h > 0$ sufficiently small. Notice that all c_k in (3.4) are nonnegative. Hence, for each $c_k > 0$ we have $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(h/\rho^k) = 0$ for all $h > 0$ sufficiently small. This implies that $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(h) = 0$ for all $h > 0$ sufficiently small and consequently $(m_k, g_k, a_k) \in \mathcal{S} \setminus \mathcal{S}_1$.

(c) Let $\tilde{\mathcal{S}}_1 := T(\mathcal{S}_1) \setminus \mathcal{S}_1$. Then $\tilde{\mathcal{S}}_1$ is finite since $\mathcal{S}_1$ is. Therefore, there exists $h_o > 0$ such that $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ for all $0 < h < h_o$ and for all $(m, g, a) \in \tilde{\mathcal{S}}_1$. For each $(m, g, a) \in \tilde{\mathcal{S}}_1$, we expand $\Phi^{(\alpha)}_{(m,g,a)}(h)$ as in (3.4). Then it follows that for each $c_k > 0$, $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(h/\rho^k) = 0$ for all $0 < h < h_o$, which implies that $\Phi^{(\alpha)}_{(m_k,g_k,a_k)}(h) = 0$ for all $0 < h < h_o$. The conclusion now follows by induction. □

Note that $\operatorname{supp}(\mu)$ lies in the closed ball $B_R(0)$ of radius R where

$$R := \frac{\max_i |b_i|}{1 - \rho^{n_1}}.$$

Consequently, if $|a| > 2R$, then $\Phi^{(\alpha)}_{(m,g,a)}(h) = 0$ for all $h > 0$ sufficiently small. Hence

$$\mathcal{S}_1 \subseteq \big\{(m, g, a) \in \mathcal{S} : \ |a| \leq 2R\big\}. \tag{3.5}$$

We will denote by T_1 the restriction of T on $\mathcal{S}_1$. In order to compute the L^2-dimension of μ we will require S_1 to be a finite set. The following proposition provides a class of iterated function systems that satisfy this condition and have the WSP. Recall that a *Pisot number* is an algebraic integer greater than 1 whose algebraic conjugates are all in modulus less than 1.

Proposition 3.3. *Suppose ρ^{-1} is a Pisot number, G is a finite group, and*

$$G\{b_i:\ 1\le i\le q\}\subseteq r_1\mathbb{Z}[\rho^{-1}]\times\cdots\times r_d\mathbb{Z}[\rho^{-1}]$$

for some $r_1,\dots,r_d\in\mathbb{R}$, where $\mathbb{Z}[\rho^{-1}]:=\{\sum_{j=0}^n z_j\rho^{-j}:\ \ z_j\in\mathbb{Z},\ n\ge 0\}$. Then $\{S_i\}_{i=1}^q$ has the WSP and $\mathcal{S}_1$ is a finite set.

Proof. That $\{S_i\}_{i=1}^q$ has the WSP is known; the finiteness of $\mathcal{S}_1$ can be obtained by a standard argument (see [NW1, Theorem 2.5]). We include a proof here for completeness.

In view of (3.5), it suffices to prove that the set $\mathcal{S}_o$ defined below is finite:

$$\mathcal{S}_o:=\big\{(m,g,a)\in\mathcal{S}:\ |a|\le 2R\big\}.$$

Each $(m,g,a)\in\mathcal{S}_o$ satisfies $0\le m\le n_q$, $g\in G$, and $|a|\le 2R$. Since G is a finite group, we only need to show that the number of a's is finite. From the proof of Proposition 3.1 we see that each a is of the form

$$a=\sum_{j=1}^t\rho^{k_j}g_jb_{\ell_j},$$

where $g_j\in G$, k_j is an integer less than m, and $1\le\ell_j\le q$. Write $a=(a_1,\dots,a_d)\in\mathbb{R}^d$. By assumption,

$$g_jb_{\ell_j}\in r_1\mathbb{Z}[\rho^{-1}]\times\cdots\times r_d\mathbb{Z}[\rho^{-1}].$$

Hence each a_i is of the form

$$r_i\rho^{k_i}p(\rho^{-1}),$$

where $0\le k_i<m$ and p is a polynomial with integer coefficients and with height uniformly bounded by a constant depending only on G and $\{b_1,\dots,b_q\}$. (The *height* of a polynomial $\sum_{k=0}^n c_kx^k$ is defined to be the number $\max\{|c_k|:0\le k\le n\}$.) Let p_1,p_2 be any two such polynomials. Since ρ^{-1} is a Pisot number, we have

$$\text{either}\quad p_1(\rho^{-1})=p_2(\rho^{-1})\quad\text{or}\quad \big|p_1(\rho^{-1})-p_2(\rho^{-1})\big|\ge C,$$

where C is an absolute constant depending only on ρ^{-1} and the height of the polynomial $p_1(x)-p_2(x)$ (see [G, Lemma 1.51]). Hence, there are only finitely many distinct values a_i satisfying $|a_i|\le 2R$, and the proposition follows. □

4. Proof of Theorem 1.1

Throughout this section we assume that $\mathcal{S}_1$ is finite and order the functions $\Phi^{(\alpha)}_{(m,g,a)}(h)$ for $(m,g,a)\in\mathcal{S}_1$ in vector form as

$$\big(\Phi^{(\alpha)}_1(h),\dots,\Phi^{(\alpha)}_\ell(h)\big) := \mathbf{\Phi}^{(\alpha)}(h).$$

Propositions 3.1 and 3.2 imply that there exists $h_o>0$ such that

$$\mathbf{\Phi}^{(\alpha)}(h)=\sum_{k=n_1}^{n_q} A_k\mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho^k}\right) \quad \text{for } 0<h<h_o, \tag{4.1}$$

where

$$A_k = A_k(\alpha)=\begin{bmatrix} a^{(k)}_{11} & \dots & a^{(k)}_{1\ell}\\ \vdots & & \vdots\\ a^{(k)}_{\ell 1} & \dots & a^{(k)}_{\ell\ell}\end{bmatrix}.$$

Note that $a^{(k)}_{ij}=a^{(k)}_{ij}(\alpha)\ge 0$ for all i,j and k, and each nonzero $a^{(k)}_{ij}(\alpha)$ is a strictly increasing function of α. Let $h=e^{-x}$ and therefore $\log h=-x$. Define

$$f_i(x) := \Phi^{(\alpha)}_i(e^{-x}),\quad i=1,\dots,\ell$$
$$\mathbf{f}(x) := \big(f_1(x),\dots,f_\ell(x)\big).$$

Let $x_o=-\log h_o$. Then we can write (4.1) as

$$\mathbf{f}(x)=\sum_{k=n_1}^{n_q} A_k\mathbf{f}(x+k\log\rho) \quad \text{for } x>x_o. \tag{4.2}$$

Equation (4.2) can be written as a convolution equation. Let μ_{ji} be the discrete measure supported on $\{-k\log\rho\}_{k=n_1}^{n_q}$ with mass $a^{(k)}_{ij}$ at $-k\log\rho$. Then

$$(f_j*\mu_{ji})(x)=\sum_{k=n_1}^{n_q} a^{(k)}_{ij} f_j(x+k\log\rho). \tag{4.3}$$

Using (4.2) and (4.3) we have, for $1\le i\le\ell$ and $x>x_o$,

$$\begin{aligned} f_i(x) &= \sum_{k=n_1}^{n_q}\sum_{j=1}^{\ell} a^{(k)}_{ij} f_j(x+k\log\rho)\\ &= \sum_{j=1}^{\ell}\sum_{k=n_1}^{n_q} a^{(k)}_{ij} f_j(x+k\log\rho)\\ &= \sum_{j=1}^{\ell}(f_j*\mu_{ji})(x) \qquad \text{(by (4.3))}. \end{aligned}$$

Consequently, (4.2) becomes

$$\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha \quad \text{for } x>x_o, \tag{4.4}$$

where

$$\mathbf{M}_\alpha = \begin{bmatrix} \mu_{11} & \dots & \mu_{1\ell} \\ \vdots & & \vdots \\ \mu_{\ell 1} & \dots & \mu_{\ell\ell} \end{bmatrix} \quad \text{and} \quad \mathbf{f} * \mathbf{M}_\alpha := \left(\sum_{i=1}^{\ell} f_i * \mu_{i1}, \dots, \sum_{i=1}^{\ell} f_i * \mu_{i\ell} \right). \tag{4.5}$$

We can extend the convolution equation (4.4) to $\mathbb{R}$ by introducing an error function $\mathbf{z}$. Since $\operatorname{supp}(\mu)$ is compact, there exists a constant $c > 0$ (depending only on $\operatorname{supp}(\mu)$ and h_o) such that $\operatorname{supp}(\mu) \subseteq B_{ch_o}(0) \subseteq B_{ch}(0)$ for all $h \geq h_o$. Thus, for each $(m, g, a) \in \mathcal{S}$ and $h \geq h_o$,

$$\begin{aligned} \Phi^{(\alpha)}_{(m,g,a)}(h) &\leq \frac{1}{h^{d+\alpha}} \left(\int \mu\big(B_{\rho^m h}(\rho^m gx + a)\big)^2 \, dx \right)^{1/2} \left(\int \mu\big(B_h(x)\big)^2 \, dx \right)^{1/2} \\ &\leq \frac{1}{h^{d+\alpha}} \left(\int_{B_{ch}(0)} 1 \, dx \right)^{1/2} \left(\int_{B_{ch}(0)} 1 \, dx \right)^{1/2} \\ &= O(h^{-\alpha}) \quad \text{as } h \to \infty. \end{aligned}$$

It follows that for each $i = 1, \dots, \ell$, we have

$$f_i(x) = \Phi_i^{(\alpha)}(e^{-x}) = O(e^{\alpha x}) \quad \text{as } x \to -\infty. \tag{4.6}$$

We can now rewrite (4.4) as

$$\mathbf{f}(x) = (\mathbf{f} * \mathbf{M}_\alpha)(x) + \mathbf{z}(x) \quad \text{for } x \in \mathbb{R}, \tag{4.7}$$

where $\mathbf{z}(x) = 0$ for $x > x_o$ and $\mathbf{z}(x) = O(e^{\alpha x})$ as $x \to -\infty$.

Define

$$\mathbf{M}_\alpha(\infty) = \begin{bmatrix} \mu_{11}(\mathbb{R}) & \dots & \mu_{1\ell}(\mathbb{R}) \\ \vdots & & \vdots \\ \mu_{\ell 1}(\mathbb{R}) & \dots & \mu_{\ell\ell}(\mathbb{R}) \end{bmatrix}.$$

Lemma 4.1. *Let $\{S_i\}_{i=1}^q$ be defined by (1.5) and let μ be an associated self-similar measure. Assume that $\mathcal{S}_1$ is finite. Then*

(a) $\mathbf{M}_\alpha(\infty)^t = \sum_{k=n_1}^{n_q} A_k$.
(b) $\mathbf{f}$ *is continuous on* $\mathbb{R}$ *and* $\mathbf{f}(x) = O(e^{\alpha x})$ *as* $x \to -\infty$.

Proof. (a) Since each μ_{ji} is a nonnegative measure, we have, by the definition of μ_{ji},

$$\mu_{ji}(\mathbb{R}) = \sum_{k=n_1}^{n_q} a_{ij}^{(k)},$$

proving the assertion.

(b) The continuity of $\mathbf{f}$ follows from the definition of the $\Phi^{(\alpha)}_{(m,g,a)}(h)$ and the continuity of μ. The second assertion follows from (4.6). □

We need to strengthen slightly the vector-valued renewal theorem in Lau *et al.* [LWC] to suit our purposes. We introduce some additional terminology. The reader is referred to [LWC] for any unexplained terms. Let $\mathbf{F}$ be a matrix-valued Radon measure that vanishes on $(-\infty, 0)$. Write

$$\mathbf{F} = \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{\ell 1} & \dots & F_{nn} \end{bmatrix},$$

where $\mathbf{F}_{ij}(x) = \mu_{ij}(-\infty, x]$ and each μ_{ij} is a Radon measure on $\mathbb{R}$. We write $\mathbf{F}(\infty) = [F_{ij}(\infty)]$ and let $\mathbf{m} = [m_{ij}] = [\int_0^\infty x\, dF_{ij}]$ be the moment matrix. We say that each column of $\mathbf{F}$ is *nondegenerate* at 0 if

$$\sum_{i=1}^n F_{ij}(0) < \sum_{i=1}^n F_{ij}(\infty) \quad \text{for } 1 \le j \le n.$$

In this case, there exists some $\delta > 0$ such that the vector

$$\left[\sum_{i=1}^n \big(F_{i1}(\infty) - F_{i1}(\delta)\big), \dots, \sum_{i=1}^n \big(F_{in}(\infty) - F_{in}(\delta)\big)\right]$$

is coordinatewise positive. For the measures μ_{ij} in $\mathbf{F}$ and for any *path* $\gamma = (i_1, \dots, i_k)$ with $i_j \in \{1, \dots, n\}$, we use the notation

$$\mu_\gamma = \mu_{i_1 i_2} * \mu_{i_2 i_3} * \dots * \mu_{i_{k-1} i_k}.$$

Such a γ is a called a *cycle* if $i_1 = i_k$ and a *simple cycle* if it is a cycle and $i_1, \dots, i_{k-1}$ are distinct. Let $\mathbb{R}_{\mathbf{F}}$ denote the closed subgroup generated by

$$\bigcup \big\{\mathrm{supp}(\mu_\gamma) : \ \gamma \text{ is a simple cycle on } \{1, \dots, n\}\big\}.$$

The following theorem is modified from Theorem 4.3 in [LWC], where the error function $\mathbf{z}$ vanishes on $(-\infty, 0)$. In our case, $\mathbf{z}$ does not satisfy this condition. Instead, we have $\mathbf{z}(x) = o(e^{ax})$ as $x \to -\infty$ for some $a > 0$. Also, in [LWC] each entry of $\mathbf{F}$ is assumed to be nondegenerate at 0. Although this condition is not satisfied in our case, it can be easily replaced by the condition that each column of $\mathbf{F}$ is nondegenerate at 0.

Theorem 4.2. *Let $\mathbf{F}$ be an $n \times n$ matrix-valued Radon measure defined on $\mathbb{R}$ that vanishes on $(-\infty, 0)$ and assume that each column of $\mathbf{F}$ is nondegenerate at 0. Suppose $\mathbf{F}(\infty)$ is irreducible and has maximal eigenvalue 1. Let $\mathbf{U} = \sum_{k=0}^\infty \mathbf{F}^{*k}$ and let $\mathbf{z}$ be a directly Riemann integrable function on $\mathbb{R}$ with $\mathbf{z}(x) = o(e^{ax})$ as $x \to -\infty$ for some $a > 0$. Then $\mathbf{f} = \mathbf{z} * \mathbf{U}$ is a bounded continuous solution of*

$$\mathbf{f}(x) = (\mathbf{f} * \mathbf{F})(x) + \mathbf{z}(x), \quad x \in \mathbb{R}, \tag{4.8}$$

and it is unique in the class of continuous solutions satisfying $\lim_{x\to-\infty} \mathbf{f}(x) = \mathbf{0}$. Furthermore

(a) *If* $\mathbb{R}_{\mathbf{F}} = \mathbb{R}$*, then*

$$\lim_{x\to\infty} \mathbf{f}(x) = \left(\int_{-\infty}^{\infty} \mathbf{z}(t)\,dt\right)\mathbf{A},$$

where

$$\mathbf{A} = \frac{1}{\gamma}\begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}, \qquad \gamma = [v_1,\dots,v_n]\begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{u} = (u_1,\dots,u_n), \mathbf{v} = (v_1,\dots,v_n)$ *are the unique normalized positive right and left 1-eigenvectors of* $\mathbf{F}(\infty)$*, respectively.*

(b) *If* $\mathbb{R}_{\mathbf{F}} = \langle\lambda\rangle$ *for some* $\lambda > 0$*, then for each* $x > 0$*,*

$$\lim_{k\to\infty}\left[f_1(x+a_{11}+k\lambda),\dots,f_n(x+a_{1n}+k\lambda)\right] = \left(\sum_{k=-\infty}^{\infty} \mathbf{z}(x+k\lambda)\right)\mathbf{A},$$

where $a_{1j} \in supp(\mu_{\gamma(1,j)})$ *and* $\gamma(1,j)$ *is any path from* 1 *to* j *such that* $\mu_{\gamma(1,j)} \neq 0$.

Proof. The assumption that the columns of $\mathbf{F}$ are nondegenerate at 0 guarantees the existence of $\mathbf{f}$ and the conclusions (a) and (b) (see [LWC, Theorem 4.3]). It suffices to prove the uniqueness of $\mathbf{f} = \mathbf{z} * \mathbf{U}$. We use a similar argument as that in [F2]. Let $\mathbf{f}_1$ be another solution of (4.8) and let $\mathbf{g} = \mathbf{f} - \mathbf{f}_1$. Then $\mathbf{g}$ satisfies $\mathbf{g} = \mathbf{g} * \mathbf{F}$. Iterating this we have $\mathbf{g} = \mathbf{g} * \mathbf{F}^{*k}$ for all $k \in \mathbb{N}$. Fix $x \in \mathbb{R}$. Then for any $u \in \mathbb{R}$ we have

$$\mathbf{g}(x) = \int_0^{\infty} \mathbf{g}(x-y)\,d\mathbf{F}^{*k}(y) = \int_0^{u} \mathbf{g}(x-y)\,d\mathbf{F}^{*k}(y) + \int_u^{\infty} \mathbf{g}(x-y)\,d\mathbf{F}^{*k}(y).$$

For any given $\epsilon > 0$, we first choose u sufficiently large so that

$$\left|\int_u^{\infty} \mathbf{g}(x-y)\,d\mathbf{F}^{*k}(y)\right| \leq \sup_{v\leq x-u} \left|\mathbf{g}(v)\right| < \frac{1}{2}\epsilon.$$

(This is possible because $\lim_{x\to-\infty}\mathbf{g}(x) = \mathbf{0}$). Next, since $\mathbf{U} = \sum_{k=0}^{\infty}\mathbf{F}^{*k}$ is uniformly bounded on any interval on $\mathbb{R}$ of fixed length (see the proof of [LWC, Theorem 4.2]), we have $\lim_{k\to\infty}\mathbf{F}^{*k}(0,u] = 0$. Therefore we can choose k sufficiently large so that

$$\left|\int_0^{u} \mathbf{g}(x-y)\,d\mathbf{F}^{*k}(y)\right| < \frac{1}{2}\epsilon.$$

Thus $|g(x)| < \epsilon$. This proves that $\mathbf{g}(x) = 0$ and the uniqueness follows. □

Proof of Theorem 1.1. Since each nonzero entry on $\mathbf{M}_\alpha(\infty)$ is a strictly increasing continuous function of α, which tends to 0 as α tends to $-\infty$ and tends to ∞ as α tends to ∞, it follows that there exists a unique α_o such that the spectral radius of $\mathbf{M}_{\alpha_o}$ is 1 (see [Mi, Theorem 2.1]).

Next, we observe from the definition of $\mathbf{M}_\alpha$ that μ is a point mass if and only if the spectral radius of $\mathbf{M}_\alpha(\infty)$ equals 1 when $\alpha = 0$. This is clear if μ is a point mass. To see the converse, we suppose that μ is not a point mass and $\alpha = 0$. Then the construction of $\mathbf{M}_\alpha(\infty)$ implies that each of its irreducible components has row sum not exceed 1 and has at least one row with row sum strictly less than 1. Hence, the spectral radius of $\mathbf{M}_\alpha(\infty)$ is less than 1. Thus, we can assume $\alpha_o > 0$.

To prove that $\dim_2(\mu) = \alpha_o$, we first assume that $\mathbf{M}_{\alpha_o}(\infty)$ is irreducible. From (4.7) we get

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_{\alpha_o} + \mathbf{z},$$

where $\mathbf{f}$ is continuous and $\lim_{x\to\infty} \mathbf{f}(x) = \mathbf{0}$ (Lemma 4.1). The error function $\mathbf{z}$ is nonzero and nonnegative. Moreover, $\mathbf{z}(x) = 0$ for $x > x_o$ and for any $a < \alpha_o$, $\mathbf{z}(x) = o(e^{ax})$ as $x \to -\infty$. We observe that $\mathbb{R}_{\mathbf{M}_{\alpha_o}} = \langle -\log\rho\rangle$, the closed subgroup generated by $-\log\rho$. Also, it follows from the derivation of the iteration formula in the proof of Proposition 3.1 that the columns of $\mathbf{M}_{\alpha_o}(\infty)$ are nondegenerate at 0. Hence, by the vector-valued renewal theorem (Theorem 4.2), for all $i = 1, \dots, \ell$,

$$0 < \overline{\lim_{x\to\infty}} f_i(x) < \infty,$$

which implies that for all $(m, g, a) \in \mathcal{S}_1$,

$$0 < \overline{\lim_{h\to 0^+}} \Phi^{(\alpha_o)}_{(m,g,a)}(h) < \infty.$$

Since one of the $(m, g, a) \in \mathcal{S}_1$ is $(0, I, 0)$ we have

$$0 < \overline{\lim_{h\to 0^+}} \Phi^{(\alpha_o)}_{(0,I,0)}(h) < \infty.$$

That is,

$$0 < \overline{\lim_{h\to 0^+}} \frac{1}{h^{d+\alpha_o}} \int \mu\big(B_h(x)\big)^2\, dx < \infty,$$

which implies that $\dim_2(\mu) = \alpha_o$.

We now consider the case $\mathbf{M}_{\alpha_o}(\infty)$ is reducible. First, we notice that if $\beta < \alpha_o$, then each nonzero entry of $\mathbf{M}_\beta(\infty)$ is strictly less than that of $\mathbf{M}_{\alpha_o}(\infty)$. It follows that the maximal eigenvalue of $\mathbf{M}_\beta(\infty)$ is less than 1 (see [Mi, Corollary 2.2] for the irreducible case; the reducible case follows easily by considering the irreducible components). Hence by [LWC, Remark 4.4 and Theorem 4.5],

$$\lim_{x\to\infty} f_i(x) = 0 \quad \text{for } i = 1, \dots, \ell.$$

In particular, $\lim_{h\to 0^+} \Phi^{(\alpha_o)}_{(0,I,0)}(h) = 0$ and thus $\dim_2(\mu) \ge \alpha_o$.

On the other hand, since $\mathbf{M}_{\alpha_o}(\infty)$ has maximal eigenvalue 1, by [LWC, Theorem 4.5], there exists some $i \in \{1, \ldots, \ell\}$ such that

$$\overline{\lim_{h\to 0^+}} f_i(x) > 0.$$

That is, there exists some $m \geq 1$, $g \in G$, and $a \in \mathbb{R}^d$ such that

$$\overline{\lim_{h\to 0^+}} \Phi^{(\alpha_o)}_{(m,g,a)}(h) > 0.$$

By Hölder's inequality,

$$\Big(\Phi^{(\alpha_o)}_{(m,g,a)}(h)\Big)^2 \leq \rho^{m\alpha_o}\Phi^{(\alpha_o)}_{(0,I,0)}(\rho^m h)\Phi^{(\alpha_o)}_{(0,I,0)}(h).$$

Therefore, $\overline{\lim_{h\to 0^+}} \Phi^{(\alpha_o)}_{(0,I,0)}(h) > 0$ and thus $\dim_2(\mu) \leq \alpha$. This completes the proof. □

5. Examples

In this section we will compute the L^2-dimension of the self-similar measures defined by the family of IFS's

$$S_1(x) = \rho x, \qquad S_2(x) = \rho^2 x + 1 - \rho^2, \tag{5.1}$$

where $1/2 < \rho < 1$, $\rho + \rho^2 \geq 1$, and ρ^{-1} is a Pisot number. By Proposition 3.3, each of the IFS's has the WSP and $\mathcal{S}_1$ is a finite set. Note that

$$S_1^{-1}(x) = \frac{x}{\rho}, \qquad S_2^{-1}(x) = \frac{x}{\rho^2} - \frac{1-\rho^2}{\rho^2}.$$

Since no rotations are involved, we denote a state (m, g, a) simply by (m, a). For $a \in \mathbb{R}$, define

$$\Phi^{(\alpha)}_{(0,a)}(h) = \frac{1}{h^{1+\alpha}} \int \mu\big(B_h(x+a)\big)\mu\big(B_h(x)\big)\, dx$$

$$\Phi^{(\alpha)}_{(1,a)}(h) = \frac{1}{h^{1+\alpha}} \int \mu\big(B_{\rho h}(\rho x+a)\big)\mu\big(B_h(x)\big)\, dx.$$

Proposition 5.1. *For any $a \in \mathbb{R}$, $\alpha > 0$, and $h > 0$,*

(a)

$$\Phi^{(\alpha)}_{(0,a)}(h) = \frac{p_1^2}{\rho^\alpha}\Phi^{(\alpha)}_{(0,\frac{a}{\rho})}\left(\frac{h}{\rho}\right) + \frac{p_2^2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(0,\frac{a}{\rho^2})}\left(\frac{h}{\rho^2}\right)$$
$$+ \frac{p_1p_2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(1,\frac{a}{\rho}+\frac{1-\rho^2}{\rho})}\left(\frac{h}{\rho^2}\right) + \frac{p_1p_2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(1,\frac{-a}{\rho}+\frac{1-\rho^2}{\rho})}\left(\frac{h}{\rho^2}\right).$$

(b)

$$\Phi^{(\alpha)}_{(1,a)}(h) = \frac{p_1^3}{\rho^\alpha}\Phi^{(\alpha)}_{(0,\frac{a}{\rho^2})}\left(\frac{h}{\rho}\right) + \frac{p_2}{\rho^{\alpha}}\Phi^{(\alpha)}_{(1,\frac{-a}{\rho}+\frac{1-\rho^2}{\rho})}\left(\frac{h}{\rho}\right) + \frac{p_1p_2^2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(0,\frac{a}{\rho^3})}\left(\frac{h}{\rho^2}\right)$$
$$+ \frac{p_1^2p_2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(1,\frac{a}{\rho^2}+\frac{1-\rho^2}{\rho})}\left(\frac{h}{\rho^2}\right) + \frac{p_1^2p_2}{\rho^{2\alpha}}\Phi^{(\alpha)}_{(1,\frac{-a}{\rho^2}+\frac{1-\rho^2}{\rho})}\left(\frac{h}{\rho^2}\right).$$

Proof. This follows from a similar derivation as that in the proof of Proposition 3.1. ☐

Remark 5.2. (a) *The identities in Proposition 5.1 can be expressed in the following forms:*

$$T(0,a;h) = \frac{p_1^2}{\rho^\alpha}\left(0,\frac{a}{\rho};\frac{h}{\rho}\right) + \frac{p_2^2}{\rho^{2\alpha}}\left(0,\frac{a}{\rho^2};\frac{h}{\rho^2}\right) + \frac{p_1p_2}{\rho^{2\alpha}}\left(1,\frac{a}{\rho}+\frac{1-\rho^2}{\rho};\frac{h}{\rho^2}\right) + \frac{p_1p_2}{\rho^{2\alpha}}\left(1,\frac{-a}{\rho}+\frac{1-\rho^2}{\rho};\frac{h}{\rho^2}\right),$$

and

$$T(1,a;h) = \frac{p_1^3}{\rho^\alpha}\left(0,\frac{a}{\rho^2};\frac{h}{\rho}\right) + \frac{p_2}{\rho^\alpha}\left(1,\frac{-a}{\rho}+\frac{1-\rho^2}{\rho};\frac{h}{\rho}\right) + \frac{p_1p_2^2}{\rho^{2\alpha}}\left(0,\frac{a}{\rho^3};\frac{h}{\rho^2}\right) + \frac{p_1^2p_2}{\rho^{2\alpha}}\left(1,\frac{a}{\rho^2}+\frac{1-\rho^2}{\rho};\frac{h}{\rho^2}\right) + \frac{p_1^2p_2}{\rho^{2\alpha}}\left(1,\frac{-a}{\rho^2}+\frac{1-\rho^2}{\rho};\frac{h}{\rho^2}\right).$$

(b) *Note that* $(0,a)\in\mathcal{S}_1$ *if and only if* $-1\le a\le 1$ *and* $(1,a)\in\mathcal{S}_1$ *if and only if* $-\rho\le a\le 1$.

Proposition 5.3. *Let* μ *be the self-similar measure defined by an IFS in (5.1) with probability weights* $\{p,1-p\}$ *and with* ρ^{-1} *being a Pisot number. If* p *is rational, then* $\dim_2(\mu)<1$ *and thus* μ *is singular.*

Proof. Note that for $p_1=p$ and $p_2=1-p$, the characteristic polynomial of the matrix $M_\alpha(\infty)$ has coefficients depending on both p and $\beta^\alpha := 1/\rho^\alpha$. Let $q(p,1/\rho^\alpha;\lambda)$ be the characteristic polynomial of the matrix $M_\alpha(\infty)$, with variable λ. Suppose, on the contrary, that $\dim_2(\mu)=1$. Then $\lambda=1$ when $\alpha=1$. Hence $\beta=1/\rho$ is a root of the polynomial $q(p,x;1)$ in the variable x. Since $q(p,x;1)$ has rational coefficients, by multiplying with the greatest common divisor if necessary, we can convert the equation $q(p,x;1)=0$ to $\tilde{q}(p,x;1)=0$, where $\tilde{q}(p,x;1)$ has integer coefficients. Thus, all algebraic conjugates β' of β, with $|\beta'|<1$ are also roots of $\tilde{q}(p,x;1)=0$. It follows that $q(p,\beta';1)=0$. This implies that the matrix $M_1'(\infty)$, obtained from $M_1(\infty)$ by replacing $\beta=1/\rho$ with β', also has maximal eigenvalue 1. This is impossible since $|\beta'|<1<\beta$ (see [Mi, Theorem 2.1]). Thus $\alpha<1$ when $\lambda=1$ and the result follows. ☐

The following known example shows that if p is irrational, then $\dim_2(\mu)$ can be 1; it also serves to illustrate our method. The IFS in this example satisfies the open set condition.

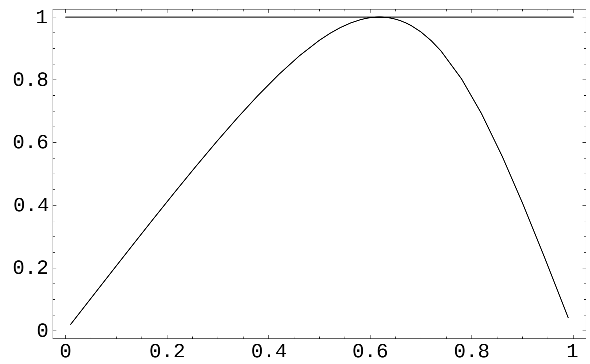

FIGURE 1. Figure showing $\dim_2(\mu)$ as a function of p_1 for the measure μ in Example 5.4, together with the horizontal line $y = 1$.

Example 5.4. *Let $\beta = \rho^{-1} = (\sqrt{5}+1)/2$ be the golden ratio ($\beta = 1.6180339887\ldots$, $\rho = 0.6180339887\ldots$). Let $\{S_1, S_2\}$ be defined as in (5.1). Then* $\dim_2(\mu) = 1$ *if and only if $p_1 = \rho$ and $p_2 = \rho^2$.*

Using the equality $\rho + \rho^2 = 1$ and Remark 5.2(a), we have

$$T(0,0;h) = \frac{p_1^2}{\rho^\alpha}\left(0,0;\frac{h}{\rho}\right) + \frac{p_2^2}{\rho^{2\alpha}}\left(0,0;\frac{h}{\rho^2}\right) + \frac{2p_1p_2}{\rho^{2\alpha}}\left(1,1;\frac{h}{\rho^2}\right).$$

Recall that T_1 is the restriction of T on $\mathcal{S}_1$. By Remark 5.2(b),

$$T_1(0,0;h) = T(0,0;h).$$

Similarly we get

$$T_1\left(1,1;\frac{h}{\rho^2}\right) = \frac{p_2}{\rho^2}\left(1,-\rho;\frac{h}{\rho}\right)$$
$$T_1\left(1,-\rho;\frac{h}{\rho}\right) = \frac{p_1^2p_2}{\rho^2}\left(1,-\rho;\frac{h}{\rho^2}\right).$$

Hence, we can order the states in $\mathcal{S}_1$ as $\{(0,0),\ (1,1),\ (1,-\rho)\}$. Moreover,

$$\mathbf{\Phi}^{(\alpha)}(h) = A_1\mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho}\right) + A_2\mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho^2}\right),$$

where $A_i = A_i(\alpha)$, $i = 1,2$, are defined as

$$A_1 = \frac{1}{\rho^\alpha}\begin{bmatrix} p_1^2 & 0 & 0 \\ 0 & 0 & p_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{\rho^{2\alpha}}\begin{bmatrix} p_2^2 & 2p_1p_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_1^2p_2 \end{bmatrix}.$$

Letting $p_1 = p$ and $p_2 = 1-p$, we get

$$\mathbf{M}_\alpha(\infty) = \begin{bmatrix} \frac{p^2}{\rho^\alpha} + \frac{(1-p)^2}{\rho^{2\alpha}} & \frac{2p(1-p)}{\rho^{2\alpha}} & 0 \\ 0 & 0 & \frac{(1-p)}{\rho^\alpha} \\ 0 & 0 & \frac{p^2(1-p)}{\rho^{2\alpha}} \end{bmatrix}.$$

We claim that for $0 < p < 1$ and $0 < \alpha \le 1$,

$$\frac{p^2}{\rho^\alpha} + \frac{(1-p)^2}{\rho^{2\alpha}} \ge \frac{p^2(1-p)}{\rho^{2\alpha}}.$$

This is clearly true if $p \le \rho$, since $1 - p \ge 1 - \rho = \rho^2 \ge p^2$. If $p > \rho$, then

$$\frac{p^2(1-p)}{\rho^{2\alpha}} < \frac{p^2(1-\rho)}{\rho^{2\alpha}} = \frac{p^2\rho^2}{\rho^{2\alpha}} < \frac{p^2}{\rho^\alpha},$$

proving the claim. Thus, $\dim_2(\mu)$ is the unique solution of

$$\frac{p^2}{\rho^\alpha} + \frac{(1-p)^2}{\rho^{2\alpha}} = 1,$$

recovering (2.1). Solving the formula yields

$$\dim_2(\mu) = \frac{\log\left((p^2 + \sqrt{4(1-p)^2 + p^4})/2\right)}{\log \rho}.$$

$\dim_2(\mu)$ is a strictly concave function of p that attains its a unique maximum value of 1 when $p = \rho$. Hence μ is absolutely continuous if and only if $p = \rho$. Figure 1 plots the graph of $\dim_2(\mu)$ as p varies.

The IFS in the following example does not satisfy the open set condition.

Example 5.5. *Let $\beta = \rho^{-1}$ be the Pisot number defined by the polynomial $x^3 - x^2 - 1 = 0$ ($\beta = 1.4655712318\ldots$, $\rho = 0.6823278038\ldots$). Let $\{S_1, S_2\}$ be defined as in (5.1). We will compute the L^2-dimension for the corresponding self-similar measures with weights $\{p_1, p_2\}$ by using Theorem 1.1.*

Observe that

$$1 - \rho = \rho^3 \quad \text{and} \quad \frac{1-\rho^2}{\rho} = \rho^2 + \rho^3.$$

Hence, starting with $a = 0$ and using the iteration formulas in Remark 5.2(a) we have

$$T(0,0;h) = \frac{p_1^2}{\rho^\alpha}\left(0,0;\frac{h}{\rho}\right) + \frac{p_2^2}{\rho^{2\alpha}}\left(0,0;\frac{h}{\rho^2}\right) + \frac{2p_1p_2}{\rho^{2\alpha}}\left(1,\rho^2+\rho^3;\frac{h}{\rho^2}\right).$$

Remark 5.2(b) implies that $T_1(0,0;h) = T(0,0;h)$.

Now we need to iterate $T(1, \rho^2 + \rho^3, h)$. By using Remark 5.2(a), we get

$$\begin{aligned} T(1,\rho^2+\rho^3;h) = {} & \frac{p_1^3}{\rho^\alpha}\left(0,1+\rho;\frac{h}{\rho}\right) + \frac{p_2}{\rho^\alpha}\left(1,-\rho+\rho^3;\frac{h}{\rho}\right) + \frac{p_1p_2^2}{\rho^{2\alpha}}\left(0,1+\frac{1}{\rho};\frac{h}{\rho^2}\right) \\ & + \frac{p_1^2p_2}{\rho^{2\alpha}}\left(1,1+\rho+\rho^2+\rho^3;\frac{h}{\rho^2}\right) + \frac{p_1^2p_2}{\rho^{2\alpha}}\left(1,-1-\rho+\rho^2+\rho^3;\frac{h}{\rho^2}\right). \end{aligned}$$

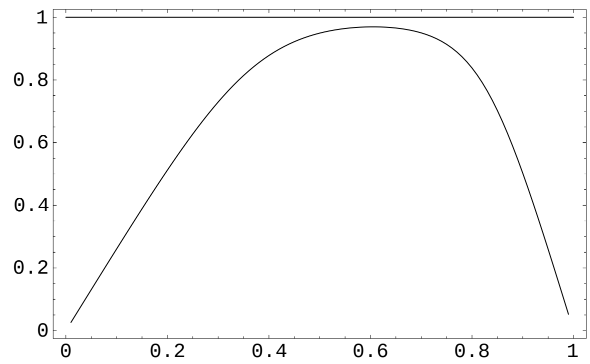

FIGURE 2. $\dim_2(\mu)$ as a function of p_1 for the measure μ in Example 5.5. The horizontal line $y = 1$ is also shown.

By Remark 5.2(b), the state in the second term is the only one satisfying the conditions to be in $\mathcal{S}_1$. Hence we have

$$T_1(1, \rho^2 + \rho^3; h) = \frac{p_2}{\rho^\alpha}\left(1, -\rho + \rho^3; \frac{h}{\rho}\right).$$

Iteration formulas for the other $T(0, a, h)$ and $T(1, a, h)$ are summarized below.

$$\begin{aligned}
T_1(1, -\rho + \rho^3; h) &= \frac{p_1^3}{\rho^\alpha}\left(0, -\rho^2 - \rho^3; \frac{h}{\rho}\right) + \frac{p_1^2 p_2}{\rho^{2\alpha}}\left(1, 0; \frac{h}{\rho^2}\right)\\
T_1(1, 0; h) &= \frac{p_1^3}{\rho^\alpha}\left(0, 0; \frac{h}{\rho}\right) + \frac{p_2}{\rho^\alpha}\left(1, \rho^2 + \rho^3; \frac{h}{\rho}\right) + \frac{p_1 p_2^2}{\rho^{2\alpha}}\left(0, 0; \frac{h}{\rho^2}\right)\\
&\quad + \frac{2p_1^2 p_2}{\rho^{2\alpha}}\left(1, \rho^2 + \rho^3; \frac{h}{\rho^2}\right)\\
T_1(0, -\rho^2 - \rho^3; h) &= \frac{p_1 p_2}{\rho^{2\alpha}}\left(1, -\rho + \rho^3; \frac{h}{\rho^2}\right).
\end{aligned}$$

To express this in matrix form, we order the states in $\mathcal{S}_1$ as

$$\big\{(0, 0),\ (0, -\rho^2 - \rho^3),\ (1, \rho^2 + \rho^3),\ (1, -\rho + \rho^3),\ (1, 0)\big\}.$$

Then we have

$$\mathbf{\Phi}^{(\alpha)}(h) = A_1 \mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho}\right) + A_2 \mathbf{\Phi}^{(\alpha)}\left(\frac{h}{\rho^2}\right),$$

where $A_i = A_i(\alpha)$, $i = 1, 2$, are defined as

$$A_1 = \frac{1}{\rho^\alpha}\begin{bmatrix} p_1^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & p_2 & 0\\ 0 & p_1^3 & 0 & 0 & 0\\ p_1^3 & 0 & p_2 & 0 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{\rho^{2\alpha}}\begin{bmatrix} p_2^2 & 0 & 2p_1p_2 & 0 & 0\\ 0 & 0 & 0 & p_1p_2 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & p_1^2p_2\\ p_1p_2^2 & 0 & 2p_1^2p_2 & 0 & 0 \end{bmatrix}.$$

It is straightforward to check that $\mathbf{M}_\alpha(\infty) = (A_1(\alpha) + A_2(\alpha))^t$ is irreducible. By Theorem 1.1, $\dim_2(\mu) = \alpha$, where α is the unique real number such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1. For $p_1 = p_2 = 1/2$ we get $\dim_2(\mu) = 0.9494876888\ldots$. For $p_1 = 2/3$, $\dim_2(\mu) = 0.9622237997\ldots$. In both cases, μ is singular by Theorem 2.4. Note that in both cases, $\prod_{i=1}^q (p_i/\rho_i)_i^p < 1$ and thus (1.3) is satisfied. Figure 2 plots $\dim_2(\mu)$ as a function of p_1. The maximum value of $\dim_2(\mu)$ can be found numerically to be $0.9695030858\ldots$ and is attained when $p = 0.6052351543\ldots$.

We do not have an algebraic proof for the singularity of all the μ in this example. It is desirable to have an algebraic technique to study the corresponding measures defined by all the Pisot numbers $\beta = \rho^{-1}$ satisfying $\rho + \rho^2 > 1$; there are finitely many of them.

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