# DEPARTMENT OF MATHEMATICAL SCIENCES

# TECHNICAL REPORT SERIES

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Number 2006-006
Submitted: January 16, 2006

# Measure of non-compactness for filters in the approach setting

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November 1, 2005

**Abstract**

New explicit descriptions of the reflections of a convergence-approach space on pseudo-approach, pre-approach and approach spaces are given. A measure of non-compactness for filters is introduced in the context of convergence-approach spaces. It is shown that this measure generalizes all the known measures of variants of compactness and can also be used to describe the reflections mentioned above.

## 1 Terminology

Let $(\mathbb{F}X, \leq)$ denote the set of filters on $X$,ordered by inclusion ( inverse to the monad order). Let $\mathbb{U}X$ be the subset of $\mathbb{F}X$ formed by ultrafilters on $X$ and, given $\mathcal{G} \in \mathbb{F}X$, let $\mathbb{U}(\mathcal{G})$ be the set of ultrafilters that are finer than $\mathcal{G}$.

Following [7] and [8], I call *convergence-approach limit* on $X$ a map $\lambda : \mathbb{F}X \to [0, \infty]^X$ which fulfills the properties:

$$\forall x \in X,\ \lambda(x)(x) = 0; \tag{CAL1}$$
$$\mathcal{G} \geq \mathcal{F} \Longrightarrow \lambda(\mathcal{F}) \geq \lambda(\mathcal{G}); \tag{CAL2}$$
$$\forall \mathcal{F}, \mathcal{G} \in \mathbb{F}X,\ \lambda(\mathcal{F} \wedge \mathcal{G}) = \lambda(\mathcal{F}) \bigvee \lambda(\mathcal{G}). \tag{CAL3}$$

$(X, \lambda)$, shortly $X$, is called a *convergence-approach space*. A map $f : X \to Y$ between two convergence-approach spaces is a *contraction* if

$$\lambda_Y(f(\mathcal{F}))\ (f(\cdot)) \leq \lambda_X(\mathcal{F})(\cdot),$$

for every $\mathcal{F} \in \mathbb{F}X$. The category with convergence-approach spaces as objects and contractions as morphisms is a cartesian-closed topological category denoted **CAP** [7]. Each convergence space $X$ can be considered as a convergence-approach space by stating

$$\lambda_X(\mathcal{F})(x) = \begin{cases} 0 & \text{if } x \in \lim_X \mathcal{F} \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, the category **Conv** of convergence spaces (and continuous maps) is included both reflectively and coreflectively in **CAP**. Indeed, if $\lambda$ is a convergence-approach, then its **Conv**-coreflection is $c(\lambda)$ defined by $x \in \lim_{c(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) = 0$, while its **Conv**-reflection is $r(\lambda)$ defined by $x \in \lim_{r(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) < \infty$. The canonical Hom-structure of **CAP** is described for example in [8]. If $X$ and $Z$ are two convergence-approach spaces, the limit $\lambda$ on the set $\mathrm{Hom}(X, Z)$ of contractions from $X$ to $Z$ is given by

$$\lambda(\mathcal{F})(f) = \bigwedge\{\alpha : \underset{\mathcal{G}\in\mathbb{F}X}{\forall} \lambda_Z(ev(\mathcal{G}\times\mathcal{F}))\circ f(\cdot) \le \lambda_X(\mathcal{G})(\cdot)\bigvee \alpha\},$$

and is called *continuous convergence-approach*. Since $\lambda$ coincides with the continuous convergence in case $X$ and $Z$ are convergences, I extend to **CAP** the notation of **Conv** and use $[X, Z]$ instead of $\lambda$.

A convergence-approach $\lambda$ is a *pseudo-approach space* [8] if

$$\forall \mathcal{F} \in \mathbb{F}X,\ \lambda(\mathcal{F}) = \bigvee_{\mathcal{U}\in\mathbb{U}(\mathcal{F})} \lambda(\mathcal{U}); \tag{PSAP}$$

and it is a *pre-approach space* [7] if $(CAL3)$ is strengthened to

$$\lambda(\bigwedge_{j\in J} \mathcal{F}_j) = \bigvee_{j\in J} \lambda(\mathcal{F}_j),\ \text{for any family } (\mathcal{F}_j)_{j\in J} \text{ of filters.} \tag{PRAP}$$

The category **PSAP** of pseudo-approach spaces (and contractions) contains the category **PsTop** of pseudotopological spaces (and continuous maps) and the category **PRAP** of pre-approach spaces contains the category **PrTop** of pretopological spaces both reflectively and coreflectively (via the restrictions of $c$ and $r$).

A map $\mathcal{G}(\cdot) : X \to \mathbb{F}(X)$ is called a *selection of filters on* $X$. If $\mathcal{F} \in \mathbb{F}(X)$ and $\mathcal{G}(\cdot)$ is a selection of filters, the *contour of* $\mathcal{G}(\cdot)$ along $\mathcal{F}$ is the filter

$$\int_{\mathcal{F}} \mathcal{G} = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} \mathcal{G}(x).$$

If more generally $\phi$ is a map from $X$ to a complete lattice $(L, \vee, \wedge)$ and $\mathcal{F} \subset 2^X$, we also call

$$\int_{\mathcal{F}} \phi = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} \phi(x)$$

the *contour of* $\phi$ *along* $\mathcal{F}$. We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set $X$ *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We write $A\#\mathcal{B}$

for $\{A\}\#\mathcal{B}$. The *grill of a family* $\mathcal{A}$ of subsets of $X$ is$\mathcal{A}^{\#} = \{H \subset X : H\#\mathcal{A}\}$. Note that [6]

$$\bigwedge_{F\in\mathcal{F}} \bigvee_{x\in F} \phi(x) = \int_{\mathcal{F}^{\#}} \phi.$$

In particular, if $\mathcal{F}$ is an ultrafilter, then $\bigwedge_{F\in\mathcal{F}} \bigvee_{x\in F} \phi(x) = \int_{\mathcal{F}^{\#}} \phi = \int_{\mathcal{F}} \phi = \bigvee_{F\in\mathcal{F}} \bigwedge_{x\in F} \phi(x)$.

An *approach space* is a pre-approach space fulfilling

$$\begin{aligned} \text{for any } \mathcal{F} &\in \mathbb{F}X \text{ and any selection } \mathcal{G}(\cdot) \text{ of filters,} \\ \lambda(\int_{\mathcal{F}} \mathcal{G})(\cdot) &\leq \lambda(\mathcal{F})(\cdot) + \bigvee_{x\in X} \lambda(\mathcal{G}(x))(x). \end{aligned} \tag{AP}$$

The category **Top** of topological spaces (with continuous maps) is a reflective and coreflective (via the restrictions of $r$ and $c$) subcategory of the category **AP** of approach spaces [10]. There are several other equivalent descriptions of **AP** and **PRAP** (see [11] and [5] for details).

## 2 Concrete endoreflectors of CAP

S. Dolecki presented in [3] a unified treatment of several important concrete endoreflectors and endocreflectors of **Conv**. In particular, given a class $\mathbb{J}$ of filters, he defined the modifications $\mathrm{Adh}_{\mathbb{J}}\,\xi$ and $\mathrm{Base}_{\mathbb{J}}\,\xi$ of a convergence $\xi$ on $X$ as follows:

$$\lim_{\mathrm{Adh}_{\mathbb{J}}\,\xi} \mathcal{F} = \bigcap_{\mathbb{J}\ni\mathcal{J}\#\mathcal{F}} \mathrm{adh}_{\xi}\,\mathcal{J},$$

and

$$\lim_{\mathrm{Base}_{\mathbb{J}}\,\xi} \mathcal{F} = \bigcup_{\mathbb{J}\ni\mathcal{J}\leq\mathcal{F}} \lim_{\xi}\,\mathcal{J}.$$

If the class $\mathbb{J}$ is independent of the convergence, stable by finite infimum and stable by relation ([1]), then $\mathrm{Adh}_{\mathbb{J}}$ is (the restriction to objects of) a reflector and $\mathrm{Base}_{\mathbb{J}}$ is (the restriction to objects of) a coreflector. In particular, when $\mathbb{J}$ is respectively the class $\mathbb{F}$ of all filters, the class $\mathbb{F}_{\omega}$ of countably based filters and the class $\mathbb{F}_1$ of principal filters, then $\mathrm{Adh}_{\mathbb{J}}$ is the reflector from **Conv** onto the category of pseudotopological, paratopological and pretopological spaces respectively; and $\mathrm{Base}_{\mathbb{J}}$ is the identity functor of **Conv**, the coreflector from **Conv** onto first-countable convergence spaces and the coreflector from **Conv** onto finitely generated convergence spaces, respectively.

As observed in [13], the definitions of the reflectors $\mathrm{Adh}_{\mathbb{J}}$ and of the coreflectors $\mathrm{Base}_{\mathbb{J}}$ extend from **Conv** to **CAP** via

$$\left(\mathrm{Adh}_{\mathbb{J}}\lambda\right)(\mathcal{F})(x) = \bigvee_{\mathbb{J}\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_{\lambda}\,\mathcal{H}(x), \tag{1}$$

[1] see [3] for more general conditions.

where

$$\operatorname{adh}_\lambda \mathcal{H}(\cdot) = \bigwedge_{\mathcal{G}\#\mathcal{H}} \lambda(\mathcal{G})(\cdot) = \bigwedge_{\mathcal{U}\in\mathbb{U}(\mathcal{H})} \lambda(\mathcal{U})(\cdot); \tag{2}$$

and

$$(\mathrm{Base}_{\mathbb{J}}\lambda)(\mathcal{F})(\cdot) = \bigwedge_{\mathbb{J}\ni\mathcal{G}\leq\mathcal{F}} \lambda(\mathcal{G})(\cdot). \tag{3}$$

When $\mathbb{J}$ is respectively the class of all filters and of principal filters, $\mathrm{Adh}_{\mathbb{J}}$ is respectively the reflector on **PSAP** and on **PRAP**. This last fact, which was used without proof in [13] and which was proved in unpublished notes ([2]), is not so obvious and gives a new explicit description of the reflector on **PRAP**. Theorem 1 below provides a (new) proof as well as additional new characterizations of the **PRAP**-reflection of a **CAP**-object. Moreover, the category **PARAP** of para-approach spaces is introduced as the category of fixed points for $\mathrm{Adh}_{\mathbb{J}}$ with the class $\mathbb{J}$ of countably based filters.

Notice that (1) gives an explicit description of the reflection of a **CAP**-object on **PSAP**, **PARAP** or **PRAP**, but not on **AP**. Recall ([11] for approach systems) that a collection $(\mathcal{A}(x))_{x\in X}$ of ideals in $[0,\infty]^X$ is a *pre-approach system* if for all $x \in X$, $\mathcal{A}(x)$ is saturated ([3]) and $\phi(x) = 0$ for every $\phi \in \mathcal{A}(x)$. If moreover

$$\begin{aligned} \forall\phi \quad &\in \quad \mathcal{A}(x), \forall\varepsilon>0, \forall\omega<\infty, \exists\, (\phi_z)_{z\in X} \in \prod_{z\in X}\mathcal{A}(z) : \\ \forall z,y \quad &\in \quad X : \phi(y)\wedge\omega \leq \phi_x(z)+\phi_z(y)+\varepsilon, \end{aligned} \tag{4}$$

then $\mathcal{A}(x)$ is called an *approach system.*

An (pre-)approach system $(\mathcal{A}(x))_{x\in X}$ determines an (**PR**)**AP**-object $(X,\lambda_{\mathcal{A}})$ via ([11] for the **AP** case)

$$\lambda_{\mathcal{A}}(\mathcal{F})(x) = \bigvee_{\phi\in\mathcal{A}(x)} \int_{\mathcal{F}^\#} \phi. \tag{5}$$

Indeed,

$$\begin{aligned} \lambda_{\mathcal{A}}(\bigwedge_{j\in J}\mathcal{F}_j)(x) &= \bigvee_{\phi\in\mathcal{A}(x)} \int_{\left(\bigwedge_{j\in J}\mathcal{F}_j\right)^\#} \phi = \bigvee_{\phi\in\mathcal{A}(x)} \int_{\bigcup_{j\in J}\mathcal{F}_j^\#} \phi \\ &= \bigvee_{j\in J}\bigvee_{\phi\in\mathcal{A}(x)} \int_{\mathcal{F}_j^\#} \phi = \bigvee_{j\in J} \lambda_{\mathcal{A}}(\mathcal{F}_j)(x) \end{aligned}$$

so that $(PRAP)$ is satisfied.

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[2] See http://www.cs.georgiasouthern.edu/faculty/mynard_f/APPRO.pdf

[3] $\phi \in [0,\infty]^X$ *is dominated by* $\mathcal{A}\subset[0,\infty]^X$ if for every $\varepsilon>0$ and every $\omega<\infty$, there exists $\phi_\varepsilon^\omega \in \mathcal{A}$ such that $\phi\wedge\omega \leq \phi_\varepsilon^\omega+\varepsilon$. $\mathcal{A}\subset[0,\infty]^X$ is *saturated* if every function dominated by $\mathcal{A}$ belongs to $\mathcal{A}$.

Conversely, an (**PR**)**AP**-object $(X,\lambda)$ determines an (pre-)approach system $(\mathcal{A}_\lambda(x))_{x\in X}$ via

$$\begin{aligned}\mathcal{A}_\lambda(x) &= \left\{\phi\in[0,\infty]^X : \forall\mathcal{U}\in\mathbb{U}(X), \int_{\mathcal{U}}\phi\leq\lambda(\mathcal{U})(x)\right\} \qquad (6)\\ &= \left\{\phi\in[0,\infty]^X : \forall\mathcal{F}\in\mathbb{F}(X), \int_{\mathcal{F}\#}\phi\leq\lambda(\mathcal{F})(x)\right\}.\end{aligned}$$

Moreover, $\mathcal{A}_{\lambda_\mathcal{A}}(x)=\mathcal{A}(x)$ and $\lambda_{\mathcal{A}_\lambda}=\lambda$.

Notice that (6) is meaningful for any **CAP**-object $(X,\lambda)$, and we have

**Theorem 1** *Let $(X,\lambda)$ be a convergence-approach space. Denote by $P\lambda$ the **PRAP**-reflection of $(X,\lambda)$.*

1. *The family $(\mathcal{A}_\lambda(x))_{x\in X}$ defined by (6) is a pre-approach system. Moreover $(\mathcal{A}_\lambda(x))_{x\in X}=(\mathcal{A}_{P\lambda}(x))_{x\in X}$ and*

$$(P\lambda)(\mathcal{F})(x)=\bigvee_{\phi\in\mathcal{A}_\lambda(x)}\int_{\mathcal{F}\#}\phi.$$

2. $$(P\lambda)(\mathcal{F})(x)=\bigvee_{A\#\mathcal{F}}\operatorname{adh}_\lambda A(x).$$

**Proof.** (1) As observed before $\lambda_{\mathcal{A}_\lambda}(\mathcal{F})(x)=\bigvee_{\phi\in\mathcal{A}_\lambda(x)}\int_{\mathcal{F}\#}\phi$ is a pre-approach structure. Moreover, $\lambda_{\mathcal{A}_\lambda}\leq\lambda$ by definition of $\mathcal{A}_\lambda$. If $\upsilon$ is a pre-approach structure, then $\lambda_{\mathcal{A}_\upsilon}=\upsilon$. Therefore, if additionally $\upsilon\leq\lambda$ then $\mathcal{A}_\upsilon(x)\subset\mathcal{A}_\lambda(x)$ and $\lambda_{\mathcal{A}_\upsilon}(\mathcal{F})(x)=\upsilon(\mathcal{F})(x)\leq\lambda_{\mathcal{A}_\lambda}(\mathcal{F})(x)$. Hence $\lambda_{\mathcal{A}_\lambda}(\mathcal{F})(x)=(P\lambda)(\mathcal{F})(x)$.

(2) Let $\widetilde{\lambda}\mathcal{F}(x)=\bigvee_{A\#\mathcal{F}}\operatorname{adh}_\lambda A(x)$. It is clear that $\widetilde{\lambda}\leq\lambda$ because $\operatorname{adh}_\lambda A(x)\leq\lambda\mathcal{F}(x)$ whenever $A\#\mathcal{F}$. Moreover $\widetilde{\lambda}$ is a pre-approach structure because

$$\begin{aligned}\widetilde{\lambda}\left(\bigwedge_{j\in J}\mathcal{F}_j\right)(x) &= \bigvee_{A\in\left(\bigwedge_{j\in J}\mathcal{F}_j\right)^\#}\operatorname{adh}_\lambda A(x)=\bigvee_{A\in\bigcup_{j\in J}\mathcal{F}_j^\#}\operatorname{adh}_\lambda A(x)\\ &= \bigvee_{j\in J}\bigvee_{A\#\mathcal{F}_j}\operatorname{adh}_\lambda A(x)=\bigvee_{j\in J}\widetilde{\lambda}\left(\mathcal{F}_j\right)(x).\end{aligned}$$

Finally, notice that if $\upsilon$ is a pre-approach structure, then $\widetilde{\upsilon}=\upsilon$. Indeed,

$$\widetilde{\upsilon}(\mathcal{F})(x)=\bigvee_{A\#\mathcal{F}}\bigwedge_{\mathcal{W}\in\mathbb{U}(A)}\upsilon(\mathcal{W})(x)\geq\bigvee_{A\#\mathcal{F}}\bigwedge_{\mathcal{W}\in\mathbb{U}(A)}\int_{\mathcal{W}}\phi$$

for every $\phi \in \mathcal{A}_\upsilon(x)$. But $\int_{\mathcal{W}} \phi = \bigvee_{W\in\mathcal{W}} \bigwedge_{y\in W} \phi(y) \geq \bigwedge_{y\in A} \phi(y)$ for every $\mathcal{W} \in \mathbb{U}(A)$,so that

$$\widetilde{\upsilon}(\mathcal{F})(x) \geq \bigvee_{\phi\in\mathcal{A}_\upsilon(x)} \bigvee_{A\#\mathcal{F}} \bigwedge_{y\in A} \phi(y) = \bigvee_{\phi\in\mathcal{A}_\upsilon(x)} \int_{\mathcal{F}^\#} \phi = \upsilon(\mathcal{F})(x).$$

In particular, if $\upsilon$ is a pre-approach structure such that $\upsilon \leq \lambda$ then $\upsilon = \widetilde{\upsilon} \leq \widetilde{\lambda}$. ■

**Proposition 2** *Let $(X, \lambda)$ be a convergence-approach space and let $A \subset X$.*

$$\operatorname{adh}_\lambda A(x) = \operatorname{adh}_{P\lambda} A(x) = \bigvee_{\phi\in\mathcal{A}_\lambda(x)} \bigwedge_{y\in A} \phi(y).$$

**Proof.** since $P\lambda \leq \lambda$, it is clear that $\operatorname{adh}_{P\lambda} A(x) \leq \operatorname{adh}_\lambda A(x)$. Moreover, $\operatorname{adh}_\lambda A(x) \leq \operatorname{adh}_{P\lambda} A(x)$ because $(P\lambda)(\mathcal{F})(x) = \bigvee_{A\#\mathcal{F}} \operatorname{adh}_\lambda A(x)$, so that

$$\forall \mathcal{F}\#A,\ \ (P\lambda)(\mathcal{F})(x) \geq \operatorname{adh}_\lambda A(x).$$

Now,

$$\begin{aligned} \operatorname{adh}_\lambda A(x) &= \bigwedge_{\mathcal{G}\#A} \lambda\mathcal{G}(x) = \bigwedge_{\mathcal{G}\#A} (P\lambda)(\mathcal{G})(x) = \bigwedge_{\mathcal{G}\#A} \bigvee_{\phi\in\mathcal{A}_\lambda(x)} \int_{\mathcal{G}^\#} \phi \\ &= \bigvee_{\phi\in\mathcal{A}_\lambda(x)} \bigwedge_{\mathcal{G}\#A} \int_{\mathcal{G}^\#} \phi = \bigvee_{\phi\in\mathcal{A}_\lambda(x)} \bigwedge_{y\in A} \phi(y). \end{aligned}$$

Indeed $\bigvee_{\phi\in\mathcal{A}_\lambda(x)} \bigvee_{B\#\mathcal{G}} \bigwedge_{y\in B} \phi(y) \geq \bigvee_{\phi\in\mathcal{A}_\lambda(x)} \bigwedge_{y\in A} \phi(y)$ for every $\mathcal{G}\#A$. Moreover, for $a \in A$ and $\mathcal{G} = \{a\}^\uparrow$, we have $\int_{\mathcal{G}^\#} \phi = \phi(a)$ so that $\bigwedge_{\mathcal{G}\#A} \int_{\mathcal{G}^\#} \phi \leq \bigwedge_{y\in A} \phi(y)$. ■

If $\{(X, \lambda_i) : i \in I\}$ is a family of **CAP**-objects, let

$$\widehat{\mathcal{A}}_{\{\lambda_i : i\in I\}}(x) = \left\{ \begin{array}{c} \phi \in [0,\infty]^X : \forall \mathcal{F} \in \mathbb{F}X, \forall \mathcal{G}(\cdot) : X \to \mathbb{F}X, \\ \int_{\left(\int_{\mathcal{F}} \mathcal{G}\right)^\#} \phi \leq \bigwedge_{i\in I} \left( \lambda_i(\mathcal{F})(x) + \bigvee_{y\in X} \lambda_i\left(\mathcal{G}(y)\right)(y) \right) \end{array} \right\}.$$

Notice that $\widehat{\mathcal{A}}_{\{\lambda\}}(x) \subset \mathcal{A}_\lambda(x)$.

Given a **CAP**-object $(X, \lambda)$, define the following decreasing transfinite sequence of pre-approach spaces:

$$\begin{aligned} \lambda^1 &= \lambda_{\mathcal{A}_\lambda} = P\lambda; \\ \lambda^\alpha &= \lambda_{\widehat{\mathcal{A}}_{\{\lambda^\beta : \beta<\alpha\}}}. \end{aligned}$$

Evidently, if $\alpha$ is not a limit ordinal bigger than 1, then $\lambda^\alpha = \lambda_{\widehat{\mathcal{A}}_{\{\lambda^{\alpha-1}\}}}$. There exists the smallest ordinal $\alpha$ such that $\lambda^\alpha = \lambda^{\alpha+1}$. Denote (momentarily) this pre-approach limit by $\overline{\lambda}$. Notice that

$$\widehat{\mathcal{A}}_{\{\overline{\lambda}\}}(x) = \mathcal{A}_{\overline{\lambda}}(x),$$

because if $\phi \in \mathcal{A}_{\overline{\lambda}}(x)$ then for every $\mathcal{F} \in \mathbb{F}X$ and every selection $\mathcal{G}(\cdot)$ of filters, we have

$$\int_{(\int_{\mathcal{F}} \mathcal{G})^{\#}} \phi \leq \overline{\lambda}\left(\int_{\mathcal{F}} \mathcal{G}\right) = \bigvee_{\Phi \in \widehat{\mathcal{A}}_{\{\overline{\lambda}\}}(x)} \int_{(\int_{\mathcal{F}} \mathcal{G})^{\#}} \Phi \leq \overline{\lambda}(\mathcal{F})(x) + \bigvee_{y \in X} \overline{\lambda}\left(\mathcal{G}(y)\right)(y)$$

so that $\phi \in \widehat{\mathcal{A}}_{\{\overline{\lambda}\}}(x)$.

**Theorem 3** *Given a* ***CAP****-object* $(X, \lambda)$*, the space* $(X, \overline{\lambda})$ *is the* ***AP****-reflection of* $(X, \lambda)$*.*

**Proof.** It is clear that $\lambda \geq \overline{\lambda}$ and that $\lambda_1 \geq \lambda_2 \Longrightarrow \overline{\lambda_1} \geq \overline{\lambda_2}$. Moreover, $(X, \overline{\lambda})$ is an approach space. Indeed, (1) is satisfied because

$$\overline{\lambda}(\int_{\mathcal{F}} \mathcal{G})(x) = \bigvee_{\phi \in \mathcal{A}_{\overline{\lambda}}(x)} \int_{(\int_{\mathcal{F}} \mathcal{G})^{\#}} \phi \leq \overline{\lambda}(\mathcal{F})(x) + \bigvee_{y \in X} \overline{\lambda}\left(\mathcal{G}(y)\right)(y).$$

Notice that $\overline{\upsilon} = \upsilon^1 = \upsilon$, whenever $(X, \upsilon)$ is an approach space. Therefore, $\overline{\lambda}$ is the finest approach structure coarser than $\lambda$. Indeed, if $\upsilon$ is an approach structure such that $\lambda \geq \upsilon$ then $\overline{\lambda} \geq \overline{\upsilon} = \upsilon$. ■

From now on, we denote $\overline{\lambda}$ by $T\lambda$, where $T$ is the reflector from **CAP** onto **AP**.

# 3 Compactness in CAP

The *measure of non* $\mathbb{D}$*-compactness of a filter* $\mathcal{F}$ *at* $\mathcal{A} \subset [0, \infty]^X$ is

$$c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F}) = \bigvee_{\mathbb{D} \ni \mathcal{D} \# \mathcal{F}} \bigvee_{\phi \in \mathcal{A}} \bigwedge_{x \in X} \left(\mathrm{adh}_{\lambda} \mathcal{D} + \phi\right)(x). \tag{7}$$

This definition is motivated by the special case where $\lambda = r\lambda$ and $\mathcal{A} \subset 2^X$ via the identification of $A \subset X$ with the *characteristic function* $\theta_A$ *of* $A$ taking the value 0 on $A$ and $\infty$ on $A^c$. In this case, a filter $\mathcal{F}$ is $\mathbb{D}$-compactoid at $\mathcal{A}$ (in the sense of [4]) if and only if $c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F}) = 0$. By a convenient abuse of notation, we will write $c_{\mathbb{D}}^{A}(\mathcal{F})$ for $c_{\mathbb{D}}^{\{\theta_A\}}(\mathcal{F})$ whenever $A \subset X$. Notice that

$$c_{\mathbb{D}}^{A}(\mathcal{F}) = \bigvee_{\mathbb{D} \ni \mathcal{D} \# \mathcal{F}} \bigwedge_{a \in A} \mathrm{adh}_{\lambda} \mathcal{D}(a).$$

**Example 4** *(measures of compactness of sets and variants)*
*If* $\mathcal{F} = \{X\}$*,* $\mathbb{D} = \mathbb{F}$ *and* $\mathcal{A} = \{\theta_X\}$*,* $c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F})$ *corresponds to the measure of non-compactness* $m(X)$ *introduced in [9]. Recall that if* $X$ *is a topological approach space, then* $X$ *is compact if and only if* $m(X) = 0$ *[9, Theorem 4.3]; if* $X$ *is an extended pseudo-quasi-metric approach space, then* $X$ *is totally bounded if and only if* $m(X) = 0$ *[9, Theorem 4.4]; if* $X$ *is a pseudo-quasi-metric approach*

*space, then $X$ is bounded if and only if $m(X) < \infty$ [9, Proposition 4.5].*
*The measures of relative compactness $\overline{C}(A,X)$, relative countable compactness, relative sequential compactness introduced in [2] are also instances of measures of the type $c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F})$ for $\mathcal{F} = \{A\}^{\uparrow}$, $\mathcal{A} = \{\theta_X\}$ (where $A \subset X$) and $\mathbb{D}$ is respectively the class $\mathbb{F}$ of all filters, the class $\mathbb{F}_\omega$ of countably based filters, and $\mathbb{F}_\omega$ again but the measure is taken in $\mathrm{Base}_{\mathbb{E}}\,\lambda$ instead of $\lambda$, where $\mathbb{E}$ is the class of filters generated by sequences. In the same way, the measure of Lindelöf of an approach space in the sense of [1] is of the type $c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F})$ for $\mathcal{F} = \{X\}$, $\mathcal{A} = \{\theta_X\}$ and $\mathbb{D}$ the class $\mathbb{F}_{\wedge\omega}$ of countably deep filters ([4]).*

On the other hand, the following section shows that measures of non-principal filters are also of fundamental interest.

## 3.1 Measure of non-compactness and approach limits in reflections of a CAP-object

Measures of non $\mathbb{D}$-compactness for filters generalizes both usual measure of non-compactness for sets and approach limits. It is this very fact that allows to derive a variety of corollaries from any result on the measure of non $\mathbb{D}$-compactness of filters.

**Theorem 5** *Let $(X,\lambda)$ be a convergence-approach space and $\mathbb{J}$ be a composable class of filters. Then*

1. $$(T\lambda)\mathcal{F}(x) = c_{\mathbb{F}_1}^{\mathcal{A}_{T\lambda}(x)}(\mathcal{F}).$$

2. $$(\mathrm{Adh}_{\mathbb{J}}\,\lambda)(\mathcal{F})(x) = c_{\mathbb{J}}^{\{x\}}(\mathcal{F}).$$

**Proof.** (1).

$$\begin{aligned}(T\lambda)\mathcal{F}(x) &= \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \int_{\mathcal{F}^{\#}} \phi = \bigvee_{A\#\mathcal{F}} \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{y\in A} \phi(y) \\ &= \bigvee_{A\#\mathcal{F}} \mathrm{adh}_{T\lambda}\, A(x)\end{aligned}$$

by Proposition 2. Moreover,

$$\mathrm{adh}_{T\lambda}\, A(x) = \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{y\in X} (\mathrm{adh}_\lambda\, A + \phi)(y).$$

Indeed,

$$\mathrm{adh}_{T\lambda}\, A(x) = \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{y\in A} (\mathrm{adh}_\lambda\, A + \phi)(y) \geq \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{z\in X} (\mathrm{adh}_\lambda\, A + \phi)(z)$$

[4] a filter $\mathcal{F}$ is *countably deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

is clear and the reverse inequality follows from the fact that $(\mathcal{A}_{T\lambda}(x))_{x\in X}$ satisfies (4). Indeed, for a fixed $\phi_0 \in \mathcal{A}_{T\lambda}(x)$, a fixed $\varepsilon > 0$ and a fixed $\omega > 0$, there exists $(\phi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}_{T\lambda}(z)$ such that

$$\phi_0(y) \wedge \omega \leq \phi_x(z) + \phi_z(y) + \varepsilon \tag{8}$$

for every $y, z \in X$. Notice that for a fixed $z_0$, we have $\operatorname{adh}_\lambda A(z_0) \geq \operatorname{adh}_{T\lambda} A(z_0) \geq \bigwedge_{y\in A} \phi_{z_0}(y)$. Therefore,

$$\begin{aligned}\bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} (\operatorname{adh}_\lambda A + \phi)(z_0) + \varepsilon &\geq \varepsilon + \phi_x(z_0) + \bigwedge_{y\in A} \phi_{z_0}(y) = \bigwedge_{y\in A} \left(\phi_x(z_0) + \phi_{z_0}(y) + \varepsilon\right) \\ &\geq \bigwedge_{y\in A} \phi_0(y) \wedge \omega,\end{aligned}$$

by (8). Hence

$$\bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{z\in X} (\operatorname{adh}_\lambda A + \phi)(z) + \varepsilon = \bigwedge_{z\in X} \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} (\operatorname{adh}_\lambda A + \phi)(z) + \varepsilon \geq \bigwedge_{y\in A} \phi_0(y) \wedge \omega$$

for every $\phi_0 \in \mathcal{A}_{T\lambda}(x)$, every $\varepsilon > 0$ and every $\omega > 0$, so that

$$\bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{z\in X} (\operatorname{adh}_\lambda A + \phi)(z) \geq \bigvee_{\phi\in\mathcal{A}_{T\lambda}(x)} \bigwedge_{y\in A} \phi_0(y) = \operatorname{adh}_{T\lambda} A(x)$$

which completes the proof.

(2) follows directly from the definitions (1) and (7). ■

## 3.2 Invariance properties

Let $f : X \to Y$ and let $\mathcal{A} \subset [0,\infty]^X$. Then

$$f\mathcal{A} = \left\{ g \in [0,\infty]^Y : \exists h \in \mathcal{A} : \forall y \in Y, g(y) = \bigwedge_{x\in f^{-1}(y)} h(x) \right\}.$$

Notice that if $\mathcal{A} \subset 2^X$ is identified with $\{\theta_A : A \in \mathcal{A}\}$, then $f\mathcal{A} = \{f(A) : A \in \mathcal{A}\}$.

The following lemma is folklore and easy to verify.

**Lemma 6** *Let $f : (X, \lambda_X) \to (Y, \lambda_Y)$ be a contraction and let $\mathcal{D}$ be a filter on $Y$. Then, for every $y \in Y$*

$$\operatorname{adh}_Y \mathcal{D}(y) \leq \bigwedge_{x\in f^{-1}(y)} \operatorname{adh}_X f^-\mathcal{D}(x).$$

**Theorem 7** *Let $f:(X,\lambda_X)\to(Y,\lambda_Y)$ be a contraction, let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters, let $\mathcal{A}\subset[0,\infty]^X$ and let $\mathcal{F}$ be a filter on $X$. Then*

$$c_{\mathbb{D}}^{f\mathcal{A}}(f(\mathcal{F}))\leq c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F}).$$

**Proof.** Let $\mathcal{D}$ be a $\mathbb{D}$-filter that meshes with $f(\mathcal{F})$, equivalently, $f^-\mathcal{D}\#\mathcal{F}$. Hence

$$\bigvee_{\phi\in\mathcal{A}}\bigwedge_{x\in X}\left(\mathrm{adh}_X\, f^-\mathcal{D}+\phi\right)(x)\leq c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F}),$$

because $f^-\mathcal{D}\in\mathbb{D}(X)$ by $\mathbb{F}_1$-composability.

But

$$\begin{aligned}\bigvee_{\phi\in\mathcal{A}}\bigwedge_{x\in X}\left(\mathrm{adh}_X\, f^-\mathcal{D}+\phi\right)(x) &= \bigvee_{\phi\in\mathcal{A}}\bigwedge_{y\in Y}\bigwedge_{x\in f^{-1}(y)}\left(\mathrm{adh}_X\, f^-\mathcal{D}+\phi\right)(x)\\ &= \bigvee_{\Psi\in f\mathcal{A}}\bigwedge_{y\in Y}\left(\Psi(y)+\bigwedge_{x\in f^{-1}(y)}\mathrm{adh}_X\, f^-\mathcal{D}(x)\right)\end{aligned}$$

so that

$$c_{\mathbb{D}}^{\mathcal{A}}(\mathcal{F})\geq\bigvee_{\phi\in\mathcal{A}}\bigwedge_{x\in X}\left(\mathrm{adh}_X\, f^-\mathcal{D}+\phi\right)(x)\geq\bigvee_{\Psi\in f\mathcal{A}}\bigwedge_{y\in Y}\left(\Psi+\mathrm{adh}_Y\,\mathcal{D}\right)(y)=c_{\mathbb{D}}^{f\mathcal{A}}(f(\mathcal{F})).$$

■

**Corollary 8** *(see [2, Theorem 3.15]) The measures of compactness, countable compactness, sequential compactness, Lindelöf as defined in Example 4 decrease under contraction.*

**Corollary 9** *Let $f:(X,\lambda_X)\to(Y,\lambda_Y)$ be a contraction and let $\mathbb{D}$ be an $\mathbb{F}_1$-composable class of filters. Then for every filter $\mathcal{F}$ and every $x\in X$,*

$$\mathrm{Adh}_{\mathbb{D}}\,\lambda_X(\mathcal{F})(x)\geq\mathrm{Adh}_{\mathbb{D}}\,\lambda_Y(f(\mathcal{F}))(f(x)),$$

*which essentially means that $\mathrm{Adh}_{\mathbb{D}}$ is a concrete endofunctor of **CAP**.*

If $(X_i)_{i\in I}$ is a family of sets and $\mathcal{A}_i\subset[0,\infty]^{X_i}$ for every $i$, then denote

$$\bigvee_{i\in I}\mathcal{A}_i=\left\{f\in[0,\infty]^{\prod_{i\in I}X_i}:\exists(f_i)_{i\in I}\in\prod_{i\in I}\mathcal{A}_i:\forall(x_i)_{i\in I}\in\prod_{i\in I}X_i, f\left((x_i)_{i\in I}\right)=\bigvee_{i\in I}f_i(x_i)\right\}.$$

**Theorem 10 (Generalized Tychonoff)** *Let $(X_i,\lambda_i)_{i\in I}$ be a family of convergence-approach spaces, let $\mathcal{A}_i\subset[0,\infty]^{X_i}$ and let $\mathcal{F}$ be a filter on $\prod_{i\in I}X_i$. Then*

$$c^{\bigvee_{i\in I}\mathcal{A}_i}(\mathcal{F})=\bigvee_{i\in I}c^{\mathcal{A}_i}(p_i\mathcal{F})$$

*where $p_i:\prod_{j\in I}X_j\to X_i$ is the $i^{th}$-projection.*

**Proof.** Let $\mathcal{U} \in \mathbb{U}(\mathcal{F})$. Then, by definition,

$$\bigvee_{f\in \bigvee_{i\in I}\mathcal{A}_i}\ \bigwedge_{(x_i)_{i\in I}\in \prod_{i\in I}X_i} \left(\left(\prod_{i\in I}\lambda_i\right)(\mathcal{U})+f\right)(x_i)_i = \bigvee_{f\in \bigvee_{i\in I}\mathcal{A}_i}\ \bigwedge_{(x_i)_{i\in I}\in \prod_{i\in I}X_i} \left(\bigvee_{i\in I}\lambda_i\left(p_i\mathcal{U}\right)(x_i)+\bigvee_{i\in I}f_i\left(x_i\right)\right).$$

Hence,

$$\begin{aligned}\bigvee_{f\in \bigvee_{i\in I}\mathcal{A}_i}\ \bigwedge_{(x_i)_{i\in I}\in \prod_{i\in I}X_i} \left(\left(\prod_{i\in I}\lambda_i\right)(\mathcal{U})+f\right)(x_i)_i &= \bigvee_{f\in \bigvee_{i\in I}\mathcal{A}_i}\ \bigwedge_{(x_i)_{i\in I}\in \prod_{i\in I}X_i}\ \bigvee_{i\in I}\left(\lambda_i\left(p_i\mathcal{U}\right)+f_i\right)(x_i)\\ &= \bigvee_{i\in I}\ \bigvee_{f_i\in\mathcal{A}_i}\ \bigwedge_{x_i\in X_i}\left(\lambda_i\left(p_i\mathcal{U}\right)+f_i\right)(x_i).\end{aligned}$$

Therefore $c^{\bigvee_{i\in I}\mathcal{A}_i}(\mathcal{F}) = \bigvee_{i\in I} c^{\mathcal{A}_i}(p_i\mathcal{F})$. ■

**Corollary 11** *[9, Theorem 6.7], [2, Theorem 3.21] Let $(X_i)_{i\in I}$ be a family of (convergence-)approach spaces and let $A_j \subset X_j$. Then*

$$\begin{aligned} m(\prod_{i\in I}X_i) &= \bigvee_{i\in I} m(X_i);\\ \overline{C}(\prod_{i\in I}A_i,\prod_{i\in I}X_i) &= \bigvee_{i\in I}\overline{C}(A_i,X_i).\end{aligned}$$

On the other hand, Theorem 10 combined with Theorem 5 leads to:

**Corollary 12** *The reflector* $\mathrm{Adh}_{\mathbb{F}}$ *on pseudo-approach spaces commutes with arbitrary product.*

As quotient and product commute in **CAP**, we obtain as an immediate corollary of Corollary 12

**Corollary 13** *An arbitrary product of quotient map in **PSAP** is quotient in **PSAP**.*

Notice that quotient maps in **PSAP** between two topological spaces are exactly biquotient maps in the sense of Michael [12].

The behavior of measures of non $\mathbb{D}$-compactness under product for classes $\mathbb{D}$ different from the class $\mathbb{F}$ of all filters is also of great interest and has a wide range of consequences. This is the subject of a forthcoming paper.

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