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Variational Refinement I: Optimality Conditions and Applications

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VARIATIONAL REFINEMENT I: OPTIMALITY CONDITIONS AND APPLICATIONS

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Abstract. A non-uniform, variational refinement scheme is presented for computing piecewise linear curves that minimize a certain discrete energy functional subject to convex constraints on the error from interpolation. Optimality conditions are derived for both the fixed and free-knot problems. These conditions are expressed in terms of jumps in certain (discrete) derivatives. A computational algorithm is given that applies to constraints whose boundaries are either piecewise linear or spherical. The results are applied to closed periodic curves, open curves with various boundary conditions, and (approximate) Hermite interpolation.

1. Introduction

In the last couple decades, subdivision and other curve refinement schemes have gained prominence, in part due to their connection to wavelets in Approximation Theory, and in part to their applications in areas such as Geometric Design and Computer Graphics. In Approximation Theory, one typically considers real-valued functions, whereas, in Geometric Design, one considers vector-valued functions, i.e., parametric curves and surfaces. In both fields, the subdivision schemes that appear in the literature are most-often uniform (and often stationary). These uniform schemes lead to elegant formulations and analysis in terms of refinement relations and subdivision masks.

In this paper, we consider a non-uniform, variational method for refining curves subject to convex set constraints that is a generalization of the uniform, interpolatory refinement scheme in [10]. We derive optimality conditions, including conditions for optimal free knots, and we use these to develop computational algorithms. To emphasize the need for non-uniform refinement, one can compare the two curves in Figure 1.1. Here, the left curve was generated by interpolatory refinement with a uniform parametrization (like in [10]), and the right image using the non-uniform refinement methods described here. Indeed, the need for

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a non-uniform refinement and subdivision schemes in geometric modeling is akin to the need for non-uniform B-spline curves over uniform splines, for example.

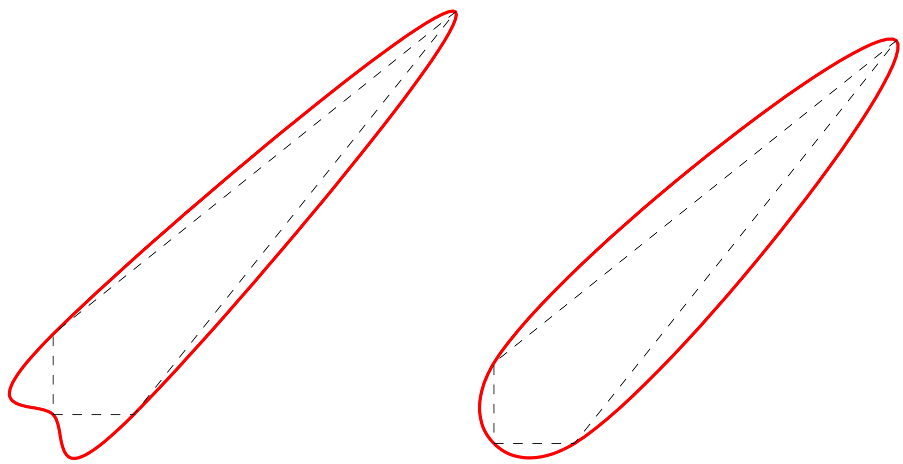

FIGURE 1.1. Uniform and non-uniform interpolatory variational refinement.

In this paper we generalize 'uniform interpolation' to 'non-uniform near-interpolation'. In particular, we assume that the near-interpolatory constraints are convex. Perhaps the first use of such constraints in spline curve interpolation was in [12] and [5]. The results in [12] include stationary conditions for general convex sets, for fixed knots, and an algorithm that applies to constraints with piecewise linear boundaries. Similar results are derived in [5], based on a particular construction that leads to fixed and free-knot optimality conditions, and a computational algorithm. It should be noted, that, since these problems are convex, there are certainly general-purpose optimization algorithms that can be used to compute the curves. The algorithm given here is easy to program, and good for computing approximate solutions.

This paper proceeds as follows: In Section 2, the "energy" of curves is represented in both matrix form, and in terms of certain jumps in third divided differences. An orthogonality condition is derived for the free-knot problem, represented in terms of these jumps. In Section 3, optimality conditions are derived for: interpolation, near-interpolation to balls, smoothing, near-interpolation to general convex constraints, and Hermite near-interpolation. Conditions are given for open and closed curves. In Section 4, an algorithm is given for computing approximate solutions. It applies to the case that the constraints are given by balls or convex sets with piecewise linear boundaries. Although we derive optimality conditions for free-knots, these conditions are not easy to implement in computation. Therefore, we prefer to use other methods to update the knots (so-called parameter updates).

A preliminary version of this paper appear in the unpublished manuscript [7]. Optimality conditions for spline curves under similar constraints as in this paper were given in [5]. Smoothness of the refined curves has been investigated in [9], and was also investigated in [8] for a certain class of parametrizations. The near-interpolatory refinement scheme considered in this paper is generalized to surfaces in [6].

2. The "energy" of piecewise linear curves

Let $f(t)$ be a closed-periodic piecewise linear (B-spline) curve $f(t) = \sum_{i=1}^{n+1} p_i\, N_{i,1}(t)$ with knots $t_0, \cdots, t_{n+2}$ and coefficients $p_i = f(t_i)$ in $\mathbb{R}^d$. Let $h_i := t_{i+1} - t_i$ and $h_{i,j} := t_{i+j} - t_i$. In particular, $h_{i,1} = h_i$ and $h_{i,2} = t_{i+2} - t_i = h_{i+1} + h_i$. Since f is closed, $p_{n+1} = p_1$, and since it is periodic, $t_{n+2} = t_{n+1} + h_0$ and $t_0 = t_1 - h_{n+1}$. Let $\Delta_{i,k} f$ denote the k-th divided difference $\Delta_{i,k} f := [t_i, \ldots, t_{i+k}]\, f$ of f at knots t_i. In particular, $2\,\Delta_{i-1,2} f = 2\,[t_{i-1} t_i, t_{i+1}]\, f$ is the central difference, centered about t_i. We define the *energy* in the curve as

$$E(f) := \frac{1}{2} \sum_{i=1}^{n} \int_{\frac{t_i+t_{i-1}}{2}}^{\frac{t_i+t_{i+1}}{2}} |2\,\Delta_{i-1,2} f|^2\, \mathrm{d}t = \sum_{i=1}^{n} |\Delta_{i-1,2} f|^2\, h_{i-1,2}$$

This is a discretization for the energy functional $\frac{1}{2}\int_{t_1}^{t_{n+1}} |D^2 f(t)|^2\, \mathrm{d}t$ that is used to characterize best C^2 cubic spline interpolants. The functional $E(f)$ differs from that considered in [11] in the extra term $h_{i-1,2}$ that results from the discretization of the measure in the integral. As it turns out, this term is important in deriving certain conditions in the next section.

The setup is illustrated in Figure 2.1. The energy of the piecewise linear curves is evaluated by summing over the second-divided differences squared across the dashed lines. On the left, the curve is closed, hence we add an additional point $p_{n+1} = p_1$. On the right, the curve is open, and two of the spans over which the curve is integrated have been removed. Open curves are considered later in Section 3.

We are assuming (for the first part of this paper) that $f(t)$ is closed with periodic knots. To handle higher order divided differences near the end points, we extend the knot sequence to

$$(\ldots, t_{-1}, t_0, \ldots, t_{n+2}, t_{n+3}, \ldots),$$

with the requirement that it wraps periodically, and the coefficient sequence by the requirement $p_{n+k+1} := p_{1+k}$ for $k = 0, \pm 1, \pm 2, \ldots$.

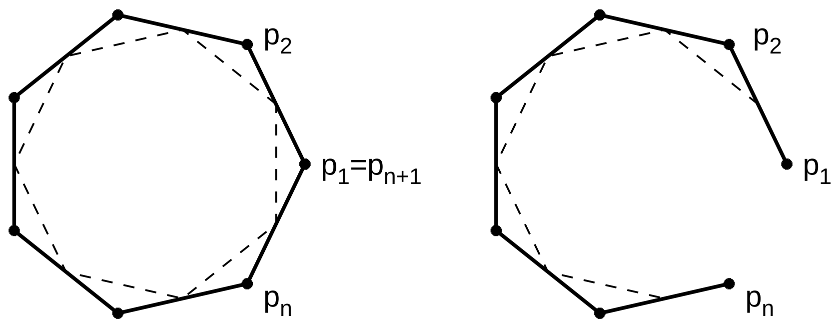

FIGURE 2.1. Energy partitions for closed and open curves.

2.1. **The subdivision mask.** The usual way to represent a subdivision scheme is by its *mask*. For the setup above, we can write the energy in our subdivided curves as

$$E(f) = \frac{1}{2} \sum_{i=1}^{n} K(p_i)^2$$

with

$$K(p_i) := \alpha_{i,i-1} p_{i-1} + \alpha_{i,i} p_i + \alpha_{i,i+1} p_{i+1}$$

and

$$\begin{aligned} \alpha_{i,i-1} &= \frac{\sqrt{2}}{h_{i-1}\sqrt{h_{i-1,2}}} \\ \alpha_{i,i} &= -\alpha_{i,i-1} - \alpha_{i,i+1} \\ \alpha_{i,i+1} &= \frac{\sqrt{2}}{h_i\sqrt{h_{i-1,2}}}. \end{aligned}$$

We define $\alpha_{1,-1} := \alpha_{1n}$ and $\alpha_{n,n+1} := \alpha_{n1}$. Note that $\alpha_{ij} \neq \alpha_{ji}$, in general.

The energy functional is now in the context of the variational subdivision scheme given by L. Kobbelt in [10]. In particular, the subdivision mask is interpolatory since $\sum_{j=i-1}^{i+1} \alpha_{i,j} = 0$ for each i, but non-uniform since α_{ij} depends on i (when the knots are not uniform), and non-stationary since the α_{ij} may change at each iteration as the knots change. As is done in [10], the matrix H can be written out explicitly, with i-th row:

$$\begin{aligned} H(i,:) \;=\; [\;\ldots,\; & \alpha_{i-1,i-2}\,\alpha_{i-1,i},\;\; \alpha_{i-1,i-1}\,\alpha_{i-1,i} + \alpha_{i,i-1}\,\alpha_{i,i},\;\; \alpha_{i-1,i}^2 + \alpha_{i,i}^2 + \alpha_{i+1,i}^2, \\ & \alpha_{i,i}\,\alpha_{i,i+1} + \alpha_{i+1,i}\,\alpha_{i+1,i+1},\;\; \alpha_{i+1,i}\,\alpha_{i+1,i+2},\;\; \ldots\;]. \end{aligned}$$

In matrix form, $K(p_i) = (A\,p)_i$ with

$$A := \begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 & 0 & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & \dots & 0 \\ 0 & \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_{n-2,1} & \alpha_{n-2,2} & \alpha_{n-2,3} & 0 \\ 0 & 0 & \dots & 0 & \alpha_{n-1,1} & \alpha_{n-1,2} & \alpha_{n-1,3} \\ \alpha_{n1} & 0 & 0 & \dots & 0 & \alpha_{n,n-1} & \alpha_{n,n} \end{bmatrix}.$$

Therefore,

$$E(f) = \frac{1}{2} p^T H p \tag{2.1}$$

with $H := A^T A$. Since $H^T = (A^T A)^T = A^T A = H$, it follows that H is a symmetric matrix and that $E(f)$ is positive semi-definite (indeed, we already knew that much from the expression for the functional $E(f)$ as a sum of squared terms). Moreover, if $E(f) = 0$, then all second divided differences of the vertices must vanish. That is, $\ker E$ is contained in the space of linear polynomials, which is of dimension two. Since the curves are periodic, the only linear polynomial with $p_1 = p_{n+1}$ is a constant polynomial, and so $\ker E = \{\text{constant functions}\}$. I.e., $E(f) = 0$ iff $p_{i+1} = p_i$ for all i. Therefore, $\dim(\ker H) = 1$. Later, when we discuss open curves, the matrix H in that setup will have kernel of dimension two. Note also that H is almost penta-diagonal for closed periodic curves, and it is penta-diagonal for open curves. Therefore, (almost-) banded matrix solvers efficiently solve the linear systems given later in this paper. Iterative solvers are not needed.

2.2. **Jumps in the third divided differences.**

Lemma 2.1.

$$|\Delta_{i,2} f|^2 = \Delta_{i,2} f \cdot \Big(\frac{p_i}{h_i h_{i,2}} - \frac{p_{i+1}}{h_i h_{i+1}} + \frac{p_{i+2}}{h_{i,2} h_{i+1}} \Big).$$

Proof. This result follows by formula 25.1.4 in [1]. Let $\pi_{i,n}(t) := \prod_{k=0}^{n} (t - t_{i+k})$. Then

$$\Delta_{i,n} f = \sum_{k=0}^{n} \frac{p_{i+k}}{\pi'_{i,n}(t_{i+k})}.$$

For $n=2$, $\pi_{i,2}(t)=(t-t_i)(t-t_{i+1})(t-t_{i+2})$. Therefore, $\pi'_{i,2}(t_i)=(t_i-t_{i+1})(t_i-t_{i+2})$, $\pi'_{i,2}(t_{i+1})=(t_{i+1}-t_i)(t_{i+1}-t_{i+2})$ and $\pi'_{i,2}(t_{i+2})=(t_{i+2}-t_i)(t_{i+2}-t_{i+1})$, and so

$$\begin{aligned}
|\Delta_{i,2}f|^2 &= \Delta_{i,2}f\cdot\Delta_{i,2}f \\
&= \Delta_{i,2}f\cdot\textstyle\sum_{k=0}^{2}\frac{p_{i+k}}{\pi'_{i,2}(t_{i+k})} \\
&= \Delta_{i,2}f\cdot\left(\frac{p_i}{(t_i-t_{i+1})(t_i-t_{i+2})}+\frac{p_{i+1}}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})}+\frac{p_{i+2}}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}\right) \\
&= \Delta_{i,2}f\cdot\left(\frac{p_i}{h_ih_{i,2}}-\frac{p_{i+1}}{h_ih_{i+1}}+\frac{p_{i+2}}{h_{i,2}h_{i+1}}.\right)
\end{aligned}$$

□

Let

$$\mathrm{jmp}_{t_i}(D^3\,f):=\frac{h_{i-1,3}}{h_i}\,\Delta_{i-1,3}\,f-\frac{h_{i-2,3}}{h_{i-1}}\,\Delta_{i-2,3}\,f,$$

and let $\mathrm{jmp}_t(D^3f):=(\mathrm{jmp}_{t_i}(D^3f):i{=}1{:}n)$. This is a discrete version of the third derivative jump across a knot.

Theorem 2.2.

$$E(f)=\sum_{i=1}^{n}p_i\cdot\mathrm{jmp}_{t_i}(D^3f)=p^T\,\mathrm{jmp}_t(D^3f)$$

Proof. By Lemma 2.1, and using well-known properties of divided differences, we have

$$\begin{aligned}
E(f) &= \textstyle\sum_{i=1}^{n}|\Delta_{i-1,2}f|^2\,h_{i-1,2} \\
&= \textstyle\sum_{i=1}^{n}\Delta_{i-1,2}f\cdot\left(\frac{p_{i-1}}{h_{i-1}h_{i-1,2}}-\frac{p_i}{h_{i-1}h_i}+\frac{p_{i+1}}{h_{i-1,2}h_i}\right)h_{i-1,2} \\
&= \textstyle\sum_{i=1}^{n}p_i\cdot\left(\frac{\Delta_{i-2,2}f}{h_{i-1}}-\frac{h_{i-1,2}\Delta_{i-1,2}f}{h_{i-1}h_i}+\frac{\Delta_{i,2}f}{h_i}\right) \\
&= \textstyle\sum_{i=1}^{n}p_i\cdot\left(\frac{\Delta_{i-2,2}f}{h_{i-1}}-\frac{\Delta_{i-1,2}f}{h_{i-1}}-\frac{\Delta_{i-1,2}f}{h_i}+\frac{\Delta_{i,2}f}{h_i}\right) \\
&= \textstyle\sum_{i=1}^{n}p_i\cdot\left(-\frac{h_{i-2,3}}{h_{i-1}}\,\Delta_{i-2,3}\,f+\frac{h_{i-1,3}}{h_i}\,\Delta_{i-1,3}\,f\right) \\
&= \textstyle\sum_{i=1}^{n}p_i\cdot\mathrm{jmp}_{t_i}(D^3f) \\
&= p^T\,\mathrm{jmp}_t(D^3f).
\end{aligned}$$

□

On comparing equation (2.1) with the result of Theorem 2.2, we arrive at the following:

Corollary 2.3.

$$\mathrm{jmp}_t(D^3f)=\frac{1}{2}H\,p.$$

That is,

$$\mathrm{jmp}_{t_i}(D^3f)=\frac{1}{2}(H\,p)_i\quad\text{for } i{=}1{:}n.$$

2.3. **Variation with respect to knots.** Let $f'(t_i^-) := [t_{i-1}, t_i]f$ and $f'(t_i^+) := [t_i, t_{i+1}]f$, and let ∂_{t_i} be the partial differential operator with respect to t_i.

Lemma 2.4.

$$\partial_{t_i} E(f) = -\frac{h_{i-1,3}}{h_i}\,\Delta_{i-1,3}f \cdot f'(t_i^+) + \frac{h_{i-2,3}}{h_{i-1}}\,\Delta_{i-2,3}f \cdot f'(t_i^-)$$

Proof.

$$\begin{aligned}
\partial_{t_i} E(f) &= \partial_{t_i} \textstyle\sum_{i=1}^{n} |\Delta_{i-1,2}f|^2\,(t_{i+2}-t_{i-1}) \\
&= \textstyle\sum_{j=i-1}^{j+1} \partial_{t_i}\,(|\Delta_{j-1,2}f|^2\,(t_{j+1}-t_{j-1})) \\
&= |\Delta_{i-2,2}f|^2 - |\Delta_{i,2}f|^2 + (-\Delta_{i-2,2}f - \tfrac{\Delta_{i-1,1}f}{h_{i-1}}) \cdot \Delta_{i-2,2}f \\
&\quad + (\tfrac{[t_i,t_{i+1}]f}{h_i} + \tfrac{[t_{i-1},t_i]f}{h_{i-1}}) \cdot \Delta_{i-1,2}f + (\Delta_{i,2}f - \tfrac{[t_i,t_{i+1}]f}{h_i}) \cdot \Delta_{i,2}f \\
&= -\tfrac{[t_{i-1},t_i]f}{h_{i-1}} \cdot \Delta_{i-2,2}f + (\tfrac{[t_i,t_{i+1}]f}{h_i} + \tfrac{[t_{i-1},t_i]f}{h_{i-1}}) \cdot \Delta_{i-1,2}f - \tfrac{[t_i,t_{i+1}]f}{h_i} \cdot \Delta_{i,2}f \\
&= \tfrac{[t_{i-1},t_i]f}{h_{i-1}}(\Delta_{i-1,2}f - \Delta_{i-2,2}f) - \tfrac{[t_i,t_{i+1}]f}{h_i}(\Delta_{i,2}f - \Delta_{i-2,2}f) \\
&= -\tfrac{t_{i+2}-t_{i-1}}{h_i}\,\Delta_{i-1,3}f \cdot [t_i,t_{i+1}]f + \tfrac{t_{i+1}-t_{i-2}}{h_i}\,\Delta_{i-2,3}f \cdot [t_{i-1},t_i]f \\
&= -\tfrac{h_{i-1,3}}{h_i}\,\Delta_{i-1,3}f \cdot f'(t_i^+) + \tfrac{h_{i-2,3}}{h_{i-1}}\,\Delta_{i-2,3}f \cdot f'(t_i^-).
\end{aligned}$$

□

Suppose that the coefficients p_i are fixed, as they would be for interpolation. Then, the knots can be varied freely without violating the interpolation conditions. If one wants to choose these knots to minimize $E(f)$, then, necessarily, $\delta_{t_i} E(f) = 0$ for i=1:n. Hence, by Lemma 2.4, we arrive at the following result:

Corollary 2.5. *Let $f(t)$ be any piecewise linear curve with fixed coefficients p_i, and with variable knots t_i chosen to minimize $E(f)$. Then,*

$$-\frac{h_{i-1,3}}{h_i}\,\Delta_{i-1,3}f \cdot f'(t_i^+) + \frac{h_{i-2,3}}{h_{i-1}}\,\Delta_{i-2,3}f \cdot f'(t_i^-) = 0 \quad \text{for } i\text{=1:}n.$$

Assuming that the subdivision schemes considered later in this paper produce C^1 curves, the result in Corollary 2.5 can be simplified. Indeed, after subdividing to a few levels, we may assume that, at each vertex, the left and right derivatives, $f'(t_i^+)$ and $f'(t_i^-)$, are approximately equal. Therefore, it is reasonable to define the derivative at a vertex as the average

$$Df(t_i) := \frac{1}{2}(f'(t_i^-) + f'(t_i^+)) = \frac{1}{2}([t_{i-1}, t_i]f + [t_i, t_{i+1}]f). \tag{2.2}$$

In the limit, this average will converge to the derivative of the limit curve.

Perhaps a better choice for this derivative follows from a construction analogous to *Bessel interpolation* (see [2]). Here, one defines the derivatives as that of the quadratic polynomial that interpolates the vertex and its two neighbors. To derive it, we start with the Lagrange polynomial interpolant

$$p(t) = \frac{(t-t_i)(t-t_{i+1})}{h_{i-1}h_{i-1,2}}\, p_{i-1} - \frac{(t-t_{i-1})(t-t_{i+1})}{h_{i-1}h_i}\, p_i + \frac{(t-t_{i-1})(t-t_i)}{h_{i-1,2}h_i}\, p_{i+1}$$

to (t_{i-1}, p_{i-1}), (t_i, p_i) and (t_{i+1}, p_{i+1}). Then, we define:

$$Df(t_i) := p'(t_i) = -\frac{h_{i-1}+h_{i-1,2}}{h_{i-1}h_{i-1,2}}\, p_{i-1} + \frac{h_{i-1,2}}{h_{i-1}h_i}\, p_i - \frac{h_{i-1}}{h_{i-1,2}h_i}\, p_{i+1}. \tag{2.3}$$

Whether by (2.2) or (2.3), $Df(t_i)$ provides a good approximation to $f'(t_i^-)$ and $f'(t_i^+)$. On replacing these one-sided derivatives by $Df(t_i)$ in Lemma 2.4, it follows that

$$\partial_{t_i} E(f) \approx -Df(t_i) \cdot \mathrm{jmp}_{t_i}(D^3 f) \quad \text{for } i{=}1{:}n,$$

with $\mathrm{jmp}_{t_i}(D^3 f)$ as defined above. Combining this with Corollary 2.5, we arrive at the following approximate orthogonality condition:

Corollary 2.6. *Suppose the subdivision scheme produces C^1 limit curves. Then, under the hypotheses of Corollary 2.5,*

$$Df(t_i) \cdot \mathrm{jmp}_{t_i}(D^3 f) \approx 0 \quad \text{for } i{=}1{:}n.$$

This orthogonality condition is analogous to conditions for smooth cubic piecewise polynomials with free knots. By Corollaries 2.3 and 2.6, we arrive at an equivalent matrix formulation for this orthogonality condition.

Corollary 2.7. *Suppose the subdivision scheme produces C^1 limit curves. Then, under the hypotheses of Corollary 2.5,*

$$Df(t_i) \cdot (H\,p)_i \approx 0 \quad \text{for } i{=}1{:}n.$$

3. Variational subdivision

Let $g(t)$ be a (periodic) spline curve with coefficients $q_1, \ldots, q_{n+1}$, and knots $u_0, \ldots, u_{n+2}$. The goal is to subdivide $g(t)$. To subdivide one level, one would typically bisect each of the segments $\overline{q_i q_{i+1}}$ and $\overline{u_i u_{i+1}}$ in two, then smooth by variational criteria. Here, rather than bisecting, we split each segment into $k{+}1$ sub-segments (k intermediate points), then smooth. Call this a k-level subdivision. At this level, there are $n_k + 1$ coefficients with

$$n_k := (k+1)\,n.$$

Let $f(t)$ be a spline curve at the k-th level of subdivision, with coefficients $p_1, \ldots, p_{n_k+1}$ and knots $t_0, \ldots, t_{n_k+2}$. In particular, at $k = 0$, $f(t)$ is just the original curve $g(t)$. Let

$$i_k := (k+1)(i-1) + 1.$$

Note that, for interpolatory subdivision, $p_{i_k} = q_i$ for $i{=}1{:}n$ (for near-interpolation, $p_{i_k} \approx q_i$). The following is a paradigm for one k-level subdivision.

1) Split: For $i{=}1{:}n{-}1$ and $j{=}0{:}k{+}1$,

$$\begin{aligned} p_{i_k+j} &:= \tfrac{k+1-j}{k+1} q_i + \tfrac{j}{k+1} q_{i+1}, \\ t_{i_k+j} &:= \tfrac{k+1-j}{k+1} u_i + \tfrac{j}{k+1} u_{i+1}. \end{aligned} \tag{3.1}$$

2) Average: Smooth $f(t)$ by variational methods.

There are a few reasons why we allow additional intermediate points, rather than follow the standard binary subdivision scheme. The most apparent reason is that "one k-level subdivision" would typically produce a smoother curve than "k one-level subdivisions" (albeit, possibly at an increase in computation). A more pragmatic reason, with regard to near-interpolation, is that the constraints are typically only prescribed on the original points. Hence, we near-interpolate for one k-level subdivision, then interpolate for subsequent levels. The larger k permits a better placement of the coefficients p_i that meet the constraints. A third reason for allowing additional intermediate points is to allow for derivative constraints in Hermite (near-)interpolation. Here, one needs the additional degrees of freedom to constrain these (approximate) derivatives.

The variational problems given below are defined for fixed knots, however, free-knot optimality conditions are derived as well. These are based on orthogonality conditions derived in the previous section. Since these optimal knot conditions are not easy to enforce in computation, we prefer to use standard methods to update the knots, such as the centripetal parametrization.

3.1. **Interpolatory subdivision.**

To force interpolation, one constrains $p_{i_k} = q_i$ for $i{=}1{:}n$. We define the variational problem of *best interpolatory subdivision* as:

$$\underset{p}{\text{minimize}}\ \{E(f) : p_{i_k} = q_i,\ i{=}1{:}n\}. \tag{3.2}$$

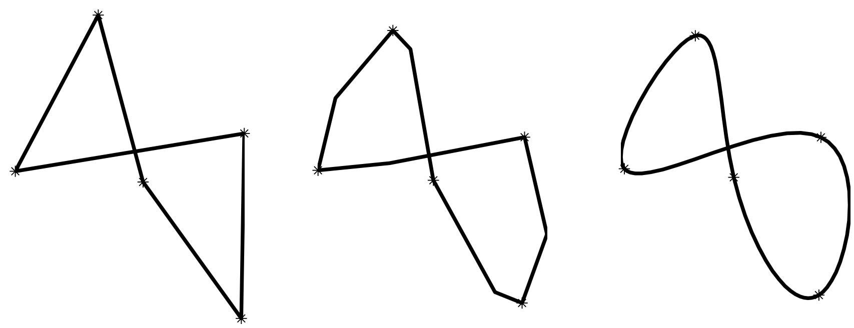

FIGURE 3.1. Interpolatory subdivision.

In the following theorem, the first set of conditions are derived for fixed knots; the last condition is optimal for variational knots, thereby extending (3.2) to the free-knot problem.

Theorem 3.1. *Suppose that $f(t)$ solves problem 3.2 with coefficients p_i, for fixed knots t_i. Then,*

$$\begin{aligned} p_{i_k} &= q_i, && \text{for } i{=}1{:}n, \\ (H\,p)_{i_k+j} &= 0, && \text{for } i{=}1{:}n,\ j{=}1{:}k. \end{aligned} \tag{3.3}$$

With $Df(t_i)$ chosen by either (2.2) or (2.3), the variable knot optimality condition is:

$$(H\,p)_i \cdot Df(t_i) \approx 0, \quad \text{for } i{=}1{:}n_k.$$

Proof. The first condition forces interpolation at the vertices p_{i_k}. The remaining coefficients, p_{i_k+j} for $j{=}1{:}k$, are unconstrained, and so the minimization problem dictates that $\partial_{p_{i_k+j}} E(f) = 0$. By (2.1), $E(f) = \frac{1}{2} p^T H\,p$ with H a symmetric matrix, and so

$$\partial_{p_{i_k+j}} E(f) = (H\,p)_{i_k+j} = 0.$$

For the variable knot problem, we note that, with p_i fixed, the knots can be varied freely without violating the interpolation conditions. Therefore, it must be that $\partial_{t_i} E(f) = 0$ for $i{=}1{:}n_k$. Hence, the third condition follows directly by Corollary 2.7. □

The equations in (3.3) lead to a linear system of dimension $(n_k - n) \times (n_k - n) = (nk) \times (nk)$ for the unknown coefficients (those not interpolated). An example of an interpolated data set is given in Figure 3.1. The original data set, on the left, is subdivided one level, in the middle, and then several more levels, on the right.

3.2. **Near-interpolation to balls.** Here, we relax the interpolatory constraint in (3.2) to *near-interpolation* to balls in $\mathbb{R}^d$. That is,

$$p_{i_k} \in B_{\varepsilon_i}(q_i), \quad i{=}1{:}n,$$

for closed balls $B_{\varepsilon_i}(q_i)$ of radius ε_i, centered about q_i. In addition to the data points q_i, one prescribes positive *tolerances* ε_i as well. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. Then, we state the problem of near-interpolation to balls as:

$$\underset{p}{\text{minimize}}\ \{E(f) : |p_{i_k} - q_i| \le \varepsilon_i,\ i{=}1{:}n\}. \tag{3.4}$$

In the next theorem, W is a diagonal matrix of dimension $n_k \times n_k$, with *Lagrange multipliers* w_i located at i_k on the main diagonal, and all other terms zero. These multipliers are *weights* when viewed as a problem of smoothing, as described in the next section. Let $\tilde{q}$ be the column vector of coefficients q_i at the location i_k, with all other entries zero. (Note, more precisely, that $\tilde{q}$ is of dimension $n_k \times d$ when $q_i \in \mathbb{R}^d$).

Theorem 3.2. *Suppose that f solves problem 3.4 with coefficients p_i, for fixed knots t_i. Then, for some non-negative multipliers w_i,*

$$\begin{aligned} &(H+W)p = W\,\tilde{q} \\ &w_i\,(|p_{i_k} - q_i| - \varepsilon_i) = 0, \qquad \text{for } i{=}1{:}n. \end{aligned} \tag{3.5}$$

With $Df(t_i)$ chosen by either (2.2) or (2.3), the variable knot optimality condition is:

$$\begin{aligned} &(H\,p)_i \cdot Df(t_i) \approx 0 \quad \text{for } i{=}1{:}n_k; \\ &w_i\,(p_{i_k} - q_i) \cdot Df(t_{i_k}) \approx 0, \quad \text{for } i{=}1{:}n. \end{aligned}$$

Proof. As in the proof of Theorem 3.2, we can vary the unconstrained vertices freely. This leads to the second set of equations in that theorem. I.e.,

$$(H\,p)_{i_k+j} = 0, \quad \text{for } i{=}1{:}n,\ j{=}1{:}k.$$

To obtain conditions for the constrained vertices, it is perhaps easiest to consider the problem from the point of view of optimization. To this end, let

$$L(p,w) := E(f) + \frac{1}{2}\sum_{i=1}^{n} w_i\,(|p_{i_k} - q_i|^2 - \varepsilon_i^2)$$

be the *Lagrangian* corresponding to (3.4). The (non-negative) numbers w_i are *Lagrange multipliers*. We know, from optimization theory, that if p solves (3.4), then (p,w) is a saddle

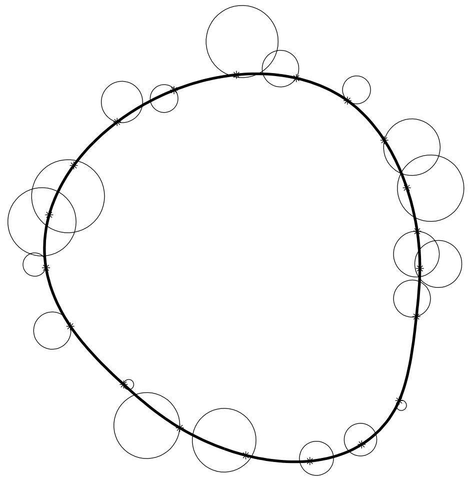

FIGURE 3.2. Near-interpolatory subdivision. Tolerances are given by circles with given center and radii, here chosen at random. The "starred" points are constrained to lie within the closed balls. A constraint is inactive if the point is in the interior; active if on the boundary.

point of $L(p, w)$. Since L is differentiable with respect to p_i, we require that $\partial_{p_{i_k}} L(p, w) = 0$. This leads to the equations

$$(H\,p)_{i_k} + w_i\,(p_{i_k} - q_i) = 0, \quad \text{for } i{=}1{:}n.$$

Combining this set of equations for the near-interpolated vertices, with the above conditions for free vertices, gives the first system of equations in the theorem.

By investigating the saddle point properties of $L(p, w)$ more carefully, one arrives at an additional set of conditions. Namely, the *slack*-conditions corresponding to the constraints – the second set of conditions given in the theorem.

It remains to obtain the free-knot condition. As in Theorem 3.2, we arrive at the condition $(H\,p)_i \cdot Df(t_i) \approx 0$. But, by the first system of equations in this theorem, we have that $(H\,p)_{i_k} = -w_i\,(p_{i_k} - q_i)$. Combining these results, gives the last set of orthogonality conditions, valid at the constrained vertices. □

The last orthogonality condition is perhaps recognizable from the problem of non-linear least squares, or near-interpolation by free knots. Note also, by the slack conditions (the second conditions in the theorem), that if a constraint is inactive, meaning that $|p_{i_k} - q_i| < \varepsilon_i$, then $w_i = 0$. It follows, by the first set of conditions, that for these inactive constraints $(H\,p)_{i_k} = 0$. By Corollary 2.3, this implies that the third "derivative" jump vanishes across this knot. Hence, we have the following result.

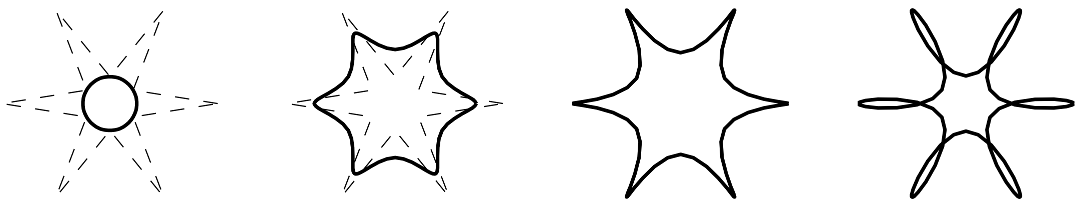

FIGURE 3.3. Decreasing the tolerances; cusps and loops. The constraints are defined with centers at the vertices of the star, and with given radius. This radius is decreased from left to right image, until interpolation (tolerances=0) at the right.

Corollary 3.3. *Suppose that f solves (3.4), and that, for some i, $|f(t_{i_k}) - q_i| < \varepsilon_i$. Then, f is "C^3" at t_{i_k} in the sense that* $\mathrm{jmp}_{t_{i_k}}(D^3 f) = 0$.

Figure 3.2 shows an example of near-interpolation to balls in $\mathbb{R}^2$. Note that, at the points p_i where the curve meets the constraints in their interior, those corresponding knots are inactive in the sense given by Corollary 3.3.

Figure 3.3 shows the effect of the tolerances ε_i. As tolerances are decreased, not only do the curves come closer to meeting the data, but cusps and loops are introduced. One can conclude that, although the limiting curves are differentiable component-wise, they may not be smooth, geometrically.

3.3. **Smoothing.** The problem of spline smoothing is closely connected to the problem of near-interpolation. Here, the smoothing spline is extended to subdivided curves. Rather than computing weights (Lagrange multipliers) based on tolerances, as in near-interpolation, one chooses these weights *a priori*. Suppose that t_i and q_i have also been prescribed. Then, we define the problem of smoothing subdivided curves as:

$$\underset{p}{\text{minimize}} \;\; E(f) + \frac{1}{2} \sum_{i=1}^{n} w_i \, |p_{i_k} - q_i|^2. \tag{3.6}$$

Note the similarity of this minimizing functional with the Lagrangian $L(p, w)$ in the proof of Theorem 3.2. It follows that, on taking variations, we arrive at the same result, minus the slack conditions. Hence, we have:

Theorem 3.4. *Suppose that f solves problem 3.6 with coefficients p_i, for fixed knots t_i. Then,*

$$(H + W)p = W\,\tilde{q}.$$

With $Df(t_i)$ chosen by either (2.2) or (2.3), the variable knot optimality condition is:

$$\begin{aligned} &(H\,p)_i \cdot Df(t_i) \approx 0 \quad \text{for } i{=}1{:}n_k;\\ &w_i\,(p_{i_k} - q_i) \cdot Df(t_{i_k}) \approx 0, \quad \text{for } i{=}1{:}n. \end{aligned}$$

3.4. **Near-interpolatory subdivision with convex constraints.** Let g_{ij} be functions on $\mathbb{R}^d$ such that the sets

$$K_{ij} := \{x \in \mathbb{R}^d : g_{ij}(x) \le 0\}$$

are closed and convex in $\mathbb{R}^d$ for $i{=}1{:}n$, and let

$$K_i := \bigcap_{j=1}^{m_i} K_{ij} \quad \text{with} \quad K_{ij} := \{x \in \mathbb{R}^d : g_{ij}(x) \le 0\}.$$

Here, m_i is the number functions needed to define constraint at q_i. We assume that the sets K_{ij} are convex, with non-empty interior, and with piecewise smooth boundary. In particular, each function g_{ij} is differentiable. We then wish to solve the following problem:

$$\operatorname*{minimize}_{p}\ \{E(f) : g_{ij}(p_{i_k}) \le 0,\ i{=}1{:}n,\ j{=}1{:}m_i\}. \tag{3.7}$$

In the next theorem, ∇g_{ij} denotes the gradient of g_{ij}, and w_{ij} are Lagrange multipliers.

Theorem 3.5. *Suppose that f solves problem 3.7 with coefficients p_i, for fixed knots t_i. Then, for some non-negative multipliers w_{ij},*

$$\begin{aligned} &(H\,p)_{i_k} + \textstyle\sum_{j=1}^{m_i} w_{ij} \nabla g_{ij}(p_{i_k}) = 0, && \text{for } i{=}1{:}n;\\ &(H\,p)_{i_k+j} = 0 && \text{for } i{=}1{:}n,\ j{=}1{:}k;\\ &w_{ij}\, g_{ij}(p_{i_k}) = 0, && \text{for } i{=}1{:}n. \end{aligned} \tag{3.8}$$

With $Df(t_i)$ chosen by either (2.2) or (2.3), the variable knot optimality condition is:

$$\begin{aligned} &(H\,p)_i \cdot Df(t_i) \approx 0, && \text{for } i{=}1{:}n_k;\\ &\textstyle\sum_{j=1}^{m_i} w_{ij} \nabla g_{ij}(p_{i_k}) \cdot Df(t_i) \approx 0, && \text{for } i{=}1{:}n. \end{aligned}$$

Proof. The Lagrangian for this problem is

$$L(p, w) := E(f) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m_i} w_{ij}\, g_{ij}(p_{i_k}).$$

On taking variations with respect to p_{i_k}, and p_{i_k+j}, we arrive immediately at the first two conditions, and the third is the slack condition for the constraints. The third and fourth conditions follow as in near-interpolation. □

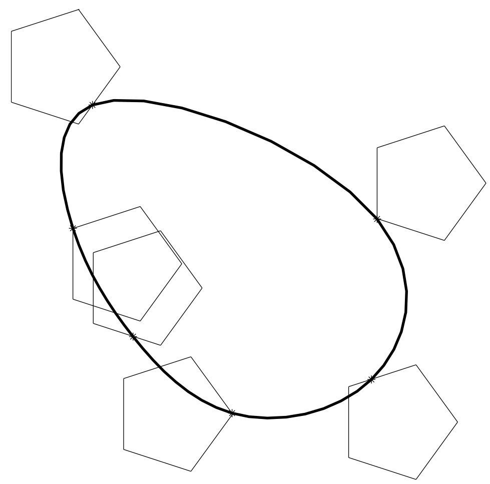

FIGURE 3.4. Near-interpolatory subdivision with convex constraints.

Note that, by the third set of conditions in the theorem, $w_{ij} = 0$ when $g_{ij}(p_{i_k}) < 0$. It follows that, when all constraints g_{ij} are inactive for some i, then $w_{ij} = 0$ for all these j, and by the first condition, $(H\,p)_{i_k} = 0$. This implies, by Corollary 2.3, that $\mathrm{jmp}_{t_{i_k}}(D^3 f) = 0$. Hence, we have the following result.

Corollary 3.6. *Suppose that f solves (3.7), and that, for some i, $g_{ij}(p_{i_k}) < 0$ for all j. Then, f is "C^3" at t_{i_k} in the sense that* $\mathrm{jmp}_{t_{i_k}}(D^3 f) = 0$.

In Figure 3.4, the constraints all have piecewise linear boundaries. For this planar curve, either two of the constraints g_{ij} are active at a vertex (when the curve meets a corner), or just one is active (where it meets an edge), or none of the g_{ij} are active (when p_{i_k} is in the interior of K_i). In the latter case, the curve is "C^3" at p_{i_k}.

3.5. **Hermite Interpolation.** The above variational problems can be generalized to near-Hermite interpolation by constraining derivatives. Since our curves are only piecewise linear, the derivatives must be approximated. For this, we define $Df(t_i)$ by either (2.2) or (2.3). Denote this approximate derivative by p_i'. For these tangent constraints, we are particularly interested in constructing cones using piecewise linear constraints. This, and other constraints, are discussed below.

Suppose first that the constraints are given by balls. As before, $p_{i_k} \in B_{\varepsilon_i}(q_i)$. For the tangent constraints, we require that $p_{i_k}' \in B_{\varepsilon_i^1}(v_i)$ for some prescribed vector v_i and tolerance e_i^1. Then, best near-Hermite interpolation is given by the problem:

$$\text{(3.9)} \qquad \underset{p}{\text{minimize}} \; \{E(f) : |p_{i_k} - q_i| \le \varepsilon_i, \; |p_{i_k}' - v_i| \le \varepsilon_i^1, \; i{=}1{:}n\}.$$

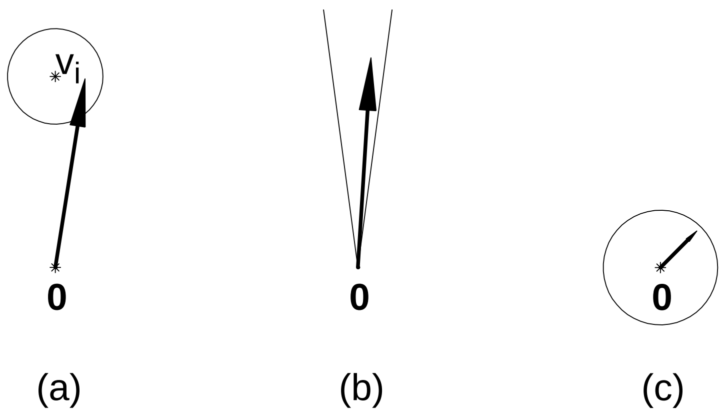

FIGURE 3.5. Various tangent constraints. On the left, the tangent vector is constrained in direction and magnitude, so that the tip of the vector lies in the circle. Here, the near-interpolation constraints are balls. In the middle, the tangent constraint is only on direction; length of the vector can vary. On the right, only the magnitude of the vector is constrained.

For arbitrary convex sets with piecewise smooth boundaries, given by functions $g_{ij}(x) \le 0$ and $g^1_{ij}(x) \le 0$, we consider the problem:

$$\underset{p}{\text{minimize}} \ \{E(f) : g_{ij}(p_{i_k}) \le 0,\ i{=}1{:}n,\ j{=}1{:}m_i,\ \ g^1_{ij}(p'_{i_k}) \le 0,\ i{=}1{:}n,\ j{=}1{:}m^1_i\}. \tag{3.10}$$

The fixed-knot optimality conditions can be derived as above, hence we will skip the details here. However, it should be noted that the free-knot optimality conditions do not follow as above. The reason is that one cannot vary t_i freely with p_i fixed and hope to maintain the derivative constraints. This is because the tangent vectors depend on the knots t_i as well as p_i. We leave it to the reader to derive conditions that are valid.

Moving on to practical considerations, we are interested in imposing various constraints on the tangent vectors, and judging their effects. In Figure 3.5, three configurations are considered. In (a), p'_{i_k} is constrained to lie in some ball of radius ε^1_i and center v_i. This constraint is good if one has a pretty good idea of the magnitude of the desired tangent vector, as well as direction. However, if one wants to determine an "optimal" magnitude for these vectors, it is better to constrain only the direction, not the length. This is accomplished in (b), using piecewise linear constraints. The effect is a "geometric" or "visual" tangent constraint, whereby only the direction is constrained. In (c), v_i is taken to be the origin. Therefore, any vector of length at most ε^1_i will satisfy the constraint. As one shrinks ε^1_i, one shrinks the magnitude of the tangent vector, accordingly. The effect is tension.

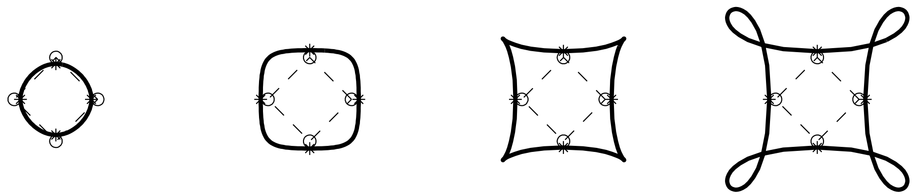

FIGURE 3.6. Tangents modeled as in Figure 3.5 (a).

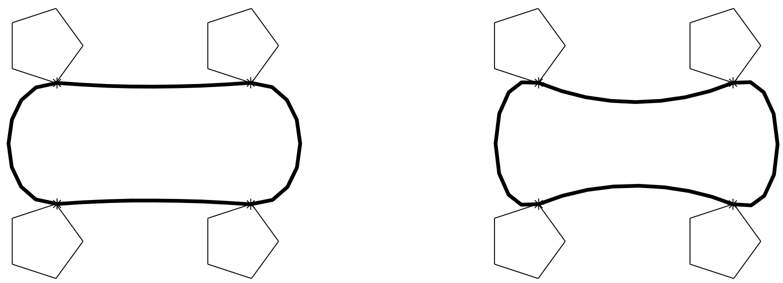

FIGURE 3.7. Tangents modeled as in Figure 3.5 (b).

The tangent constraints illustrated in Figure 3.5 are applied to Hermite subdivision in Figures 3.6 to 3.8. In Figure 3.6, the tangent vectors are constrained by balls, like in Figure 3.5 (a). Here, v_i is either horizontal or vertical. The problem with this construction, is that the length of the tangent vector is constrained by the length of v_i, which may not be optimal. Hence, as we reduce the tolerance ε_i^1, as we do left-to-right in the figure, the lengths of the tangents p'_{i_k} are forced to be approximately that of v_i. In this case, since the magnitude of the v_i are probably too large, the curve develops loops. Hence, unless one has a pretty good idea of the desired magnitude of the tangent vector, this is probably not a good constraint.

If one knows nothing about the length of the desired tangent vector, or one wants a geometric Hermite interpolant, then it is better to constrain only the direction of the tangent. This can be accomplished using the configuration in Figure 3.5 (b). Such constraints are used in Figure 3.7. Here, near-horizontal tangents are prescribed (directed either right or left), constructed by constraints with piecewise linear boundary, as in Figure 3.5 (b). On the left, these "wedges" are centered about the horizontal; on the right curve, the wedges are angled downward (right two points) or upward (left points), moving counter-clockwise. For both curves, the length of the tangents are optimal for the variational problem.

The tangent vectors for the Hermite near-interpolants in Figure 3.8 are constrained to lie in a ball of radius ε_i^1 about the origin, as in Figure 3.5 (c). On the left, the tolerances are large (the balls have large radii), hence those tangent constraints are inactive. As the

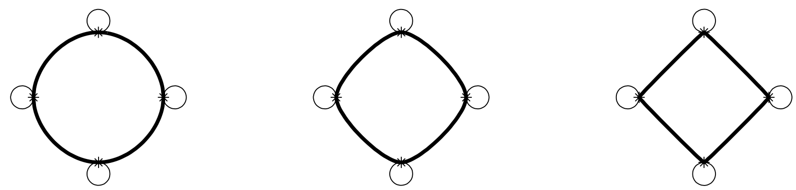

FIGURE 3.8. Tangents modeled as in Figure 3.5 (c).

tolerance shrinks, the near-zero tangent constraints become active. The tangent vectors for the far right curve are nearly zero at the points p_{i_k}, leading to a sharp corner. The tolerances in this case (with $v_i = 0$), control the *tension* of the curve.

3.6. **Open curves – Boundary conditions.** To handle open curves, we eliminate the vertex p_{n+1} and the knot t_{n+2}, as illustrated in Figure 2.1. Hence, the breakpoints (data sites) for the piecewise linear curves are $t_1, \dots, t_n$. We also assume that the knot sequence has interpolatory ends, meaning that $t_0 = t_1$ and $t_{n+1} = t_n$. The energy of our functional is

$$E(f) := \frac{1}{2} \sum_{i=2}^{n-1} \int_{\frac{t_i + t_{i-1}}{2}}^{\frac{t_i + t_{i+1}}{2}} |2\, \Delta_{i-1,2} f|^2 \, \mathrm{d}t = \sum_{i=2}^{n-1} |\Delta_{i-1,2} f|^2 \, h_{i-1,2}.$$

All that has changed is the indexing, which now begins at $i = 2$ and ends at $n-1$. It follows that $E(f) = p^T H p$ with $H = A^T A$ and

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 & \dots & 0 \\ 0 & \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_{n-2,1} & \alpha_{n-2,2} & \alpha_{n-2,3} & 0 \\ 0 & 0 & \dots & 0 & \alpha_{n-1,1} & \alpha_{n-1,2} & \alpha_{n-1,3} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

As stated before, $E(f) = 0$ exactly when the second divided differences of all vertices vanish, and since the curves are now open, $\dim(\ker H) = 2$.

In Figure 3.9, the curves solve the problem of near-interpolatory subdivision (3.4) with this modified matrix H. In the figure, the tolerances are decreased from the left curve fit to more closely meet the data, then increased at two points to reduce the overshoot.

The boundary conditions for this new system can determined by analyzing the optimization problem. The Lagrangian for (3.4) is

$$L(p,w) := E(f) + \frac{1}{2}\sum_{i=1}^{n} w_i\,(|p_{i_k} - q_i|^2 - \varepsilon_i^2),$$

which, with our new matrix H, leads to

$$\partial_{t_1} L(p,w) = \frac{1}{h_1}\,\Delta_{1,2}\,p + w_1\,(p_1 - q_1) = 0.$$

If the first constraint is inactive, then $w_1 = 0$, in which case $\Delta_{1,2}\,p = 0$. That is, the second-divided difference vanished across p_2. Hence, in this case one gets a *natural*-boundary condition. An analogous statement can be made on the right side of the curve.

Another boundary condition of interest is *clamped*. Given tangent vectors v_1 and v_n, we require that $Df(t_1) = v_1$ and $Df(t_n) = v_n$. This is similar to Hermite interpolation earlier, only here, the curve is not closed. It seems natural to choose $Df(t_1) := f'(t_1^+) = [t_1, t_2]f$ and $Df(t_n) := f'(t_n^-) = [t_{n-1}, t_n]f$, however, as an alternative, we could use the Bessel end condition, as in (2.3). Here, one interpolates the first three (resp., last three) points by a polynomial, and then evaluates the derivative of this polynomial. Only, here we evaluate the derivative at the left(resp., right) end points, not the middle. The result is

$$-\frac{h_1 + h_{1,2}}{h_1 h_{1,2}}\,p_1 + \frac{h_{1,2}}{h_1 h_2}\,p_2 - \frac{h_1}{h_{1,2} h_2}\,p_3 \;= v_1$$

at the left end of the curve, and

$$\frac{h_{n-1}}{h_{n-2} h_{n-2,2}}\,p_{n-2} - \frac{h_{n-2,2}}{h_{n-2} h_{n-1}}\,p_{n-1} + \frac{h_{n-2,2} + h_{n-1}}{h_{n-1} h_{n-2,2}}\,p_n \;= v_n$$

on the right. In deference to spline function theory, we call the corresponding curve a *complete subdivided curve.*

Following the theme in this paper, our end conditions may be *near-clamped*, rather than clamped. In this case, they are constrained just as in near-Hermite interpolation. For example, if the constraints are given by balls, we would require that $|f'(t_1) - v_1| \le e_i^1$ and $|f'(t_n) - v_n| \le e_n^1$ for some tolerances ε_i^1. In this way, only minor changes are needed to the formulations given earlier.

4. Computation

As stated earlier, one can only assume that constraints are prescribed on the vertices of the original curves. Indeed, for arbitrary convex constraints, there does not seem a reasonable choice of constraints on the intermediate points after each level of subdivision. Therefore, the

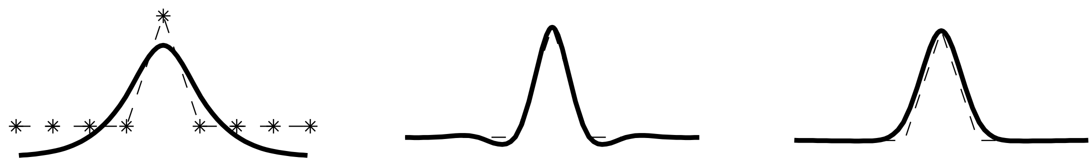

FIGURE 3.9. Open curves.

approach we take here is to enforce near-interpolation at only the first k-level subdivision, then interpolate the vertices at subsequent iterations. This will guarantee that the curves meet the constraints.

The algorithm given next assumes that the constraints are given by balls in $\mathbb{R}^d$. This is an extension of the algorithm given in [3] and [4]. Following this, we describe modifications necessary to handle convex constraints with piecewise linear boundaries, as an extension of the algorithm given in [5]. We leave it to the reader to generalize these algorithms to Hermite subdivision.

4.1. **Algorithm.**

1) Near-interpolatory subdivision (just one k-level step).
   i) Input initial spline curve $g(t)$ with coefficients $q_1, \ldots, q_n$ and knots $u_{-1}, \ldots, u_{n+1}$. For periodic, add another knot u_{n+2} and coefficient $q_{n+1} = q_1$.
   ii) Choose subdivision level k. Initialize the coefficients p_i and knots t_i by (3.1). Initialize weights $w_{i_k} = 1$ for i=1:n, else $w_i = 0$.
   iii) Find the near-interpolant for fixed knots:
      a) Solve the first system in (3.5) for coefficients p_i.
      b) Update the weights to satisfy the second equations in (3.5):
      $$w_i := w_i \frac{|p_{i_k} - q_i|}{\varepsilon_i}$$
      c) Iterate several times.
   iv) Update the knots to centripetal: $t_{i+1} := t_i + |p_{i+1} - p_i|^e$ with $e = 1/2$.
   v) Iterate between steps iii and iv a few times.
2) Interpolatory subdivision (arbitrary number of steps).
   i) Re-initialize: $g(t) := f(t)$, $q_i := p_i$, $u_i := t_i$.
   ii) Choose subdivision level k. Initialize p_i and t_i by (3.1).
   iii) Solve the linear system in (3.3).
   iv) Update knots as in iv, part 1.

v) Iterate between iii and iv a few times.
vi) Go to i, part 2, to further subdivide.

We have had good results using the centripetal parameter update in step iv, parts 1 and 2. If one prefers a uniform subdivision, we would choose $e = 0$, and $e = 1$ for chordal. As experience has shown, the main problem in using a chordal update of the knots is that the scheme does not converge to a smooth function. This problem has been investigated in [8]. The variational parametrization (i.e., one that satisfies the free-knot optimality conditions derived earlier) does not seem practical to enforce. It seems debatable whether one should go much effort to achieve optimal parametrizations, since the computation required may be significant. The centripetal parametrization seems justifiable, since results can be much better than uniform, and the added computation is not prohibitive.

The key to generalizing the algorithm to convex constraints with piecewise linear boundaries is in step (iii)(b) of part 1. Recall that the goal of the iteration in (iii)(b) is to satisfy the slack conditions in (3.5). Hence, for convex constraints, we need to satisfy the slack conditions in (3.8). To construct the sets K_i, first let K_{ij} be the set of points on one side (including boundary) of a hyperplane with outward unit normal N_{ij}, and with some point x_{ij} on the hyperplane. Then, the constraints can be written $g_{ij}(p_i) \leq 0$ with $g_{ij}(p_i) := (p_i - x_{ij}) \cdot N_{ij}$, and $\nabla g_{ij} = N_{ij}$. Now replace the set K_{ij} by a slot of width $2\,\varepsilon_{ij}$, as drawn in Figure 4.1 (a). Then, with p_i the i-th spline coefficient,

$$w_{ij} := w_{ij} \frac{|p_i - z_{ij}|}{\varepsilon_{ij}}.$$

If p_i is outside the slot, the multiplier (weight) w_{ij} is increased; if inside, it is decreased. In particular, if p_i is inside the slot for several iterations, the multiplier w_i is decreased, and tends to zero. This is what one would expect for an inactive point, which corresponds with the slack condition. Now, consider the setup in Figure 4.1 (b). Here, the constraint has a piecewise linear boundary, in 2-D. For each boundary edge, we can choose a tolerance ε_{ij} so that the slot K_{ij} between the boundary edge and its parallel edge contains the entire set K_i, i.e., $K_i \subset K_{ij}$. The intersection of all these slots is the set K_i. That is, $K_i = \cap K_{ij}$. To satisfy the slack conditions, we use the above weight-iteration on each of these slots. After these multipliers are updated, the new coefficients are computed by solving the linear system given by the first two sets of equations in (3.8). This replaces (3.5) in step (iii)(a) of part 1.

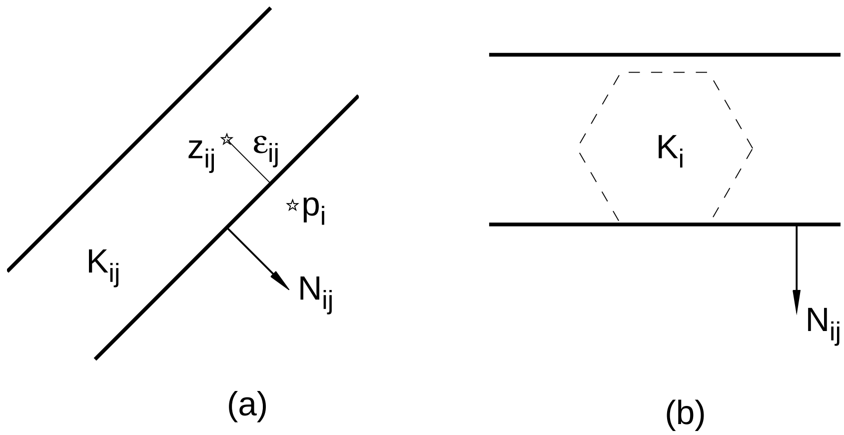

Figure 4.1. Piecewise linear constraints.

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