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Variational Refinement II: Smoothness Conditions

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Abstract. Sufficient conditions are given for C^1 and C^2 (subdivision) curves generated by a particular non-uniform, interpolatory, variational refinement scheme. The 'energy' functional being minimized is a discretization of the standard linearized spline functional over piecewise linear curves – a generalization of the minimizing functional used for the uniform scheme in [10]. The conditions used are uniform bounds on either the energy functional or certain divided differences, along with a condition on the knots, forcing them to be dense and uniform in the limit. To establish C^2, a certain 'bootstrap' argument is applied. The argument is based on a generalization of a result in [6] used to show smoothness of curves generated by nonuniform corner cutting.

§1. Introduction

In this paper we investigate the smoothness of the limiting curves generated by a particular non-uniform, interpolatory, variational refinement scheme. The problem is a generalization of the uniform scheme given in [10] to non-uniform subdivision, while the analysis is more in-line with that given in [6] and [4]. The major difference between the analysis in [6] and here is that our scheme is not convex, as they are in corner cutting. To get around this, we exploit the variational nature of the problem explicitly to establish contraction of certain differences. To do so, we require that either the functional or certain divided differences are uniformly bounded, and that the knots become nearly uniform during the refinement process. This paper does not investigate whether these bounds are achievable in practice, or the effect on the knots due to the bounds. We leave those questions for future work.

To motivate this paper, we remind the reader that non-uniform parametrizations provide better shape preserving control than uniform. This idea carries over to subdivision, as one observes in comparing the left and right images in Figure 1. In each image, the original (piecewise linear)

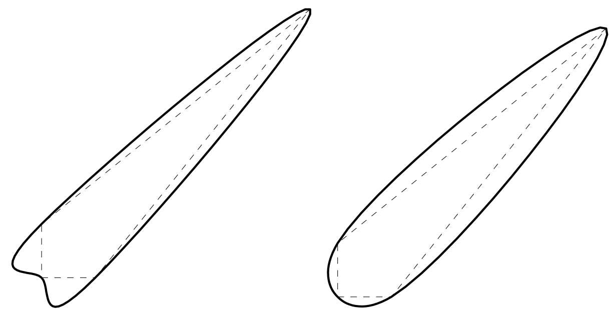

Fig. 1. Uniform and non-uniform parametrizations.

curve is displayed by a dashed line style and the "smooth" curves were generated by several iterations of variational subdivision of the type described here and in [7] and [8]. For the curve in the left image, the parametrization is uniform. In this case, the variational subdivision scheme reduces to the uniform scheme originally presented in [10]. As a result of the uniform parametrization, there is some "overshoot". This is a result of forcing the curve to interpolate unequally-spaced data at equally-spaced knots. Geometrically, the curve does not preserve convexity. For the curve in the right image, the knots depend on the spacing of the points using a *centripetal* parametrization (similar to *chordal*). In this case, the subdivided curve preserves convexity of the original piecewise linear curve.

§2. Variational Refinement

Let $f_1, \ldots, f_n$ be the coefficients of a closed periodic piecewise linear curve with knots $t_0, \ldots, t_{n+1}$. Let $h_i := t_{i+1} - t_i$ and $h_{i,j} := t_{i+j} - t_i$ be forward knot differences . To make the curve closed, we let $f_{n+1} = f_1$, and for periodicity, let $t_{n+2} = t_{n+1} + h_1$ and $t_0 = t_1 - h_n$. As needed, we assume that knot sequences are extended to

$$(\ldots, t_{-1}, t_0, \ldots, t_{n+2}, t_{n+3}, \ldots),$$

with the requirement that it wraps periodically, and the coefficient sequence by the requirement $f_{n+j+1} := f_{1+j}$ for $j = 0, \pm 1, \pm 2, \ldots$. The curves are defined on the closed interval $[a, b]$ with $a := t_1$ and $b := t_{n+1}$.

Let:

0th:	f_i	$f(t) := \sum_{i=1}^{n+1} f_i\, N_i(t)$
1st (forward):	$d_i := \frac{f_{i+1}-f_i}{h_i}$	$d(t) := \sum_{i=1}^{n} d_i\, N_i(t)$
2nd:	$s_i := \frac{d_{i+1}-d_i}{h_{i,2}}$	$s(t) := \sum_{i=1}^{n-1} s_{i-1}\, N_i(t)$
3rd:	$v_i := \frac{s_{i+1}-s_i}{h_{i,3}}$	
1st (symmetric):	$\tilde{d}_i := \frac{h_{i-1}d_i+h_i d_{i-1}}{h_{i-1,2}}$	$\tilde{d}(t) := \sum_{i=1}^{n} \tilde{d}_i\, N_i(t)$
2nd:	$\tilde{s}_i := \frac{\tilde{d}_{i+1}-\tilde{d}_i}{h_i} = s_i + s_{i-1}$	$\tilde{s}(t) := \sum_{i=1}^{n-1} \tilde{s}_{i-1}\, N_i(t)$

Here, $f(t)$ is the original curve represented in the B-spline basis, with knots $t_0, \cdots, t_{n+2}$, and coefficients f_i. The piecewise linear B-spline (degree 1) with knots t_{i-1}, t_i, t_{i+1}, centered about t_i, is denoted $N_i(t)$ throughout. Derived from this are the first and second *divided difference curves* $d(t)$ and $s(t)$, as well as the symmetric versions $\tilde{d}(t)$ and $\tilde{s}(t)$, respectively. As used in [7], we define the *energy* in the curve $f(t)$ as

$$E(f) := \sum_{i=1}^{n} |s_{i-1}|^2\, h_{i-1,2}. \tag{1}$$

This is a discretization of the linearized thin-beam functional $\int_a^b |f''(t)|^2\, \mathrm{d}t$.

In variational refinement (or subdivision in general), one generates a sequence of broken lines $f^0(t), f^1(t), \ldots$, with coefficients f_i^k and knots t_i^k (we assume here that $k = 0$ is the first level, i.e., the original data). At the next level, there are twice the coefficients and twice the knots. Our general framework for interpolatory subdivision is:

$$\left\{ \begin{array}{l} f_{2i} = f_i \\ t_{2i} = t_i \\ f_{2i+1} = \mathcal{F}(f_i, t_i) \\ t_{2i+1} = \mathcal{G}(f_i, t_i) \end{array} \right\}. \tag{2}$$

Here, the indices $2*$ indicate one subdivision level higher than $*$. So, for example, if $f_i = f_i^{k-1}$ then $f_{2i} = f_{2i}^k$. The function $\mathcal{F}()$ is determined by solving the variational problem

$$\underset{f_i^{k+1}}{\text{minimize}}\ \{E(f^{k+1}) : f_{2i}^{k+1} = f_i^k,\ i{=}1{:}n\}, \tag{3}$$

and $\mathcal{G}()$ is defined in some way that at least maintains monotonicity of the knots; specific requirements on the knots (hence $\mathcal{G}()$) are given later. As is typical with variational schemes, the support of $\mathcal{F}()$ is not local with respect to knots or coefficients.

§3. Assumptions and Lemmas

We will refer to the following assumptions and lemmas for various results in the remainder of this paper.

A1: (f^k) is sequence of piecewise linear curves generated by (2).
A2: $\max\{h^{k+1}_{2i-1}, h^{k+1}_{2i}\} \le \alpha h^k_i$ for some $1/2 \le \alpha < 1$.
A3: $E(f^k)$ is uniformly bounded as $k \to \infty$.
A4: $|s^k_i|^2 h^k_{i,2}$ are uniformly bounded for all i as $k \to \infty$.
A5: $|s^k_i|$ are uniformly bounded for all i as $k \to \infty$.
A6: $|v^k_i|$ are uniformly bounded for all i as $k \to \infty$.
A7: $|\mathrm{jmp}_{t_i}(D^3 f^k)|$ are uniformly bounded for all i as $k \to \infty$, with

$$\mathrm{jmp}_{t_i}(D^3 f) := h_{i-1,3}\, v_{i-1}\,/h_i - h_{i-2,3}\, v_{i-2}\,/h_{i-1}.$$

A8: $\frac{h^k_i h^k_{i+1}}{h^k_{i-2} h^k_{i-1}} \to 1$ as $k \to \infty$.

Lemma 1. *Assume A2. Then, for all* i*,*

$$\min\{h^{k+1}_{2i-1},\, h^{k+1}_{2i}\} \ge (1-\alpha)\, h^k_i$$

and

$$(1-\alpha)^k \, \min_j\{h^0_j\} \;\le\; h^k_i \;\le\; \alpha^k\,(b-a)$$

with $a = t_1$ *and* $b = t_{n+1}$*. In particular,* $\max_i\{h^k_i\} \to 0$ *as* $k \to \infty$*.*

Proof: The first claim follows by $h^{k+1}_{2i-1} + h^{k+1}_{2i} = h^k_i$. Indeed,

$$\min\{h^{k+1}_{2i-1},\, h^{k+1}_{2i}\} = h^k_i - \max\{h^{k+1}_{2i-1},\, h^{k+1}_{2i}\} \ge h^k_i - \alpha\, h^k_i = (1-\alpha)\, h^k_i.$$

From this, it follows by induction that, for all i,

$$h^k_i \ge (1-\alpha)\, \min_j h^{k-1}_j \ge (1-\alpha)^2\, \min_j\{h^{k-2}_j\} \ge \cdots \ge (1-\alpha)^k\, \min_j\{h^0_j\}.$$

Likewise,

$$h^k_i \le \alpha\, \max_j h^{k-1}_j \le \alpha^2\, \max_j\{h^{k-2}_j\} \le \cdots \le \alpha^k\, \max_j\{h^0_j\} \le \alpha^k\,(b-a).$$

From this it follows that $\max_i\{h^k_i\} \to 0$ when $\alpha < 1$. □

The next lemma is a variation of the Cauchy Criterion for uniform convergence, as used in [12], Theorems 3.1 and 3.2. Here, and below, "$\xrightarrow{u}$" is uniform convergence (in the sup norm over functions on $[a,b]$).

Lemma 2. *Let* $g^k : [a,b] \to \mathbb{R}$ *be a sequence of continuous functions such that* $\|g^k - g^{k-1}\|_\infty \le \beta\, \theta^{k-1}$ *for some* $\beta > 0$ *and* $0 < \theta < 1$*, for all* k*. Then* $g^k \xrightarrow{u} g$ *for some* $g \in C([a,b])$*.*

For subdivision, we will call θ in Lemma 2 the **contraction constant**. Clearly, the smaller the θ, the faster the convergence, i.e., rate of contraction. The next results provide conditions for the uniform convergence of piecewise linear curves and their difference curves to continuous and smooth functions. The first of these results is from [6].

Lemma 3. ([6], Theorem 4.1) *Assume* $\limsup h_i^k \to 0$ *as* $k \to \infty$ *and that* $d^k \xrightarrow{u} d \in C[a,b]$. *Then* $f^k \xrightarrow{u} f$ *for some* $f \in C^1([a,b])$, *with* $f' = d$.

We would like a higher order version of Lemma 3 along the lines of:

$$\text{“}s^k \xrightarrow{u} s \in C[a,b] \implies d^k \xrightarrow{u} d \in C^1([a,b]) \text{ with } d' = s.\text{”}$$

At a glance, it may seem that this is a trivial application of Lemma 3 to higher order divided differences. However, this is not the case. The reason is that $s^k(t)$ is *not* the *first divided difference curve* to $d^k(t)$, as $d^k(t)$ is to $f^k(t)$. That is, $s_i \neq (d_{i+1} - d_i)/h_i$. Rather, $s_i = (d_{i+1} - d_i)/h_{i,2}$, which, since the term $h_{i,2}$ includes the knot t_{i+2}, is not even localized over the interval $[t_i, t_{i+1}]$. Consequently, the proof in [6], which is based on local Hermite interpolation, doesn't directly generalize to higher differences. We can, however, get around this difficulty by using the *symmetric differences* defined earlier in this paper. Here $\tilde{s}_i = (\tilde{d}_{i+1} - \tilde{d}_i)/h_i$, and so $\tilde{s}(t)$ *is* the first divided difference curve to $\tilde{d}(t)$. With this, we can establish the higher order result. The following identity is used in the proof:

$$\begin{aligned} d_i - \tilde{d}_i &= d_i - \frac{h_{i-1}d_i + h_i d_{i-1}}{h_{i-1,2}} = \frac{h_{i-1,2}d_i - h_{i-1}d_i - h_i d_{i-1}}{h_{i-1,2}} \\ &= \frac{h_i d_i - h_i d_{i-1}}{h_{i-1,2}} = h_i s_{i-1}. \end{aligned} \tag{4}$$

Lemma 4. *Assume that* $\limsup h_i^k \to 0$ *as* $k \to \infty$ *and that* $s^k \xrightarrow{u} s \in C[a,b]$. *Then* $d^k \xrightarrow{u} d$ *for some* $d \in C^1([a,b])$ *with* $d' = 2s$.

Proof: Since $s^k \xrightarrow{u} s$ and $\tilde{s}_i = s_i + s_{i-1}$, it follows that $\tilde{s}^k \xrightarrow{u} \tilde{s} := 2s$. Now, $\tilde{s}_i = (\tilde{d}_{i+1} - \tilde{d}_i)/h_i$, and so $\tilde{s}(t)$ is the first divided difference curve to $\tilde{d}(t)$. By Lemma 3, $\tilde{s}^k \xrightarrow{u} \tilde{s} \in C[a,b]$ implies that $\tilde{d}^k \xrightarrow{u} \tilde{d}$ for some $\tilde{d} \in C^1([a,b])$, with $\tilde{d}' = \tilde{s}$. By (4), $d_i^k - \tilde{d}_i^k = h_i^k s_{i-1}^k$ for all k, and so

$$\begin{aligned} \|d^k - \tilde{d}\|_\infty &\leq \|d^k - \tilde{d}^k\|_\infty + \|\tilde{d}^k - \tilde{d}\|_\infty \\ &= \max_i |d_i^k - \tilde{d}_i^k| + \|\tilde{d}^k - \tilde{d}\|_\infty \\ &= \max_i h_i^k |s_{i-1}^k| + \|\tilde{d}^k - \tilde{d}\|_\infty \to 0. \end{aligned}$$

Hence, $d^k \xrightarrow{u} \tilde{d}$, and we define $d := \tilde{d}$. We have shown that $s^k \xrightarrow{u} s$ implies $d^k \xrightarrow{u} d$ for some differentiable function d such that $d' = \tilde{s} = 2s$. This completes the proof. □

Combining Lemma 3 and 4, we have the following generalization of Theorem 4.1 in [6] to higher order differences.

Corollary 1. *Assume that* $\limsup h_i^k \to 0$ *as* $k \to \infty$ *and that* $s^k \xrightarrow{u} s \in C[a,b]$. *Then* $f^k \xrightarrow{u} f$ *for some* $f \in C^2([a,b])$ *with* $f'' = 2s$.

Proof: By Lemma 4, $d^k \xrightarrow{u} d$ for some $d \in C^1([a,b])$ with $d' = 2s$, and by Lemma 3, $f^k \xrightarrow{u} f$ for some $f \in C([a,b])$ with $f' = d$. Since $f' = d$ and $d \in C^1([a,b])$, it follows that $f \in C^2([a,b])$, and that $f'' = d' = 2s$. □

§4. C^0 Smoothness

For the remainder of this paper, recall from page 3, that we suppress the superscripts k when possible, with it understood that the subscript $2i$ indicates one level of subdivision up from i. For example, the interpolation condition $f_{2i}^k = f_i^k$ for all k may be stated $f_{2i} = f_i$. Explicit mention of k will be used sparingly.

Theorem 1. *Assume A1, A2 and A4. Then f^k converges uniformly to a continuous curve with contraction constant $\alpha^{3/2}$. Assuming A5, the constant is α^2.*

Proof: Recall that the curves $f^k(t)$ are vector-valued in $\mathbb{R}^d$ for some d. To verify continuity, and differentiability later, it is sufficient to verify convergence of the d component functions of these curves. Therefore, WLOG we will assume here that $f^k : [a,b] \to \mathbb{R}$.

Recall that $f_{2i} = f_i$ for interpolation. Therefore, for any level k, $f^k(t_{2i}) = f^{k-1}(t_{2i}) = f^{k-1}(t_i)$ and $f^k(t_{2i+2}) = f^{k-1}(t_{2i+2}) = f^{k-1}(t_{i+1})$ Since f^k and f^{k-1} are both piecewise linear functions, the distance between them on the interval $[t_{2i}, t_{2i+2}]$ is $|f^k(t_{2i+1}) - f^{k-1}(t_{2i+1})|$. Now, under Assumption A4, $|s_i^k|^2 h_{i,2}^k \le B$ for some constant $B < \infty$. Then, we have the estimate:

$$
\begin{aligned}
|f^k(t_{2i+1}) - f^{k-1}(t_{2i+1})| &= \big| f_{2i+1} - \Big(\frac{h_{2i+1}}{h_{2i,2}} f_{2i} + \frac{h_{2i}}{h_{2i,2}} f_{2i+2} \Big) \big| \\
&= \big| \frac{h_{2i+1}}{h_{2i,2}} (f_{2i+1} - f_{2i}) - \frac{h_{2i}}{h_{2i,2}} (f_{2i+2} - f_{2i+1}) \big| \\
&= \frac{h_{2i} h_{2i+1}}{h_{2i,2}} \big| \frac{f_{2i+1} - f_{2i}}{h_{2i}} - \frac{f_{2i+2} - f_{2i+1}}{h_{2i+1}} \big| \\
&= h_{2i} h_{2i+1} |s_{2i}| = \frac{h_{2i} h_{2i+1}}{\sqrt{h_{2i,2}}} |s_{2i}| \sqrt{h_{2i,2}} \\
&\le \frac{h_{2i} h_{2i+1}}{\sqrt{h_{2i,2}}} \sqrt{B} \quad \text{(by A4)} \\
&\le \frac{(\alpha h_{2i,2})^2}{\sqrt{h_{2i,2}}} \sqrt{B} \quad \text{(by A2)} \\
&= \alpha^2 (h_i)^{3/2} \sqrt{B} \quad (\text{note} : h_i \text{ is at level } k-1)
\end{aligned}
$$

$$\leq \alpha^2(\alpha^{k-1}(b-a))^{3/2}\sqrt{B} \quad \text{(by Lemma 1)}$$
$$\leq \alpha^2(\alpha^{3/2})^{k-1}(b-a))^{3/2}\sqrt{B} =: \beta\,\theta^{k-1}$$

with $\beta := \alpha^2(b-a)^{3/2}\sqrt{B}$ and $\theta := \alpha^{3/2}$. Since this is independent of i,

$$\|f^k - f^{k-1}\|_\infty \leq \beta(\theta)^{k-1}.$$

By Lemma 2, f^k converges uniformly to a continuous function with contraction constant $\theta = \alpha^{3/2}$. This proves the first part of the Theorem.

For the stronger assumption A5, $|s_i^k|^2 \leq B < \infty$, and we have the estimate:

$$\begin{aligned} |f^k(t_{2i+1}) - f^{k-1}(t_{2i+1})| &= h_{2i}h_{2i+1}|s_{2i}| \\ &\leq (\alpha h_{2i,2})^2\sqrt{B} \\ &\leq (\alpha^2)^k((b-a))^2\sqrt{B}. \end{aligned}$$

Hence, we have a faster contraction rate with constant $\theta = \alpha^2$. □

Corollary 2. *Assume A1, A2 and A3. Then* $f^k \xrightarrow{u} f \in C^0([a,b])$.

Proof: Since $E(f^k) = \sum_i |s_i^k|^2 h_{i,2}^k$ is a sum of positive terms, boundedness of $E(f^k)$ implies boundedness of $|s_i^k|^2 h_{i,2}^k$. Hence, assumption A3 implies assumption A4. The result then follows by Theorem 1. □

§5. C^1 Smoothness

Recall that $d(t) = \sum_i d_i N_i(t)$ is a piecewise linear curve with coefficients d_i, the first divided differences of the points f_i at knots t_i.

Theorem 2. *Assume A1, A2 and A4. Then* $d^k \xrightarrow{u} d$ *for some continuous curve* d *with contraction constant* $\alpha^{1/2}$*. Assuming A5, the constant is* α*. Moreover,* $f^k \xrightarrow{u} f$ *with* $f' = d$.

Proof: The goal here is to show that $\|d^k - d^{k-1}\|_\infty \to 0$ as $k \to \infty$. Let f_{2i}, d_{2i} and h_{2i} correspond to level k, and f_i, d_i and h_i to level $k-1$. Fix i. On $[t_{2i}, t_{2i+1}]$, $d^{k-1}(t) - d^k(t) =$

$$\big(\frac{t-t_i}{h_i}d_{i+1} + \frac{t_{i+1}-t}{h_i}d_i\big) - \big(\frac{t-t_{2i}}{h_{2i}}d_{2i+1} + \frac{t_{2i+1}-t}{h_{2i}}d_{2i}\big)$$

and on $[t_{2i+1}, t_{2i+2}]$, $d^{k-1}(t) - d^k(t) =$

$$\big(\frac{t-t_i}{h_i}d_{i+1} + \frac{t_{i+1}-t}{h_i}d_i\big) - \big(\frac{t-t_{2i+1}}{h_{2i+1}}d_{2i+2} + \frac{t_{2i+2}-t}{h_{2i+1}}d_{2i+1}\big).$$

Both of these functions are linear, therefore their max and min values occur at the end points. That is, at t_{2i} or t_{2i+1} for the first, and at t_{2i+1} or t_{2i+2} for the second. At t_{2i} we have

$$\begin{aligned} d^{k-1}(t_{2i}) - d^k(t_{2i}) &= d_i - d_{2i} = \frac{f_{2i+2} - f_{2i}}{h_{2i,2}} - d_{2i} \\ &= \frac{h_{2i+1}}{h_{2i,2}} d_{2i+1} + \frac{h_{2i}}{h_{2i,2}} d_{2i} - d_{2i} \\ &= \frac{h_{2i+1}}{h_{2i,2}} (d_{2i+1} - d_{2i}) + \frac{h_{2i}}{h_{2i,2}} (d_{2i} - d_{2i}) = h_{2i+1} s_{2i}. \end{aligned}$$

Symmetrically, $d^{k-1}(t_{2i+2}) - d^k(t_{2i+2}) = h_{2i+3} s_{2i+2}$. At t_{2i+1} we have

$$\begin{aligned} d^{k-1}(t_{2i+1}) - d^k(t_{2i+1}) &= \big(\frac{h_{2i+1}}{h_{2i,2}} d_i + \frac{h_{2i}}{h_{2i,2}} d_{i+1}\big) - d_{2i+1} \\ &= \big(\frac{h_{2i+1}}{h_{2i,2}} \frac{f_{2i+2} - f_{2i}}{h_{2i,2}} + \frac{h_{2i}}{h_{2i,2}} \frac{f_{2i+4} - f_{2i+2}}{h_{2i+2,2}}\big) - d_{2i+1} \\ &= \frac{h_{2i+1}}{h_{2i,2}} \big(\frac{h_{2i}}{h_{2i,2}} d_{2i} + \frac{h_{2i+1}}{h_{2i,2}} d_{2i+1}\big) \\ &+ \frac{h_{2i}}{h_{2i,2}} \big(\frac{h_{2i+2}}{h_{2i+2,2}} d_{2i+2} + \frac{h_{2i+3}}{h_{2i+2,2}} d_{2i+3}\big) - d_{2i+1} \\ &= \frac{h_{2i}}{h_{2i,2}} \frac{h_{2i+3}}{h_{2i+2,2}} (d_{2i+3} - d_{2i+2}) - \frac{h_{2i+1}}{h_{2i,2}} \frac{h_{2i}}{h_{2i,2}} (d_{2i+1} - d_{2i}) \\ &+ \frac{h_{2i+1}}{h_{2i,2}} d_{2i+1} + \frac{h_{2i}}{h_{2i,2}} d_{2i+2} - d_{2i+1} \\ &= \frac{h_{2i} h_{2i+3}}{h_{2i,2}} s_{2i+2} - \frac{h_{2i+1} h_{2i}}{h_{2i,2}} s_{2i} + \frac{h_{2i} h_{2i+1,2}}{h_{2i,2}} s_{2i+1} \\ &= \frac{h_{2i}}{h_{2i,2}} \big(h_{2i+3} s_{2i+2} + h_{2i+1,2} s_{2i+1} - h_{2i+1} s_{2i}\big) \end{aligned}$$

By Lemma 1 and A4,

$$|d^{k-1}(t_{2i}) - d^k(t_{2i})| = h_{2i+1} s_{2i} \le \sqrt{h_{2i+1}} s_{2i} \sqrt{h_{2i,2}} \le (\sqrt{\alpha})^k \sqrt{B(b-a)}$$

and

$$\begin{aligned} &|d^{k-1}(t_{2i+1}) - d^k(t_{2i+1})| \\ &\quad = \big|\frac{h_{2i}}{h_{2i,2}} \big(h_{2i+3} s_{2i+2} + h_{2i+1,2} s_{2i+1} - h_{2i+1} s_{2i}\big)\big| \\ &\quad \le |h_{2i+3} s_{2i+2} + h_{2i+1,2} s_{2i+1} - h_{2i+1} s_{2i}| \\ &\quad \le \sqrt{h_{2i+3}} |s_{2i+2}| \sqrt{h_{2i+2,2}} + \sqrt{h_{2i+1,2}} |s_{2i+1}| \sqrt{h_{2i+1,2}} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{h_{2i+1}}|s_{2i}|\sqrt{h_{2i,2}} \\
&\le (\sqrt{h_{2i+3}} + \sqrt{h_{2i+1,2}} + \sqrt{h_{2i+1}})\sqrt{B} \\
&\le (\sqrt{\alpha^k} + \sqrt{\alpha^{k-1}} + \sqrt{\alpha^k})\sqrt{B\,(b-a)} \\
&\le (\sqrt{\alpha})^k(2 + 1/\sqrt{\alpha})\sqrt{B(b-a)} \\
&\le (2+\sqrt{2})(\sqrt{\alpha})^k\sqrt{B(b-a)}. \quad \text{(since } \alpha > 1/2)
\end{aligned}
$$

Since the maximum value of the error will occur at one of the knots, we have the uniform bound

$$\|d^{k-1} - d^k\|_\infty \le (2+\sqrt{2})(\sqrt{\alpha})^k\sqrt{B(b-a)}.$$

And so, by Lemma 2, d^k converges uniformly to a continuous function. The contraction constant is moreover $\sqrt{\alpha}$, which can be improved to a rate of α with assumption A5. By Lemma 3, $f \xrightarrow{u} f$ with $f' = d$. □

Like in the proof of Corollary 2, assumption A3 implies A4, and so by Theorem 2 we have the following:

Corollary 3. *Assume A1, A2 and A3. Then* $f^k \xrightarrow{u} f$ *for some* $f \in C^1([a,b])$ *with* $f' = d$.

§6. C^2 Smoothness

Lemma 5. $s_i = \dfrac{1}{h_{2i,4}}\big(h_{2i+3}h_{2i+1,3}v_{2i+1} - h_{2i}h_{2i,3}v_{2i}\big) + s_{2i+1}.$

Proof:

$$
\begin{aligned}
h_{2i,4}\, s_i &= \frac{f_{2i+4} - f_{2i+2}}{h_{2i+2,2}} - \frac{f_{2i+2} - f_{2i}}{h_{2i,2}} \\
&= \frac{f_{2i+4} - f_{2i+3}}{h_{2i+2,2}} + \frac{f_{2i+3} - f_{2i+2}}{h_{2i+2,2}} - \frac{f_{2i+2} - f_{2i+1}}{h_{2i,2}} - \frac{f_{2i+1} - f_{2i}}{h_{2i,2}} \\
&= \frac{h_{2i+3}}{h_{2i+2,2}}d_{2i+3} + \frac{h_{2i+2}}{h_{2i+2,2}}d_{2i+2} - \frac{h_{2i+1}}{h_{2i,2}}d_{2i+1} - \frac{h_{2i}}{h_{2i,2}}d_{2i} \\
&= \frac{h_{2i+3}}{h_{2i+2,2}}(d_{2i+3} - d_{2i+2}) + d_{2i+2} + \frac{h_{2i}}{h_{2i,2}}(d_{2i+1} - d_{2i}) - d_{2i+1} \\
&= h_{2i+3}s_{2i+2} + h_{2i+1,2}s_{2i+1} + h_{2i}s_{2i} \\
&= h_{2i+3}(s_{2i+2} - s_{2i+1}) - h_{2i}(s_{2i+1} - s_{2i}) + h_{2i,4}s_{2i+1} \\
&= h_{2i+3}h_{2i+1,3}v_{2i+1} - h_{2i}h_{2i,3}v_{2i} + h_{2i,4}s_{2i+1}.
\end{aligned}
$$

□

Theorem 3. *Assume either A1, A2 and A6, or A1, A2, A7 and A8. Then, s^k converges uniformly to a continuous curve.*

Proof: Recall that $s(t) = \sum_{i=1}^{n} s_{i-1}N_i(t)$. Then, on $t_{2i} \leq t \leq t_{2i+1}$, $s^{k-1}(t) - s^k(t) =$

$$\Big(\frac{t_{2i+2}-t}{h_{2i,2}}s_{i-1} + \frac{t-t_{2i}}{h_{2i,2}}s_i\Big) - \Big(\frac{t_{2i+1}-t}{h_{2i}}s_{2i-1} + \frac{t-t_{2i}}{h_{2i}}s_{2i}\Big),$$

and on $t_{2i+1} \leq t \leq t_{2i+2}$, $s^{k-1}(t) - s^k(t) =$

$$\Big(\frac{t_{2i+2}-t}{h_{2i,2}}s_{i-1} + \frac{t-t_{2i}}{h_{2i,2}}s_i\Big) - \Big(\frac{t_{2i+2}-t}{h_{2i+1}}s_{2i} + \frac{t-t_{2i+1}}{h_{2i+1}}s_{2i+1}\Big).$$

Both of these equations are linear in t, and so extrema will occur at the end points. Using Lemma 5, we have the following at the end points:

$$\begin{aligned}
&s^{k-1}(t_{2i}) - s^k(t_{2i}) = s_{i-1} - s_{2i-1}\\
&\quad= \frac{1}{h_{2i-2,4}}\big(h_{2i+1}h_{2i-1,3}v_{2i-1} - h_{2i-2}h_{2i-2,3}v_{2i-2}\big) + s_{2i-1} - s_{2i-1}\\
&\quad= \frac{1}{h_{2i-2,4}}\big(h_{2i}h_{2i+1}\frac{h_{2i-1,3}}{h_{2i}}v_{2i-1} - h_{2i-2}h_{2i-1}\frac{h_{2i-2,3}}{h_{2i-1}}v_{2i-2}\big),
\end{aligned}$$

$$\begin{aligned}
&s^{k-1}(t_{2i+1}) - s^k(t_{2i+1}) = \frac{h_{2i+1}}{h_{2i,2}}s_{i-1} + \frac{h_{2i}}{h_{2i,2}}s_i - s_{2i}\\
&\quad= \frac{h_{2i+1}}{h_{2i,2}}(\frac{h_{2i+1}h_{2i-1,3}}{h_{2i-2,4}}v_{2i-1} - \frac{h_{2i-2}h_{2i-2,3}}{h_{2i-2,4}}v_{2i-2} + s_{2i-1})\\
&\quad+ \frac{h_{2i}}{h_{2i,2}}(\frac{h_{2i+3}h_{2i+1,3}}{h_{2i,4}}v_{2i+1} - \frac{h_{2i}h_{2i,3}}{h_{2i,4}}v_{2i} + s_{2i+1}) - s_{2i}\\
&\quad= \frac{h_{2i+1}}{h_{2i,2}}(\frac{h_{2i+1}h_{2i-1,3}}{h_{2i-2,4}}v_{2i-1} - \frac{h_{2i-2}h_{2i-2,3}}{h_{2i-2,4}}v_{2i-2})\\
&\quad+ \frac{h_{2i}}{h_{2i,2}}(\frac{h_{2i+3}h_{2i+1,3}}{h_{2i,4}}v_{2i+1} - \frac{h_{2i}h_{2i,3}}{h_{2i,4}}v_{2i})\\
&\quad+ \frac{h_{2i}}{h_{2i,2}}(s_{2i+1} - s_{2i}) - \frac{h_{2i+1}}{h_{2i,2}}(s_{2i} - s_{2i-1})\\
&\quad= \frac{h_{2i+1}}{h_{2i,2}}(\frac{h_{2i+1}h_{2i-1,3}}{h_{2i-2,4}}v_{2i-1} - \frac{h_{2i-2}h_{2i-2,3}}{h_{2i-2,4}}v_{2i-2})\\
&\quad+ \frac{h_{2i}}{h_{2i,2}}(\frac{h_{2i+3}h_{2i+1,3}}{h_{2i,4}}v_{2i+1} - \frac{h_{2i}h_{2i,3}}{h_{2i,4}}v_{2i})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{h_{2i}h_{2i,3}}{h_{2i,2}}v_{2i} - \frac{h_{2i+1}h_{2i-1,3}}{h_{2i,2}}v_{2i-1} \\
&= \frac{h_{2i+1}}{h_{2i,2}h_{2i-2,4}}(h_{2i}h_{2i+1}\frac{h_{2i-1,3}}{h_{2i}}v_{2i-1} - h_{2i-2}h_{2i-1}\frac{h_{2i-2,3}}{h_{2i-1}}v_{2i-2}) \\
&+ \frac{h_{2i}}{h_{2i,2}h_{2i,4}}(h_{2i+3}h_{2i+2}\frac{h_{2i+1,3}}{h_{2i+2}}v_{2i+1} - h_{2i+1}h_{2i}\frac{h_{2i,3}}{h_{2i+1}}v_{2i}) \\
&+ \frac{h_{2i}h_{2i+1}}{h_{2i,2}}(\frac{h_{2i,3}}{h_{2i+1}}v_{2i} - \frac{h_{2i-1,3}}{h_{2i}}v_{2i-1}),
\end{aligned}
$$

and, by symmetry with the left side, $s^{k-1}(t_{2i+2}) - s^k(t_{2i+2}) =$

$$
\frac{1}{h_{2i,4}}\big(h_{2i+2}h_{2i+3}\frac{h_{2i+1,3}}{h_{2i+2}}v_{2i+1} - h_{2i+2}h_{2i+1}\frac{h_{2i,3}}{h_{2i+1}}v_{2i}\big).
$$

As before, the maximum of $|s^{k-1}(t) - s^k(t)|$ occurs at an end point, and when v_i is uniformly bounded (assumption A6), then, as before, we get the estimate $|s^{k-1}(t) - s^k(t)| = O(\alpha^k)$ due to the extra h_* in the numerator of each term above. Hence, the result is established in this case.

Now, recall the jump functionals

$$
\mathrm{jmp}_{t_i}(D^3 f) = \frac{h_{i-1,3}}{h_i}\, v_{i-1} - \frac{h_{i-2,3}}{h_{i-1}}\, v_{i-2}.
$$

By assumption A8, we have the estimates:

$$
\begin{aligned}
|s^{k-1}(t_{2i}) - s^k(t_{2i})| &= \frac{h_{2i}h_{2i+1}}{h_{2i-2,4}}\big(\frac{h_{2i-1,3}}{h_{2i}}v_{2i-1} - \frac{h_{2i-2}h_{2i-1}}{h_{2i}h_{2i+1}}\frac{h_{2i-2,3}}{h_{2i-1}}v_{2i-2}\big) \\
&\approx \frac{h_{2i}h_{2i+1}}{h_{2i-2,4}}\big(\frac{h_{2i-1,3}}{h_{2i}}v_{2i-1} - h_{2i}h_{2i+1}\frac{h_{2i-2,3}}{h_{2i-1}}v_{2i-2}\big) \\
&= \frac{h_{2i}h_{2i+1}}{h_{2i-2,4}}|\mathrm{jmp}_{t_{2i}}(D^3 f)|,
\end{aligned}
$$

and

$$
|s^{k-1}(t_{2i+1}) - s^k(t_{2i+1})| \approx \frac{h_{2i+1}h_{2i}h_{2i+1}}{h_{2i,2}h_{2i-2,4}}|\mathrm{jmp}_{t_{2i}}(D^3 f)|
$$

$$
+ \frac{h_{2i}h_{2i+3}h_{2i+1}}{h_{2i,2}h_{2i,4}}|\mathrm{jmp}_{t_{2i+2}}(D^3 f)| + \frac{h_{2i}h_{2i+1}}{h_{2i,2}}|\mathrm{jmp}_{t_{2i+1}}(D^3 f)|,
$$

and a similar result for the right side. Each of these terms is $O(\alpha^k)$ under assumption A7. By Lemma 2, s^k converges uniformly to some continuous curve. $\square$

The following is a consequence of Theorem 3 and Corollary 1.

Corollary 4. *Assume either A1, A2 and A6, or A1, A2, A7 and A8. Then,* $f^k \xrightarrow{u} f$ *for some* $f \in C^2$.

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§7. References

1. Boor, C. de, *Corner cutting always works*, Comput. Aided Geom. Design **4** (1987), 125–131.
2. Boor, C. de, Local corner cutting and the smoothness of the limiting curves, Comput. Aided Geom. Design **7** (1990), 389–397.
3. Daubechies, I., I. Guskov, and W. Sweldens, Regularity of irregular subdivision, Constr. Approx. **15** (1999), 381–426.
4. Dyn, N., and D. Levin, A 4-point interpolatory subdivision scheme for curve design, Comput. Aided Geom. Design **4** (1987), 257–268.
5. Dyn, N., and D. Levin, Subdivision schemes in geometric modeling, Acta Numerica **11** (2002), 73–144.
6. Gregory, J., and R. Qu, Nonuniform corner cutting, Comput. Aided Geom. Design **13** (1996), 763–772. (Based on a 1988 manuscript.)
7. Kersey, S., Near-interpolatory refinement curves, manuscript (2003).
8. Kersey, S., The effect of parametrization on subdivided curves, Advances in Constructive Approximation, Eds. M. Neamtu and E. Saff (2004), 243–254.
9. Kersey, S., An abstract formulation of variational refinement, J. Approx. Theory **130(2)** (2004), 150–163.
10. Kobbelt, L., A variational approach to subdivision, Comput. Aided Geom. Design **13** (1996), 743–761.
11. Kuijt, F., and R. van Damme, Shape preserving interpolatory subdivision schemes for nonuniform data, J. Approx. Theory **114** (2002), 1–32.
12. Warren, J., and H. Weimer, *Subdivision Methods for Geometric Design*, Morgan Kaufmann (New York), 2002.

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