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Inequality Approach in Topological Categories

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INEQUALITY APPROACH IN TOPOLOGICAL CATEGORIES

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Abstract. We show that many notions of topology can be interpreted via functorial inequalities and how many topological results follow from a simple calculus of such inequalities.

Contents

1. Introduction

We refer to [1] for undefined categorical notions. The aim of this paper is to present categorical methods based on functorial inequalities. These methods proved extremely useful in the category **Conv** of convergence spaces (and continuous maps) [7], [8], [13], [14], [29], [31], [28], [32], [33], [30], [34], [27]. Essential to the approach are objectwise characterizations of morphisms and functors. The spirit of the method is therefore quite different from the usual viewpoint of category theory. Of course, objectwise reduction of arguments is not always possible. Its favorable environment is that of topological categories. They are not, as one might think, subcategories of the category of topologies with continuous maps, but categories sharing some important properties with the latter.

Throughout the paper, **A** denotes a category which is *topological* over a category **X**, with the forgetful functor $|\cdot| : \mathbf{A} \to \mathbf{X}$. In other words, every structured

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source has a unique initial **A**-lift and every structured sink has a unique final **A**-lift. In particular, if $B \in Ob(\mathbf{A})$, $X \in Ob(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(X, |B|)$, then there exists the unique **A**-object $\overleftarrow{f} B$ such that $|\overleftarrow{f} B| = X$ and if $f \in \mathrm{Hom}_{\mathbf{A}}(A, B)$ ([1]) with $|A| = X$ then $i_X \in \mathrm{Hom}_{\mathbf{A}}(A, \overleftarrow{f} B)$. In other words, $\overleftarrow{f} B$ denotes the initial object (in **A**) with respect to f and B. Dually, if $A \in Ob(\mathbf{A})$, $Y \in Ob(\mathbf{X})$ and $f \in \mathrm{Hom}_{\mathbf{X}}(|A|, Y)$, then there exists the unique final **A**-object $\overrightarrow{f} A$, satisfying $|\overrightarrow{f} A| = Y$ and $i_Y \in \mathrm{Hom}_{\mathbf{A}}(\overrightarrow{f} A, B)$ whenever $f \in \mathrm{Hom}_{\mathbf{A}}(A, B)$ with $|B| = Y$ ([2]).

For each $X \in Ob(\mathbf{X})$, the fiber $\{A \in Ob(\mathbf{A}) : |A| = X\}$ is a complete lattice [1, Proposition 21.11] for a partial order ([3]) defined by

$$A_0 \geq A_1 \iff i_X \in \mathrm{Hom}_{\mathbf{A}}(A_0, A_1).$$

Therefore for arbitrary **A**-objects A and B and an **X**-morphism $f : |A| \to |B|$,

$$f \in \mathrm{Hom}_{\mathbf{A}}(A, B) \iff A \geq \overleftarrow{f} B \iff \overrightarrow{f} A \geq B. \tag{1}$$

The equivalences (1) are the essence of the *inequality approach.* They enable us to represent various diagrams in terms of inequalities, and to transform many (often complicated) arguments into a *calculus of inequalities.* This calculus usually reduces considerably the complexity of arguments.

The introduction of the notation $\overleftarrow{f} B$ and $\overrightarrow{f} A$ respectively for initial and final objects associated to a structured map are essential to the said calculus.

The aim of the paper is to show that the inequality approach allows a very convenient formalism in the following particular context. **A** is topological over **X**, all the other considered categories are subcategories of **A**, and all functors (except for the forgetful functor onto **X**) are concrete endofunctors of **A**, that is, functors $F : \mathbf{A} \to \mathbf{A}$ such that $|Ff| = |f|$ for every **A**-morphism f. They are functors that modify the **A**-structure but do not affect the underlying **X**-object. In particular (concrete endo-) functors depend only on their restrictions to objects and can be characterized via inequalities.

From now on, *functor* should be understood as concrete endofunctor of **A**.

Proposition 1. *The following are equivalent:*

(1) *A map $M : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ is (the restriction to objects of) a functor;*
(2) *M is concrete, order-preserving, and*

$$\overrightarrow{f}(MA) \geq M\left(\overrightarrow{f} A\right); \tag{2}$$

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[1] In the context of the present paper, that is, in a topological category **A** concrete over **X**, for each $\phi \in \mathrm{Hom}_{\mathbf{A}}(A, B)$, there exists exactly one $f \in \mathrm{Hom}_{\mathbf{X}}(|A|, |B|)$ such that $|\phi| = f$. Hence, one can identify [1, Remark 5.3]

$$\mathrm{Hom}_{\mathbf{A}}(A, B) \subset \mathrm{Hom}_{\mathbf{X}}(|A|, |B|).$$

Here we use an extension of this abuse of notation to write that an **X**-morphism $f : X \to Y$ is also an **A**-morphism $f : A \to B$, where $|A| = X$ and $|B| = Y$.

[2] In the sequel, we will frequently use objects like $\overrightarrow{f} A$ or $\overleftarrow{f} B$ associated to an **A**-morphism $f : A \to B$.They should be understood as associated to the structured maps $f : |A| \to Y$ and $f : X \to |B|$ respectively, where X and Y are the underlying **X**-objects of A and B. Moreover, **A** denotes a fixed topological category (concrete over **X**) and, even if various subcategories of **A** are considered, the notations $\overrightarrow{f} A$ or $\overleftarrow{f} B$ will always refer to final and initial objects in **A**.

[3] Note that [1] uses the inverse order.

(3) *M is concrete, order-preserving, and*

$$M\left(\overleftarrow{f}B\right) \geq \overleftarrow{f}(MB). \tag{3}$$

Among such functors, reflectors are those idempotent and contractive (on objects), and coreflectors are those idempotent and expansive (on objects). Moreover, if F is a functor, then the class of $\mathbf{A}$-objects A such that

$$A \leq FA \tag{4}$$

determines a reflective subcategory of $\mathbf{A}$. Analogously, the class of $\mathbf{A}$-objects A such that

$$A \geq FA \tag{5}$$

form a coreflective subcategory of $\mathbf{A}$. The associated reflectors and coreflectors can be explicitly calculated in terms of F. In particular, it is known that (Section 8) many classical coreflective notions naturally arise as objects satisfying

$$A \geq JE(A) \tag{6}$$

where $J : \mathbf{A} \to \mathbf{J}$ is a reflector and $E : \mathbf{A} \to \mathbf{E}$ is a coreflector ([4]).

Although the class of objects satisfying (6) is coreflective in $\mathbf{A}$, the functor JE is not the associated coreflector, but constitutes a natural tool to define the class.

Example 2 (Sequential spaces)**.** *Sequential topological spaces are those for which sequentially closed sets are closed. Hence, considering a topological space X as a convergence space, X is sequential if $X = T\mathrm{Seq}X$ where $T : \mathbf{Conv} \to \mathbf{T}$ is the reflector on topological spaces (and continuous maps) and* $\mathrm{Seq} : \mathbf{Conv} \to \mathbf{Seq}$ *is the coreflector on sequentially based convergence spaces ([5]). Because X is topological, the equality above is equivalent to $X \geq T\mathrm{Seq}X$, which extends the notion from topological spaces to general convergence spaces.*

Example 3 (Fréchet spaces)**.** *A topological space is Fréchet if every point in the closure of a subset is the limit of a sequence of points of that subset. The notion of a Fréchet topological space extends to convergence spaces. A space X is Fréchet if $X \geq P\mathrm{Seq}X$, where $P : \mathbf{Conv} \to \mathbf{P}$ is the reflector on pretopological spaces.*

Dually, many reflective notions can be characterized via

$$A \leq EJ(A) \tag{7}$$

where $J : \mathbf{A} \to \mathbf{J}$ is a reflector and $E : \mathbf{A} \to \mathbf{E}$ is a coreflector (Section 9).

Example 4. *Urysohn sequentially based convergence spaces are those satisfying $X \leq \mathrm{Seq}P_\omega X$ where $P_\omega : \mathbf{Conv} \to \mathbf{P}_\omega$ is the reflector on paratopological spaces (in the sense of* [7]*).*

Example 5. *Sequentially based convergence spaces whose convergence of sequences is induced by a topology are those satisfying $X \leq \mathrm{Seq}TX$.*

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[4]Here we apply the convention that the (co)reflector associated to a (co)reflective subcategory $\mathbf{J}$ of $\mathbf{A}$ is denoted by the same non bold letter J.

[5]Sequentially based convergences form a category isomorphic to the category of *sequential convergences*, for which the convergence relation is defined only for sequences.

Various types of maps can also be characterized via inequalities. In particular, we study morphisms $f : A \to B$ satisfying

$$B \geq J\left(\overrightarrow{f}A\right) \tag{8}$$

where J is a functor of $\mathbf{A}$. It was noticed in [7] that (6) is equivalent to the fact that $i : EA \to A$ satisfies (8). Instances of such maps classically used in topology include *quotient, hereditarily quotient, countably biquotient, biquotient and almost open maps,* when J ranges over the reflectors, respectively, *T on topologies, P on pretopologies, P_ω on paratopologies, S on pseudotopologies, and the identity I.*

As an example of an argument characteristic of the inequality approach, consider the following.

Theorem 6. *Let J be a reflector and E a functor (not necessarily coreflector). If $f : A \to B$ is a morphism satisfying* (8) *and A satisfies* (6)*, then B satisfies* (6).

Proof. As $A \geq JE(A)$, we have $\overrightarrow{f}A \geq \overrightarrow{f}(JEA)$. Because JE is a functor, $\overrightarrow{f}(JEA) \geq JE\left(\overrightarrow{f}A\right)$ in view of (2). Hence, $\overrightarrow{f}A \geq JE\left(\overrightarrow{f}A\right)$, so that $J\left(\overrightarrow{f}A\right) \geq JE\left(\overrightarrow{f}A\right)$ because J is idempotent. Moreover, $B \geq J\left(\overrightarrow{f}A\right)$, and $\overrightarrow{f}A \geq B$ because $f : A \to B$ is a morphism. Thus $B \geq JE(B)$. □

In particular, many classical preservation results extend from topological spaces to convergence spaces and follow from this simple scheme, for example,

The quotient image of a sequential space is sequential,

or

The hereditarily quotient image of a Fréchet space is Fréchet.

Moreover, the interpretation of topological notions like sequentiality and Fréchetness via (6) and of classes of maps like quotient and hereditarily quotient maps via (8) gives immediate reconstruction results including, among others,

Every sequential space is a quotient image of a first-countable space,

and

Every Fréchet space is a hereditarily quotient image of a first-countable space.

Other instances of problems for which a simple argument involving functorial inequalities captures a whole family of results include modified continuity (for example, sequential continuity versus continuity, Section 12), reconstruction results in terms of covering maps (Section 11), and results on cartesian-closed hulls and exponential objects (Section 13).

For example, the classical results of F. Siwiec

A topological space X is sequential (resp. Fréchet, strongly Fréchet) if and only if every sequence-covering map onto X is quotient (resp. hereditarily quotient, countably biquotient),

and of F. Siwiec, V. J. Mancuso, A. V. Arhangel'skii and E. Michael gathered below

A topological space X is a k-topology (resp. k', strongly k', locally compact) if and only if each compact-covering map onto X is quotient (resp. hereditarily quotient, countably biquotient, biquotient),

are instances of one simple result by S. Dolecki and M. Pillot [17, Theorem 5.4].

All the examples presented in this paper concern the case where $\mathbf{A}$ is the category of convergence spaces and continuous maps and reflect our research experience, but we trust that the approach may be useful in other contexts as well.

2. Morphisms, initial sources and final sinks

As already observed in the Introduction, if A and B are $\mathbf{A}$-objects and $f : |A| \to |B|$ is an $\mathbf{X}$-morphism then

$$f \in \mathrm{Hom}_{\mathbf{A}}(A, B) \iff A \geq \overleftarrow{f} B \iff \overrightarrow{f} A \geq B,$$

where $\overrightarrow{f} A$ is defined by the fact that $f : A \to B$ is final (in $\mathbf{A}$) if $B = \overrightarrow{f} A$, and $\overleftarrow{f} B$ is defined by the fact that $f : A \to B$ is initial (in $\mathbf{A}$) if $A = \overleftarrow{f} B$. More generally, $(f_i : A \to B_i)_{i \in I}$ is an initial source if and only if $A = \bigvee_{i \in I} \overleftarrow{f_i} B_i$ and $(f_i : A_i \to B)_{i \in I}$ is a final sink if and only if $B = \bigwedge_{i \in I} \overrightarrow{f_i} A_i$.

Notice that the fact that final morphisms (resp. initial morphisms) compose is expressed in a "set-like" way in our notation. If the composition $g \circ f$ makes sense, then

$$(\overrightarrow{g \circ f})A = \overrightarrow{g}(\overrightarrow{f} A); \tag{9}$$

$$(\overleftarrow{g \circ f})B = \overleftarrow{f}(\overleftarrow{g} B). \tag{10}$$

This clearly extends to final sinks (resp. initial sources) in the following way:

$$\bigwedge (\overrightarrow{g \circ f_i})A_i = \overrightarrow{g}(\bigwedge \overrightarrow{f_i} A_i); \tag{11}$$

$$\bigvee (\overleftarrow{g_i \circ f})B_i = \overleftarrow{f}(\bigvee \overleftarrow{g_i} B_i). \tag{12}$$

3. Projectors, coprojectors

As mentioned in the Introduction, our focus is on maps $M : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ that preserve the underlying $\mathbf{X}$-objects. For each $\mathbf{X}$-object X, the fiber $\{A \in Ob(\mathbf{A}) : |A| = X\}$ is a complete lattice. If the restriction of M to each fiber is order-preserving, then we call M a *modifier.* To study modifiers, we first consider order-preserving maps $g : X \to X$, where X is a complete lattice.

A subset of a complete lattice is called *projective* (respectively *coprojective*) if it is closed under suprema (respectively, under infima). Let i denote the identity map of X.

Proposition 7. *If X is a complete lattice and $g : X \to X$ is an order-preserving map, then*

$$\{g \geq i\} = \{x \in X : g(x) \geq x\}$$

is projective, and

$$\{g \leq i\} = \{x \in X : x \leq g(x)\}$$

is coprojective.

Given a map $f : X \to X$, denote by fix$f$ the set of fixed points of f. It is straightforward that f is idempotent if and only if $f(X) = \text{fix}f$ and that f is expansive (respectively, contractive) if and only if fix$f = \{x : f(x) \leq x\}$ (respectively, fix$f = \{x : f(x) \geq x\}$). The subset fix$f$ of X is projective if and only if f is order-preserving, idempotent, and contractive. In this case f is called a *projector.* Moreover, each projective subset is the set of fixed points of a unique projector.

Dually, fix$f$ is a coprojective subset of X if and only if f is order-preserving, idempotent, and contractive; then f is called a *coprojector*. Each coprojective subset is the set of fixed points of a unique coprojector.

By extension, we call a modifier $M : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ a *projector* (respectively, a *coprojector*) if its restriction to each fiber is a projector (respectively, a coprojector) in the sense above. A class of **A**-objects whose trace on each fiber is (co)projective is called (co)projective, and a modifier M is a (co)projector if and only if fix$M = \{A \in Ob(\mathbf{A}) : A = MA\}$ is (co)projective in this sense.

In Proposition 7, g need neither be the projector on $\{g \geq i\}$ nor the coprojector on $\{g \leq i\}$, but the projector and coprojector can be easily calculated from g ([6]). Let $g : X \to X$ be an order-preserving map and let α be an ordinal number. If g is contractive then g^α is defined by $g^\alpha(x) = g(\bigwedge_{\beta<\alpha} g^\beta(x))$, whereas g^α is defined by $g^\alpha(x) = g(\bigvee_{\beta<\alpha} g^\beta(x))$ if g is expansive.

Theorem 8. *Let P be a projective subset of a complete lattice X.*

(1) *The projector p on P is the greatest element of the class of order-preserving maps $g : X \to X$ such that $P = \{g \geq i\}$.*
(2) *If $g : X \to X$ is a contractive and order-preserving map such that $P = \{g \geq i\}$, then $\bigwedge_{\alpha\in\mathrm{Ord}} g^\alpha$ is the projector on P.*
(3) *If $g : X \to X$ is an order-preserving map such that $P = \{g \geq i\}$, then*
$$\bigwedge_{\alpha\in\mathrm{Ord}} (g \wedge i)^\alpha$$
is the projector on P.

Proof. 1. Let g be an order-preserving map such that $\{g \geq i\} = P$. As p is contractive, $x \geq px$ for every $x \in X$. Hence $gx \geq g(px)$. Moreover, $px \in P = \{g \geq i\}$, so that $g(px) \geq px$. Hence, $gx \geq px$ for every x, that is, $g \geq p$.

2. If g is contractive then for each $x \in X$, the transfinite sequence $(g^\alpha(x))_{\alpha\in\mathrm{Ord}}$ is decreasing. As X is a complete lattice, it cannot be strictly decreasing. Hence, there exists an ordinal α for which $g^\alpha(x) = g^{\alpha+1}(x)$. As an element of fix$g$, $g^\alpha(x)$ belongs to P and is smaller than x. Hence $p(x) \geq g^\alpha(x)$ by definition of p. Moreover, by 1., $g \geq p$, so that, by an easy induction $g^\alpha \geq p$ for every α ([7]). Hence, $p(x) = g^\alpha(x)$.

3. It is sufficient to observe that $\{g \geq i\} = \{g \wedge i \geq i\}$ and that $g \wedge i$ is contractive. □

Dually,

Theorem 9. *Let C be a coprojective subset of a complete lattice X.*

[6] In Theorem 8 and Theorem 9 below, (2) is evidently a particular instance of (3). However, we include (2) because this simplified form is most often used, and because it allows for a more transparent argument.

[7] Assume that $g^\beta(x) \geq p(x)$ for every $\beta < \alpha$. Then $\bigwedge_{\beta<\alpha} g^\beta(x) \geq p(x)$ and
$$g\left(\bigwedge_{\beta<\alpha} g^\beta(x)\right) \geq g\,(p(x))$$
because g is order preserving. But $p(x) \in P = P_g$ so that $g(p(x)) \geq p(x)$.

(1) *The coprojector* c *on* C *is the smallest element of the class of order-preserving maps* $g : X \to X$ *such that* $C = \{g \leq i\}$.
(2) *If* $g : X \to X$ *is an* expansive *and order-preserving map such that* $C = \{g \leq i\}$, *then* $\bigvee_{\alpha \in \mathrm{Ord}} g^{\alpha}$ *is the coprojector on* C.
(3) *If* $g : X \to X$ *is an order-preserving map such that* $C = \{g \leq i\}$, *then*

$$\bigvee_{\alpha \in \mathrm{Ord}} (g \vee i)^{\alpha}$$

is the coprojector on C.

4. Functors, reflectors, coreflectors

Theorem 10. *Let* $M : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ *be a modifier. The following are equivalent:*

(1) M *extends to a unique functor* $M : \mathbf{A} \to \mathbf{A}$;
(2) *for every* $\mathbf{X}$*-morphism* $f : |A| \to Y$

$$\overrightarrow{f}(MA) \geq M\left(\overrightarrow{f}A\right);$$

(3) *for every* $\mathbf{X}$*-morphism* $f : X \to |B|$

$$M\left(\overleftarrow{f}B\right) \geq \overleftarrow{f}(MB).$$

Proof. $(1 \Longrightarrow 2)$. By definition of $\overrightarrow{f}A$, the map $f : A \to \overrightarrow{f}A$ is an $\mathbf{A}$-morphism, whenever $f : |A| \to Y$ is an $\mathbf{X}$-morphism. Since $M : \mathbf{A} \to \mathbf{A}$ is a functor, $Mf : MA \to M(\overrightarrow{f}A)$ is an $\mathbf{A}$-morphism. In view of (1) and of $|Mf| = |f|$, we have $\overrightarrow{f}(MA) \geq M(\overrightarrow{f}A)$.

$(2 \Longrightarrow 3)$. Let $f : X \to |B|$ be an $\mathbf{X}$-morphism. Then $f : \overleftarrow{f}B \to B$ is an $\mathbf{A}$-morphism. In view of (1), $\overrightarrow{f}\left(\overleftarrow{f}B\right) \geq B$, and, as M is a modifier,

$$M\left(\overrightarrow{f}\left(\overleftarrow{f}B\right)\right) \geq MB.$$

By 2., $\overrightarrow{f}\left(M\left(\overleftarrow{f}B\right)\right) \geq MB$. In other words, $f : M\left(\overleftarrow{f}B\right) \to MB$ is an $\mathbf{A}$-morphism. In view of (1), we have $M\left(\overleftarrow{f}B\right) \geq \overleftarrow{f}(MB)$.

$(3 \Longrightarrow 1)$. Let $f : A \to B$ be an $\mathbf{A}$-morphism, that is, $A \geq \overleftarrow{f}B$. As M is a modifier, $MA \geq M\left(\overleftarrow{f}B\right)$. By 3., $MA \geq \overleftarrow{f}(MB)$, that is, $|f|$ underlies a unique $\mathbf{A}$-morphism $Mf : MA \to MB$. The fact that M preserves identities and compositions is clear from (10). □

Corollary 11. *A subclass* $\mathbf{B}$ *of* $Ob(\mathbf{A})$ *determines a full reflective category of* $\mathbf{A}$ *if and only if it is projective and the associated projector* $B : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ *is a functor.*

Proof. A subclass $\mathbf{B}$ of $Ob(\mathbf{A})$ determines a full reflective category of $\mathbf{A}$ if it is closed under initial sources. Recall that $(f_i : A \to B_i)_{i \in I}$ is an initial source if and only if $A = \bigvee_{i \in I} \overleftarrow{f_i}B_i$. Assume that $\mathbf{B}$ is projective, that the associated projector $B : Ob(\mathbf{A}) \to Ob(\mathbf{A})$ is a functor and that each $B_i \in \mathbf{B}$. Hence $B_i = B(B_i)$ for each $i \in I$, so that, in view of Theorem 10, $B\left(\overleftarrow{f}B_i\right) \geq \overleftarrow{f}B_i$. As B is a projector,

$\overleftarrow{f} B_i \in \mathbf{B}$. Moreover, $\mathbf{B}$ is projective, hence closed under supremum. Therefore, $A = \bigvee_{i \in I} \overleftarrow{f_i} B_i \in \mathbf{B}$. Conversely, if $\mathbf{B}$ is a full reflective subcategory of $\mathbf{A}$, then it is clear that the restriction of the associated reflector to objects is a projector and a functor. □

In particular, a modifier is a reflector if and only if it is a projector and a functor.

Of course, Corollary 11 admits a dual statement. In particular, a modifier is a coreflector if and only if it is a coprojector and a functor.

5. Projectors and reflectors in **Conv**

A *convergence* ξ on a set X is a relation between points of X and the set $\mathbb{F}X$ of filters on X, denoted $x \in \lim_\xi \mathcal{F}$ if $(x, \mathcal{F}) \in \xi$, that satisfies [8]

$$x \in \lim_\xi \{x\}^\uparrow$$

for every x in X, and

$$\mathcal{F} \geq \mathcal{G} \Longrightarrow \lim_\xi \mathcal{F} \supseteq \lim_\xi \mathcal{G}.$$

A map $f : X \to Y$ between two convergence spaces (X, ξ) and (Y, τ) is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$ for every filter $\mathcal{F}$ on X. The category **Conv** of convergence spaces and continuous maps is topological over **Set**. Consider $f : (X, \xi) \to (Y, \tau)$. A filter $\mathcal{F}$ converges to x for the initial convergence $\overleftarrow{f}\tau$ if and only if $f(x) \in \lim_\tau f(\mathcal{F})$ and $\mathcal{G}$ converges to y for the final convergence $\overrightarrow{f}\xi$ if there exists $\mathcal{F}$ on X and $x \in f^{-1}(y)$ such that $x \in \lim_\xi \mathcal{F}$ and $\mathcal{G} \geq f(\mathcal{F})$.

A topological space can be considered as a particular convergence space by declaring that a filter converges to a point if it is finer than the neighborhood filter of the point. Conversely, a topology is canonically associated to each convergence space: A subset of a convergence space (X, ξ) is *open* if it belongs to every filter that converges to one of its points. Complements of open sets are called *closed.* The family $\mathcal{O}(\xi)$ of open sets for ξ satisfies the axioms of a topology. This topology is called the *topological modification of* ξ and denoted $T\xi$. Let $\mathcal{C}(\xi)$ denote the family of closed subsets of ξ. Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a set X *mesh*, in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. When $\mathcal{A} = \{A\}$ consists of one element, we abridge $\{A\}\#\mathcal{B}$ to $A\#\mathcal{B}$.

Example 12. *The category* $\mathbf{T}$ *of topological spaces (and continuous maps) is a reflective subcategory of* **Conv** *with reflector* T. *The topological modification (or reflection) of a convergence* ξ *can be calculated explicitly:*

$$\lim_{T\xi} \mathcal{F} = \bigcap_{\mathcal{C}(\xi) \ni C \# \mathcal{F}} C. \tag{13}$$

The infimum of all the filters converging to a point x in ξ is called *vicinity filter of* x and is denoted $\mathcal{V}_\xi(x)$. A convergence ξ is *a pretopology* [5] if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for every $x \in |\xi|$. Of course, each topology is a pretopology, but not conversely. Indeed,

[8] $\{x\}^\uparrow$ is the principal ultrafilter of x. More generally, if $\mathcal{A} \subset 2^X$, then

$$\mathcal{A}^\uparrow = \{B \subset X : \exists A \in \mathcal{A}, A \subset B\}.$$

$\mathcal{V}_\xi(x)$ does not need to have a base of open sets ([9]). The family of open vicinities of x generates a filter $\mathcal{N}_\xi(x)$, called *neighborhood filter of* x. It is the neighborhood filter of x for $T\xi$.

The *adherence (*for ξ*) of a family* $\mathcal{H}$ of subsets of $|\xi|$ is the union of limits of filters meshing with it, i.e.,

$$\mathrm{adh}_\xi \mathcal{H} = \bigcup_{\mathcal{G} \# \mathcal{H}} \lim_\xi \mathcal{G}.$$

In particular, the adherence $\mathrm{adh}_\xi A$ of a subset A of $|\xi|$ is the adherence of its principal filter $\{A\}^\uparrow = \{B \subset |\xi| : A \subset B\}$.

Example 13. *The category* **P** *of pretopological spaces (and continuous maps) is a reflective subcategory of* **Conv**. *The associated reflector* P *can be characterized by*

$$\lim_{P\xi} \mathcal{F} = \bigcap_{A \# \mathcal{F}} \mathrm{adh}_\xi A. \tag{14}$$

If $\mathcal{F} \in \mathbb{F}Y$ and $\mathcal{G} : Y \to \mathbb{F}X$, then we define the *contour of* $\mathcal{G}$ *along* $\mathcal{F}$ ([10]) by

$$\int_{\mathcal{F}} \mathcal{G} = \bigvee_{F \in \mathcal{F}} \bigwedge_{y \in F} \mathcal{G}(y).$$

A convergence ξ is called *diagonal* if $x \in \lim_\xi \int_{\mathcal{F}} \mathcal{G}$ whenever $x \in \lim_\xi \mathcal{F}$ and $y \in \lim_\xi \mathcal{G}(y)$ for every $y \in |\xi|$. A convergence ξ is topological if and only if it is pretopological and diagonal (e.g., [9]).

Example 14. *The class of diagonal convergence spaces is projective but not reflective* [23]. *The corresponding projector* D *is obtained by iteration, as in Theorem 8 (2), of the contractive modifier* d *defined by* $x \in \lim_{d\xi} \mathcal{H}$ *whenever there exist a filter* $\mathcal{F}$ *with* $x \in \lim_\xi \mathcal{F}$ *and* $\mathcal{G} : X \to \mathbb{F}X$ *with* $y \in \lim_\xi \mathcal{G}(y)$ *for every* $y \in |\xi|$ *such that*

$$\mathcal{H} \geq \int_{\mathcal{F}} \mathcal{G}.$$

Example 15. *A convergence* ξ *is a* pseudotopology [5] *if a filter converges to* x *whenever each finer ultrafilter converges to* x. *The category* **S** *of pseudotopological spaces (and continuous maps) is a reflective subcategory of* **Conv**. *The associated reflector* S *can be characterized by*

$$\lim_{S\xi} \mathcal{F} = \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim_\xi \mathcal{U} = \bigcap_{\mathcal{H} \# \mathcal{F}} \mathrm{adh}_\xi \mathcal{H}, \tag{15}$$

where $\mathbb{U}(\mathcal{F})$ *is the family of ultrafilters finer than* $\mathcal{F}$.

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[9] For instance the pretopology on $\{x_n^m : n, m \in \omega\} \cup \{x_n : n \in \omega\} \cup \{\infty\}$ defined by $\mathcal{V}(x_n^m) = \{x_n^m\}^\uparrow$ for every n and m, $\mathcal{V}(x_n) = \{\{x_n\} \cup \{x_k^n : k \geq p\} : p \in \omega\}^\uparrow$ for every n, and $\mathcal{V}(\infty) = \{\{\infty\} \cup \{x_k : k \geq p\} : p \in \omega\}^\uparrow$ is non topological. Its topological modification is the usual Arens space topology.

[10] Here we follow the terminology of [12]. Contours have been used by many authors, most frequently without attributing them any name. Frolík calls them *sums of filters* [19], Cook and Fischer refer to the *compression operator* of filters [6], while many other authors refer to the *Kowalsky diagonal operation* e.g., [25].

Equations (13) , (14) and (15) can be unified by defining the map $\mathrm{Adh}_{\mathbb{J}}$ on objects of **Conv** via

$$\lim_{\mathrm{Adh}_{\mathbb{J}}\xi}\mathcal{F} = \bigcap_{\mathbb{J}\ni\mathcal{J}\#\mathcal{F}} \mathrm{adh}_{\xi}\mathcal{J},$$

where $\mathbb{J}$ is a class of filters [7]. In particular, (15) is recovered when $\mathbb{J} = \mathbb{F}$ is the class of all filters; (14) is recovered when $\mathbb{J}$ is the class $\mathbb{F}_0$ of principal filters, and (13) corresponds to the case where $\mathbb{J}$ is the class of principal filters of closed sets. In general, the class $\mathbb{J}$ can depend on the convergence. Hence, as seen before, we write $\mathbb{J}(\xi)$ for the set of $\mathbb{J}$-filters on the convergence space $(|\xi|, \xi)$. Consider the following properties of a class $\mathbb{J}$ of filters:

$$\sigma \leq \xi \Longrightarrow \mathbb{J}(\sigma) \subset \mathbb{J}(\xi) \tag{16a}$$

$$\forall_{\xi} \ \mathbb{J}(\mathrm{Adh}_{\mathbb{J}}\xi) = \mathbb{J}(\xi) \tag{16b}$$

$$\forall_{f:X\to|\tau|} \left(\mathcal{G} \in \mathbb{J}(\tau) \Longrightarrow f^{-}\mathcal{G} \in \mathbb{J}(\overleftarrow{f}\,\tau)\right) \tag{16c}$$

Proposition 16. [7]

1. *If $\mathbb{J}$ satisfies (16a) then $\mathrm{Adh}_{\mathbb{J}}$ is a contractive modifier of* **Conv**;
2. *If $\mathbb{J}$ satisfies (16a) and (16b), then $\mathrm{Adh}_{\mathbb{J}}$ is a projector;*
3. *Let $f : X \to |\tau|$. If $\mathbb{J}$ satisfies (16c), then*

$$\mathrm{Adh}_{\mathbb{J}}\left(\overleftarrow{f}\,\tau\right) \geq \overleftarrow{f}\left(\mathrm{Adh}_{\mathbb{J}}\tau\right).$$

Corollary 17. *If $\mathbb{J}$ satisfies (16a), (16b) and (16c), then $\mathrm{Adh}_{\mathbb{J}}$ is a reflector.*

A class $\mathbb{J}$ of filters is called $\mathbb{F}_0$-*composable* if $R\mathcal{J} \in \mathbb{J}(Y)$ whenever $\mathcal{J} \in \mathbb{J}(X)$ and $R : X \rightrightarrows Y$ is a relation.

Corollary 18. *Let $\mathbb{J}$ be an $\mathbb{F}_0$-composable class of filters independent of the convergence. Then $\mathrm{Adh}_{\mathbb{J}}$ is a reflector.*

If B is an object of a reflective subcategory **J** of **A**, then $\overleftarrow{f}B$ is in **J**, but in general the corresponding projector J does not commute with $\overleftarrow{f}$. However, many of the reflectors $\mathrm{Adh}_{\mathbb{J}}$ have this remarkable property:

Theorem 19. *Let $\mathbb{J}$ be an $\mathbb{F}_0$-composable class of filters independent of the convergence. Then $\mathrm{Adh}_{\mathbb{J}}$ is a reflector satisfying*

$$\mathrm{Adh}_{\mathbb{J}}\left(\overleftarrow{f}\,\tau\right) = \overleftarrow{f}\left(\mathrm{Adh}_{\mathbb{J}}\tau\right)$$

for each $f : X \to |\tau|$.

Proof. Let $x \in \lim_{\overleftarrow{f}(\mathrm{Adh}_{\mathbb{J}}\tau)}\mathcal{F}$, and let $\mathcal{J} \in \mathbb{J}$ such that $\mathcal{J}\#\mathcal{F}$. Then $f(\mathcal{J})\#f(\mathcal{F})$ and $f(\mathcal{J}) \in \mathbb{J}$ by $\mathbb{F}_0$-composability. Hence $f(x) \in \mathrm{adh}_{\tau}f(\mathcal{J})$ because $f(x) \in \lim_{\mathrm{Adh}_{\mathbb{J}}\tau}f(\mathcal{F})$. But $\overleftarrow{f}(\mathrm{adh}_{\tau}f(\mathcal{J})) = \mathrm{adh}_{\overleftarrow{f}\tau}\mathcal{J}$ (e.g., [7]). Hence, $x \in \lim_{\mathrm{Adh}_{\mathbb{J}}\left(\overleftarrow{f}\tau\right)}\mathcal{F}$. □

Corollary 20. *Let $\mathbb{J}$ be an $\mathbb{F}_0$-composable class of filters independent of the convergence. Then $\mathrm{Adh}_{\mathbb{J}}$ is a reflector preserving initial maps. In particular, if (X, ξ) is a subspace of (Y, τ), then $(X, \mathrm{Adh}_{\mathbb{J}}\xi)$ is a subspace of $(Y, \mathrm{Adh}_{\mathbb{J}}\tau)$.*

Corollary 21. *If $\mathbb{J}$ is an $\mathbb{F}_0$-composable class of filters independent of the convergence, then $\mathbf{Adh}_{\mathbb{J}}$ is an extensional* ([11]) *reflective subcategory of* **Conv**.

Proof. Note that **Conv** is extensional, so that for every $\tau = \mathrm{Adh}_{\mathbb{J}}\tau$, there exists a one point extension τ^* in **Conv** such that every $f : \xi \to \tau$, where ξ is a subspace of σ extends to a continuous $f^* : \sigma \to \tau^*$. If moreover $\xi = \mathrm{Adh}_{\mathbb{J}}\xi$ is a subspace of $\sigma = \mathrm{Adh}_{\mathbb{J}}\sigma$ in $\mathbf{Adh}_{\mathbb{J}}$, then $f^* : \sigma \to \mathrm{Adh}_{\mathbb{J}}(\tau^*)$ is continuous. Moreover, in view of Corollary 20, τ is a subspace of $\mathrm{Adh}_{\mathbb{J}}(\tau^*)$ in $\mathbf{Adh}_{\mathbb{J}}$. □

Let $\mathbb{F}_1$ denote the class of filters that admit a countable filter base. In [7] *paratopologies* were introduced as convergences ξ satisfying $\xi = \mathrm{Adh}_{\mathbb{F}_1}\xi$. Therefore the category of *paratopological spaces*, is denoted by either $\mathbf{P}_\omega$ or $\mathbf{Adh}_{\mathbb{F}_1}$.

Example 22. *The categories* **P** *of pretopological spaces (and continuous maps),* $\mathbf{P}_\omega$ *of paratopological spaces, and* **S** *of pseudotopological spaces are extensional subcategories of* **Conv**.

Finally, the condition of functoriality in Proposition 16 (3) is best possible if $\mathbb{J}$ is independent of the convergence.

Proposition 23. *Let $\mathbb{J}$ be a class of filters independent of convergence. If there exists $\mathcal{J} \in \mathbb{J}(Y)$ and $f : X \to Y$ such that $f^-\mathcal{J} \notin \mathbb{J}(X)$, then there exists a convergence τ on Y such that*

$$\mathrm{Adh}_{\mathbb{J}}\left(\overleftarrow{f}\,\tau\right) \not\geq \overleftarrow{f}\,(\mathrm{Adh}_{\mathbb{J}}\tau)\,.$$

Proof. Let $\mathcal{F}_0$ be a filter on X such that $\mathcal{F}_0 \# f^-\mathcal{J}_0$ and fix $y_0 \in Y$. Since $f^-\mathcal{J}_0 \notin \mathbb{J}(X)$, for every $\mathcal{J} \in \mathbb{J}(X)$ such that $\mathcal{J} \# \mathcal{F}_0$, there exists a filter $\mathcal{M}_{\mathcal{J}}$ such that $\mathcal{M}_{\mathcal{J}} \# \mathcal{J}$ but $\mathcal{M}_{\mathcal{J}}$ and $f^-\mathcal{J}_0$ do not mesh. Consider on Y the convergence τ in which every point but y_0 is isolated, and $y_0 \in \lim_\tau \mathcal{U}$ if there exists $\mathcal{J} \in \mathbb{J}(X)$ such that $\mathcal{J} \# \mathcal{F}_0$ and $\mathcal{U} \geq f(\mathcal{M}_{\mathcal{J}}) \wedge \{y_0\}^\uparrow$. Then, by definition of τ, $x \in \lim_{\mathrm{Adh}_{\mathbb{J}}\left(\overleftarrow{f}\,\tau\right)} \mathcal{F}_0$ for every $x \in f^-(y_0)$, but $x \notin \lim_{\overleftarrow{f}\,(\mathrm{Adh}_{\mathbb{J}}\tau)} \mathcal{F}_0$ because $\mathcal{J}_0 \in \mathbb{J}(Y)$, $\mathcal{J}_0 \# f(\mathcal{F}_0)$ and $y_0 \notin \mathrm{adh}_\tau \mathcal{J}_0$. □

In view of Propositions 16 and 23:

Corollary 24. *Let $\mathbb{J}$ be a class of filters independent of the convergence. If there exists $\mathcal{J} \in \mathbb{J}(Y)$ and $f : X \to Y$ such that $f^-\mathcal{J} \notin \mathbb{J}(X)$, then* $\mathrm{Adh}_{\mathbb{J}}$ *is a projector but not a reflector.*

Let $\mathbb{E}$ denote the class of filters generated by sequences. In view of Corollary 24, $\mathrm{Adh}_{\mathbb{E}}$ is a projector but is not a reflector.

6. Coprojectors and coreflectors in **Conv**

From the viewpoint of convergence, there is no reason to distinguish between a sequence $(x_n)_{n\in\omega}$ and the filter $\{\{x_n : n \geq k\} : k \in \omega\}^\uparrow$ generated by the family of its tails. Therefore, we denote the later also by $(x_n)_{n\in\omega}$.

[11] Recall that a category is *extensional* if for every object Y, there exists a one point extension Y^* of Y such that whenever $f : X \to Y$ is a morphism, where X is a subspace of Z, the map $f^* : Z \to Y^*$ that coincide with f on X and sends $Z \setminus X$ on the point $Y^* \setminus Y$ is a morphism.

Example 25. *The modifier* Seq *of* **Conv** *defined by*

$$\lim_{\mathrm{Seq}\xi}\mathcal{F} = \bigcup_{(x_n)_{n\in\omega}\leq\mathcal{F}} \lim_{\xi}(x_n)_{n\in\omega}$$

is a coreflector. A convergence $\xi = \mathrm{Seq}\xi$ *is called* sequentially based.

A subset K of $|\xi|$ is *compact* (for ξ) if $\lim_{\xi}\mathcal{U}\cap K \neq \varnothing$ for every ultrafilter $\mathcal{U}$ containing K. Let $\mathbb{K}(\xi)$ denote the family of (principal filters of) compact subsets of $|\xi|$.

Example 26. *A convergence* ξ *is called* locally compact *if each convergent filter contains a compact set. The class of locally compact convergence spaces is coreflective in* **Conv** *and the associated coreflector* K *is given by*

$$\lim_{K\xi}\mathcal{F} = \begin{cases} \lim_{\xi}\mathcal{F} \text{ if } \mathcal{F}\cap\mathbb{K}(\xi)\neq\varnothing \\ \varnothing \text{ otherwise} \end{cases}$$

More generally, following [7], given a class $\mathbb{J}(\cdot)$ of filters (possibly depending on the convergence) that contains principal ultrafilters, we say that a convergence ξ is $\mathbb{J}$-*based* if whenever $x \in \lim_{\xi}\mathcal{F}$ there exists a $\mathbb{J}$-filter $\mathcal{J}\leq\mathcal{F}$ such that $x\in\lim_{\xi}\mathcal{J}$. We denote by $\mathbb{J}^{\uparrow}$ the class of filters that are finer than a $\mathbb{J}$-filter. Sequentially based convergences are exactly $\mathbb{E}$-based convergences and locally compact convergences are exactly $\mathbb{K}^{\uparrow}$-based convergences, where $\mathbb{K}$ is the class of principal filters of compact sets ([12]).

For every class $\mathbb{J}$ of filters, the map $\mathrm{Base}_{\mathbb{J}}$ is defined on objects of **Conv** by

$$\lim_{\mathrm{Base}_{\mathbb{J}}\xi}\mathcal{F} = \bigcup_{\mathbb{J}(\xi)\ni\mathcal{J}\leq\mathcal{F}} \lim_{\xi}\mathcal{J}.$$

Consider the following properties of a class $\mathbb{J}$ of filters:

$$\sigma\leq\xi \Longrightarrow \mathbb{J}(\xi)\subset\mathbb{J}(\sigma) \tag{17a}$$

$$\forall_{\xi}\ \mathbb{J}(\mathrm{Base}_{\mathbb{J}}\xi) = \mathbb{J}(\xi) \tag{17b}$$

$$\forall_{f:|\xi|\to Y}\left(\mathcal{J}\in\mathbb{J}(\xi)\Longrightarrow f(\mathcal{J})\in\mathbb{J}(\overrightarrow{f}\xi)\right) \tag{17c}$$

Proposition 27. [7] *Let* $\mathbb{J}$ *be a class of filters that contains principal ultrafilters.*

1. *If* (17*a*) *is satisfied then* $\mathrm{Base}_{\mathbb{J}}$ *is an expansive modifier of* **Conv**;
2. *If* (17*a*) *and* (17*b*) *are satisfied, then* $\mathrm{Base}_{\mathbb{J}}$ *is a coprojector;*
3. *If* (17*c*) *is satisfied then*

$$\overrightarrow{f}(\mathrm{Base}_{\mathbb{J}}\xi)\geq\mathrm{Base}_{\mathbb{J}}(\overrightarrow{f}\xi).$$

Corollary 28. *If* (17*a*), (17*b*) *and* (17*c*) *are satisfied, then* $\mathrm{Base}_{\mathbb{J}}$ *is a coreflector.*

Corollary 29. *if* $\mathbb{J}$ *is an* $\mathbb{F}_0$*-composable class of filters that does not depend on the convergence, then* $\mathrm{Base}_{\mathbb{J}}$ *is a coreflector.*

Example 30. *If* $\mathbb{J}=\mathbb{F}_1$ *is the class of countably based filters, then* $\mathrm{Base}_{\mathbb{F}_1}$ *is the coreflector on* first-countable convergences, *or convergences* of countable character.

[12] In [7], a convergence is called $\mathbb{J}$-*founded* if every convergent filter admits a coarser $\mathbb{J}$-filter. Notice that $\mathbb{J}$-founded convergences are $\mathbb{J}^{\uparrow}$-based convergences.

Example 31. *If* $\mathbb{J} = \mathbb{F}_0$ *is the class of principal filters, then* $\mathrm{Base}_{\mathbb{F}_0}$ *is the coreflector on* finitely generated convergences.

Example 32. *A filter* $\mathcal{F}$ *is called* countably tight *if whenever* $A\#\mathcal{F}$ *there exists a countable subset* B *of* A *such that* $B\#\mathcal{F}$*. Let* $\mathbb{F}_{\#\omega}$ *denote the class of countably tight filters. It is easy to see that a topology is countably tight* ([13]) *if and only if all its neighborhood filters are countably tight, that is, if and only if it is* $\mathbb{F}_{\#\omega}$*-based as a convergence. By extension, we call* countably tight *convergences that are* $\mathbb{F}_{\#\omega}$*-based. They form a coreflective subcategory of* **Conv**.

Let $\mathbb{D}$ be a class of filters and let (X, ξ) be a convergence space. A family $\mathcal{F}$ of subsets of X is $\mathbb{D}$-*compact at* $A \subset X$ if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{F} \Longrightarrow \mathrm{adh}\mathcal{D} \cap A \neq \varnothing.$$

In case $A = X$, we say that $\mathcal{F}$ is *relatively* $\mathbb{D}$*-compact.* Notice that if $\mathbb{D}$ is $\mathbb{F}_0$-composable, then a continuous image of a relatively $\mathbb{D}$-compact filter is relatively $\mathbb{D}$-compact.

Example 33 (q-spaces)**.** *Recall that a topological space is a* q*-space if every point admits a sequence* $(Q_n)_{n\in\omega}$ *of neighborhoods such that every sequence* $(x_n)_{n\in\omega}$ *with* $x_n \in Q_n$ *for every* n*, has non-empty adherence. In other words, when considered as a convergence space, a* q*-space is simply a convergence based in the class* $\mathbb{Q}$ *of filters finer than countably based and relatively countably compact filters. The class* $\mathbb{Q}$ *is preserved by continuous maps, and therefore, in view of Corollary 28, the class of* $\mathbb{Q}$*-based convergences, or* q-convergences*, is coreflective. Denote by* Q *the associated coreflector.*

Example 34 (spaces of pointwise countable type)**.** *Recall that a topological space is* of pointwise countable type *if every point admits a sequence* $(Q_n)_{n\in\omega}$ *of neighborhoods such that every filter meshing with* $(Q_n)_{n\in\omega}$ *has non empty adherence. In other words, when considered as a convergence space, a space of pointwise countable type is simply a convergence based in the class* $\mathbb{Q}_K$ *of filters finer than countably based, relatively compact filters. The class* $\mathbb{Q}_K$ *is preserved by continuous maps, and therefore, in view of Corollary 28, the class of* $\mathbb{Q}_K$*-based convergences, or* convergences of pointwise countable type*, is coreflective. Denote by* Q_K *the associated coreflector.*

Finally, the functoriality condition in Proposition 27 (3) is best possible if $\mathbb{J}$ is independent of the convergence.

Proposition 35. *Let* $\mathbb{J}$ *be a class of filters independent of the convergence. If there exists* $\mathcal{J} \in \mathbb{J}(X)$ *and* $f : X \to Y$ *such that* $f(\mathcal{J}) \notin \mathbb{J}(Y)$*, then there exists a convergence* ξ *on* X *such that*

$$\overrightarrow{f}(\mathrm{Base}_{\mathbb{J}}\xi) \ngeq \mathrm{Base}_{\mathbb{J}}(\overrightarrow{f}\xi).$$

Example 36. *Let* $\mathbb{F}_U$ *be the class of ultrafilters that are either uniform* ([14]) *or principal. Then* $\mathrm{Base}_{\mathbb{F}_U}$ *is a coprojector but not a coreflector.*

[13] In the sense that if $x \in \mathrm{cl}A$ then there exists a countable subset B of A such that $x \in \mathrm{cl}B$.

[14] Recall that a filter $\mathcal{F}$ on X is *uniform* if every $F \in \mathcal{F}$ has the same cardinality as X.

7. Projectors and coprojectors in functorial context

Proposition 7 combines with Theorem 10 to the effect that

Theorem 37. *Let F be a functor of $\mathbf{A}$. Then*

$$\mathbf{F}^{\geq I}=\{A\in Ob(\mathbf{A}):FA\geq A\}$$

determines a full reflective subcategory of $\mathbf{A}$, and

$$\mathbf{F}^{\leq I}=\{A\in Ob(\mathbf{A}):FA\leq A\}$$

determines a full coreflective subcategory of $\mathbf{A}$.

Proof. As $\mathbf{F}^{\geq I}$ is projective by Proposition 7, it is closed under supremum. If F is a functor then $F\left(\overleftarrow{f}A\right)\geq\overleftarrow{f}(FA)$. If moreover $A\in\mathbf{F}^{\geq I}$ then $FA\geq A$, so that $F\left(\overleftarrow{f}A\right)\geq\overleftarrow{f}(FA)\geq\overleftarrow{f}A$. Hence $\overleftarrow{f}A\in\mathbf{F}^{\geq I}$. The proof for $\mathbf{F}^{\leq I}$ is dual. □

In particular, if F is a functor of **Conv**, convergences of $\mathbf{F}^{\geq I}$ (resp. $\mathbf{F}^{\leq I}$) are called $\mathbf{F}^{\geq I}$-*convergences* (resp. $\mathbf{F}^{\leq I}$-*convergences*). Immediate consequences of Theorems 8 and 9 are:

Corollary 38. *Let $\mathbf{G}$ be a reflective subcategory of $\mathbf{A}$.*

1. *The reflector $G:\mathbf{A}\to\mathbf{G}$ is the smallest element of the class of functors F of $\mathbf{A}$ such that $\mathbf{F}^{\geq I}=\mathbf{G}$;*
2. *If F is a functor of $\mathbf{A}$ such that $\mathbf{F}^{\geq I}=\mathbf{G}$, then*
$$G=\bigwedge_{\alpha\in\mathrm{Ord}}(F\wedge I)^{\alpha},$$
where I is the identity functor and $F^{\alpha}A=F\left(\bigwedge_{\beta<\alpha}F^{\beta}A\right)$.

If $\mathcal{F}$ is a filter on a convergence space (X,ξ), let $\mathrm{adh}_{\xi}^{\natural}\mathcal{F}$ be the filter generated by $\{\mathrm{adh}_{\xi}F:F\in\mathcal{F}\}$ and more generally let $o^{\natural}\mathcal{F}=\{o(F):F\in\mathcal{F}\}^{\uparrow}$ if $o:2^X\to 2^X$.

Example 39. *A convergence ξ is called* regular if $\lim_{\xi}\left(\mathrm{adh}_{\xi}^{\natural}\mathcal{F}\right)=\lim_{\xi}\mathcal{F}$ *for every $\mathcal{F}$* [6]. *It is easy to see that a supremum of regular convergences is again regular so that the class of regular convergence spaces is projective. Moreover, ξ is regular if and only if $\xi\leq r\xi$, where*

$$\lim_{r\xi}\mathcal{F}=\bigcup_{\mathrm{adh}_{\xi}^{\natural}\mathcal{G}\leq\mathcal{F}}\lim_{\xi}\mathcal{G}.$$

It is straightforward that $\xi\geq r\xi$ for each ξ, and $r(\overleftarrow{f}\tau)\geq\overleftarrow{f}(r\tau)$ for every $f:X\to|\tau|$, so that r is a contractive functor. In view of Corollary 38, the category $\mathbf{R}$ of regular convergence spaces is a reflective subcategory of **Conv**, *and the associated reflector R is obtained by transfinite iteration of r.*

Corollary 40. *Let $\mathbf{C}$ be a full coreflective subcategory of $\mathbf{A}$.*

1. *The coreflector $C:\mathbf{A}\to\mathbf{C}$ is the greatest element of the class of functors F of A such that $\mathbf{F}^{\leq I}=\mathbf{C}$;*

(2) *If F is a functor of* $\mathbf{A}$ *such that* $\mathbf{F}^{\leq I}=\mathbf{C}$, *then*

$$C=\bigvee_{\alpha\in\mathrm{Ord}}(F\vee I)^{\alpha},$$

where I is the identity functor and $F^{\alpha}A=F\left(\bigvee_{\beta<\alpha}F^{\beta}A\right)$.

Example 41. *A convergence is* topologically core compact [13] *if whenever* $x\in\lim_{\xi}\mathcal{F}$, *for every* $V\in\mathcal{N}_{\xi}(x)$, *there exists* $F\in\mathcal{F}$ *such that* $\{F\}^{\uparrow}$ *is compact at* V. *Consider the modifier* C_K *of* **Conv** *defined by* $x\in\lim_{C_K\xi}\mathcal{F}$ *if* $x\in\lim_{\xi}\mathcal{F}$ *and for every* $V\in\mathcal{N}_{\xi}(x)$ *there exists* $F\in\mathcal{F}$ *that is compact at* V. *The map* C_K *is an expansive concrete functor so that* $\mathrm{Fix}C_K$ *is coreflective. The corresponding coreflector is obtained by iteration as in Corollary 40.*

8. Properties of the type $X\geq JE(X)$

Recall that a topological space X is

- *sequential* if every sequentially closed subset is closed;
- *Fréchet* if whenever $x\in X$, $A\subset X$ and $x\in\mathrm{cl}A$, there exists a sequence $(x_n)_{n\in\omega}$ on A such that $x\in\lim(x_n)_{n\in\omega}$;
- *strongly Fréchet* if whenever $x\in\bigcap_{n\in\omega}\mathrm{cl}A_n$ for a decreasing sequence of subsets A_n of X, there exists $x_n\in A_n$ such that $x\in\lim(x_n)_{n\in\omega}$;
- *bisequential* if every convergent ultrafilter contains a countably based filter that converges to the same point;
- *weakly bisequential* [24] if whenever $x\in\mathrm{adh}\mathcal{F}$ where $\mathcal{F}$ is a countably deep filter ([15]), there exists a countably based filter $\mathcal{H}\#\mathcal{F}$ such that $x\in\lim\mathcal{H}$.

Hence, a topology ξ is sequential if and only if ξ and $\mathrm{Seq}\xi$ have the same closed sets, that is, $T\xi=T\mathrm{Seq}\xi$, and since. $\xi=T\xi\leq T\mathrm{Seq}\xi$ for every topology, if and only if

$$\xi\geq T\mathrm{Seq}\xi. \tag{18}$$

It is easy to see that (18) is equivalent to

$$\xi\geq T\mathrm{Base}_{\mathbb{F}_1}\xi. \tag{19}$$

Moreover, (18) and (19) are meaningful and equivalent for general convergences, and therefore can be used to extend the definition of sequential spaces from topological to convergence spaces.

Similarly, a topology ξ is Fréchet if $\mathrm{adh}_{\xi}A\subset\mathrm{adh}_{\mathrm{Seq}\xi}A$. In view of (14), this means that $\xi\geq P\mathrm{Seq}\xi$ or equivalently that $\xi\geq P\mathrm{Base}_{\mathbb{F}_1}\xi$.

More generally,

$$\mathcal{J}\in\mathbb{J}(\xi)\implies\mathrm{adh}_{\xi}\mathcal{J}\subset\mathrm{adh}_{\mathrm{Base}_{\mathbb{D}}\xi}\mathcal{J}$$

is equivalent to

$$\xi\geq\mathrm{Adh}_{\mathbb{J}}\mathrm{Base}_{\mathbb{D}}\xi.$$

In particular, the functorial inequality

$$\xi\geq\mathrm{Adh}_{\mathbb{J}}\mathrm{Base}_{\mathbb{F}_1}\xi$$

extends the notions of sequentiality, Fréchetness, strong Fréchetness, weak bisequentiality and bisequentiality from topological to convergence spaces when $\mathbb{J}$ ranges

[15] A filter $\mathcal{F}$ is *countably deep* if $\bigcap\mathcal{A}\in\mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

over the classes of principal filters of closed sets, principal filters, countably based filters, countably deep filters and all filters respectively.

Other classical notions can be characterized by functorial inequalities of the form

$$\xi \geq JE\xi \tag{20}$$

where J is a reflector and E is a coreflector.

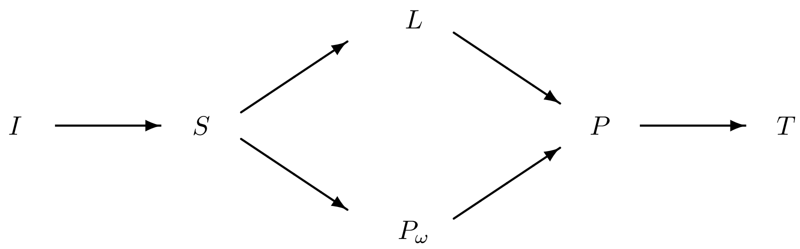
Example 42. *The family of G_δ-subsets of a topological space (X,ξ) forms a base for a (finer) topology called* G_δ-topology *of ξ. Recall that a topological space is called a P*-space *if the topology coincides with its G_δ-topology. Hence, a P-space is a space whose every neighborhood filters is countably deep. In other words, a topology ξ is a P-space if and only if $\xi = \mathrm{Base}_{\mathbb{F}_{\wedge\omega}}\xi$. It turns out that if $\xi = T\xi$ then $T\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}\xi = \mathrm{Base}_{\mathbb{F}_{\wedge\omega}}\xi$. Hence a topology is a P-space if and only if $\xi \geq \mathrm{Base}_{\mathbb{F}_{\wedge\omega}}\xi$, if and only if $\xi \geq T\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}\xi$.*

The following table (mainly from [7]) gathers classical topological properties than can be characterized via (20) .

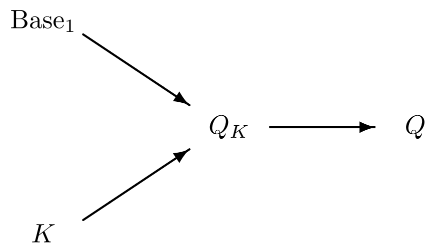
	$\mathrm{Base}_{\mathbb{F}_1}$	K	Q_K	Q	$\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
I	first-countable $\mathrm{Base}_{\mathbb{F}_1}$	locally compact K	pointwise countable type Q_K	strict q Q	P-space $\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
S	bisequential $S\mathrm{Base}_{\mathbb{F}_1}$	locally compact SK	bi-k SQ_K	bi-quasi-k SQ	P-space $S\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
P_ω	strongly Fréchet $P_\omega\mathrm{Base}_{\mathbb{F}_1}$	strongly k' $P_\omega K$	countably bi-k $P_\omega Q_K$	countably bi-quasi-k $P_\omega Q$	P-space $S\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
L	weakly bisequential $L\mathrm{Base}_{\mathbb{F}_1}$	? LK	? LQ_K	? LQ	P-space $L\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
P	Fréchet $P\mathrm{Base}_{\mathbb{F}_1}$	k'-space PK	singly bi-k PQ_K	singly bi-quasi-k PQ	P-space $P\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$
T	sequential $T\mathrm{Base}_{\mathbb{F}_1}$	k-space TK	k-space TQ_K	quasi-k TQ	P-space $T\mathrm{Base}_{\mathbb{F}_{\wedge\omega}}$

Table 1

To our knowledge, the properties of a topology corresponding to $LK^{\leq I}$-convergences, $LQ^{\leq I}$convergences and $LQ_K^{\leq I}$-convergences have not been considered before. Each row corresponds to a reflector, and each columns to a coreflector. The order relations between these functors immediately gives how these notions relate. Indeed, the reflectors involved compare as follows, where an arrow $I \to S$ means $I \geq S$.



The comparison of two notions in the same column only depends on this ordering of reflectors. For instance bisequentiality implies Fréchetness because $S \geq P$, and so, $\xi \geq P\mathrm{Base}_{\mathbb{F}_1}\xi$ whenever $\xi \geq S\mathrm{Base}_{\mathbb{F}_1}\xi$. Analogously,



The comparison of two notions in the same row depends on this ordering of coreflectors. For instance strong Fréchetness and strongly k'-ness both imply countably bi-k-ness because $\mathrm{Base}_{\mathbb{F}_1} \geq Q_K$ and $K \geq Q_K$. Hence $\xi \geq P_\omega Q_K \xi$ whenever $\xi \geq P_\omega \mathrm{Base}_{\mathbb{F}_1}\xi$ or $\xi \geq P_\omega K\xi$.

9. Properties of the type $X \leq EJ(X)$

A convergence space is called *countably Choquet* [3, page 49] or *countably pseudotopological* if a countably based filter converges to x whenever each finer ultrafilter does. Such convergences have also been called *of type* **W** in [**?**]. Evidently, (X, ξ) is countably pseudotopological if and only if

$$\xi \leq \mathrm{Base}_{\mathbb{F}_1} S\xi. \tag{21}$$

Recall that a convergence space is *Urysohn* ([16]) if a sequence converges to x whenever every subsequence of ξ has a subsequence which converges to x [11]. In [3], this property is also called *sequentially Choquet.*

Theorem 43. [15] *The convergence* $\mathrm{Seq}\xi$ *is a pseudotopology if and only if*

$$\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi.$$

In particular, the objects of $\mathbf{Seq} \cap \mathbf{S}$ *are the convergences of sequences that are the convergence of sequences for a pseudotopology.*

Theorem 44. [15] *A convergence* ξ *is Urysohn if and only if*

$$\xi \leq \mathrm{Seq}P_\omega\mathrm{Seq}\xi.$$

[16]This notion is not to be confused with that of a $T_{2\frac{1}{2}}$ topological space, sometimes also called *Urysohn space.* A topological space is $T_{2\frac{1}{2}}$ if two disjoint points always have open neighborhood with disjoint closures (e.g., [18]).

In particular, a sequentially based convergence τ *is Urysohn if and only if*

$$\tau \leq \mathrm{Seq} P_\omega \tau$$

if and only if it is the convergence of sequences for a paratopology.

While the characterization (21) of countably Choquet spaces immediately implies the observation [3] that Choquet (i.e., pseudotopological) spaces are countably Choquet, the characterization of the Urysohn property obtained above shows that paratopologies are Urysohn.

The following example improves [11, Example 5.6] which gives a sequentially based pseudotopological space that is not Urysohn. Indeed, the convergence constructed below is also Hausdorff, which was not the case of the cited example.

Example 45. *[A Hausdorff non-Urysohn sequentially based pseudotopology] Let* X *be a countably infinite set and let* ∞ *be an element of* X. *Consider a free ultrafilter* $\mathcal{W}$ *and define a convergence* ξ *on* X *in which* ∞ *is the only non-isolated point by* $\lim_\xi \mathcal{F} = \{\infty\}$ *if* $\mathcal{F}$ *is either the principal ultrafilter of* ∞ *or is free and does not mesh* $\mathcal{W}$. *The convergence* ξ *is not Urysohn, because each sequence on* X *contains a subsequence, the range of which does not belong to* $\mathcal{W}$, *hence converging to* ∞, *but the sequence that generates the cofinite filter of* X *does not converge, because it meshes* $\mathcal{W}$. *The convergence* ξ *is a pseudotopology, so that* $\mathrm{Seq}\xi$ *is also a pseudotopology in view of Corollary 43, because if* $\infty \in \lim_\xi \mathcal{U}$ *for each ultrafilter* $\mathcal{U} \geq \mathcal{F}$, *then for each such* $\mathcal{U}$ *there is* $U_\mathcal{U} \in \mathcal{U} \backslash \mathcal{W}^\#$, *hence by the compactness of* $\beta\omega$ *there exist* $n < \omega$ *and* $\mathcal{U}_1, \ldots, \mathcal{U}_n$ *finer than* $\mathcal{F}$ *such that* $U_{\mathcal{U}_1} \cup \ldots \cup U_{\mathcal{U}_1} \in \mathcal{F} \backslash \mathcal{W}^\#$ *proving that* $\infty \in \lim_\xi \mathcal{F}$.

Proposition 46. [11] *A sequentially based convergence* τ *is the convergence of sequences for a topology if and only if*

$$\tau \leq \mathrm{Seq} T \tau.$$

Proof. If $\tau \leq \mathrm{Seq}T\tau$, then $\mathrm{Seq}\tau = \mathrm{Seq}T\tau$ is the convergence of sequences for a topology. Conversely, if τ is the convergence of sequences for a topology $\sigma = T\sigma$, then $\tau = \mathrm{Seq}\sigma$.Applying the idempotent functor $\mathrm{Seq}T$ to each side of the equality, we get $\mathrm{Seq}T\tau = \mathrm{Seq}T\mathrm{Seq}T\sigma = \mathrm{Seq}\sigma$. Hence $\tau = \mathrm{Seq}T\tau$. □

Because $T \leq P_\omega$, it is obvious that such convergences must be Urysohn. Among Hausdorff convergences (but not in general), the converse is true (e.g., [11]). Hence, in view of Theorem 44, if ξ is a Hausdorff paratopology (but not if we only know that it is a Hausdorff pseudotopology) then the convergent sequences for ξ, $P\xi$ and $T\xi$ are the same.

10. Relatively quotient maps

We turn to a new notion of quotient relative to a subcategory. If **J** is a subcategory of **A**, an **X**-morphism $f : |A| \to |B|$ is called **J**-*final* if $g : |B| \to |C|$ with codomain $C \in Ob(\mathbf{J})$ is an **A**-morphism provided that $g \circ f$ is. More generally, a sink $(f_i : |A_i| \to |B|)_{i \in I}$ is called **J**-*final* if $g : |B| \to |C|$ with codomain $C \in Ob(\mathbf{J})$ is an **A**-morphism provided that $g \circ f_i$ is an **A**-morphism for each i in I. A **J**-final map need not be an **A**-morphism ([17]). Thus we shall distinguish between **J**-final maps and **J**-quotient maps, that is, **J**-final **A**-morphisms.

[17]For instance, pick a (concrete) functor F of $\mathbf{A}$ and an object A such that $A < FA$. Then the identity carried map $i : |A| \to |FA|$ is an upper F-map because $\overrightarrow{i} A = A$ but it is not a morphism.

Notice that f is quotient in $\mathbf{J}$ if and only if it is $\mathbf{J}$-quotient and has domain and codomain in $Ob(\mathbf{J})$.

This fact allows for instance to extend the notion of topological quotient to the category **Conv** of convergence spaces and continuous maps. Indeed, a quotient map between topological spaces is a quotient morphism in the category $\mathbf{T}$ of topological spaces and continuous maps. A $\mathbf{T}$-quotient need not have topological domain and codomain.

Analogously, it is known from [21] by D.C Kent that a map between topological spaces is biquotient if and only if it is quotient in the category $\mathbf{S}$ of pseudotopological spaces and that it is hereditarily quotient if and only if it is quotient in the category $\mathbf{P}$ of pretopological spaces. Hence, $\mathbf{S}$-quotient and $\mathbf{P}$-quotient extend to **Conv** the classical topological notions of biquotient and hereditarily quotient maps. Moreover **Conv**-quotients and $\mathbf{P}_\omega$-quotient extend to **Conv** the notions of almost open and countably biquotient maps [7].

The notion of final map with respect to a reflector admits an objectwise reduction too.

Proposition 47. *Let* $\mathbf{J}$ *be a concretely reflective subcategory of a topological category* $(\mathbf{A}, |\cdot|)$*. The following are equivalent:*

(1) $(f_i : A_i \to B)_{i\in I}$ *is* $\mathbf{J}$*-final in* $\mathbf{A}$*;*
(2) $B \geq J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$*;*
(3) $(Jf_i : JA_i \to JB)_{i\in I}$ *is final in* $\mathbf{J}$*.*

Proof. $1 \Longrightarrow 2$: Assume that $(f_i : A_i \to B)_{i\in I}$ is $\mathbf{J}$-final. Let g denote the identity-carried map from B to $J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$. The composites $g \circ f_i$ are morphisms because $\bigwedge_{i\in I} \overrightarrow{f_i} A_i \geq J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$. Thus g is a morphism so that $B \geq J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$.

$2 \Longrightarrow 3$: Assume that $g : JB \to C = JC$ is such that $g \circ (Jf_i) : JA \to C$ is a morphism for every i. In view of 2, $JB \geq J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$. Thus $\overrightarrow{g} JB \geq \overrightarrow{g} J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$. By Theorem 10, $\overrightarrow{g} JB \geq J\overrightarrow{g}(\bigwedge_{i\in I} \overrightarrow{f_i} A_i)$, so that $\overrightarrow{g} JB \geq J(\bigwedge_{i\in I} \overrightarrow{g \circ f_i} A_i)$ by (11). Since each $g \circ (Jf_i) : JA_i \to C$ is a morphism, $\overrightarrow{g} JB \geq C$ and g is a morphism.

$3 \Longrightarrow 1$: Assume that $g : B \to C = JC$ is such that all $g \circ f_i : A_i \to C$ are morphisms. In other words, $\bigwedge_{i\in I} \overrightarrow{g \circ f_i}(A_i) \geq C$. Since $\bigwedge_{i\in I} \overrightarrow{g \circ f_i}(JA_i) \geq \bigwedge_{i\in I} J\overrightarrow{g \circ f_i} A_i$ because of Theorem 10 and $\bigwedge_{i\in I} J\overrightarrow{g \circ f_i} A_i \geq J(\bigwedge_{i\in I} \overrightarrow{g \circ f_i} A_i)$, we conclude that $\bigwedge_{i\in I} \overrightarrow{g \circ f_i}(JA_i) \geq C$. Consequently, $\overrightarrow{g}(JB) \geq C$ because $(Jf_i : JA_i \to JB)_{i\in I}$ is final in $\mathbf{J}$. Hence, $\overrightarrow{g} B \geq C$ so that g is a morphism. $\square$

In particular,

Corollary 48. *If* $\mathbf{J}$ *is a reflective subcategory of* $\mathbf{A}$*, then* $f : X \to Y$ *is* $\mathbf{J}$*-final if and only if*

$$Y \geq J(\overrightarrow{f} X). \tag{22}$$

The inequality (22) is meaningful for any (concrete endo-) functor J of $\mathbf{A}$. An $\mathbf{X}$-morphism $f : |X| \to |Y|$ is an *upper J-map* provided that (22) holds. Notice that an upper J-map need not be an **A**-morphism. Thus, we shall distinguish between *upper J-maps* and *upper J-morphisms*. A map is an *upper J-morphism* if it is an upper J-map and an **A**-morphism.

Analogously, a **X**-sink $(f_i : |A_i| \to |B|)_{i\in I}$ is an *upper J-sink* provided that

$$B \geq J(\bigwedge_{i\in I} \overrightarrow{f_i} A_i).$$

Hence, Proposition 47 states that for a (concretely) reflective subcategory **J** of **A**, upper J-sinks and **J**-final sinks coincide. This coincidence turns out to be very useful because of the following preservation property of upper J-maps.

Theorem 49. *Let F and M be two functors of a topological category* **A** *such that $MF \geq F$. Then each upper M-map with $\mathbf{F}^{\leq I}$-domain is an upper F-map. In particular, each upper M-morphism maps $\mathbf{F}^{\leq I}$-objects on $\mathbf{F}^{\leq I}$-objects.*

Proof. Let $f : X \to Y$ with $X \geq FX$ and $Y \geq M(\overrightarrow{f} X)$. Then $\overrightarrow{f} X \geq \overrightarrow{f}(FX)$. As F is a functor, $\overrightarrow{f} X \geq F\overrightarrow{f} X$, by Theorem 10. Applying the functor M, we obtain

$$M(\overrightarrow{f} X) \geq MF(\overrightarrow{f} X) \geq F(\overrightarrow{f} X).$$

As $Y \geq M(\overrightarrow{f} X)$, we conclude that $Y \geq F(\overrightarrow{f} X)$.

If moreover f is a morphism then $\overrightarrow{f} X \geq Y$. Consequently, $Y \geq F(\overrightarrow{f} X) \geq FY$. □

Corollary 50. *Let* **J** *be a reflective and* **E** *a coreflective subcategory of a topological category* **A**. *Each* **J***-quotient image of a $\mathbf{JE}^{\leq I}$-object is also a $\mathbf{JE}^{\leq I}$-object.*

Proof. Take $M = J$ and $F = JE$ in Theorem 49. By Proposition 47, upper J-morphisms and **J**-quotients coincide. □

Recall (from Section 8) that many topological notions can be characterized as $\mathbf{JE}^{\leq I}$-objects in **Conv** for various instances of **J** and **E**. Hence, Corollary 50 takes all its sense in view of Table 1: each property of a given row is preserved by the class of maps corresponding to this row (*a fortiori* to a higher row). Hence the **Conv** counterpart [7, Theorem 4.2] of Corollary 50 recovers many classical preservation theorems.

Proposition 51. *Let* **J** *be a reflective and* **E** *a coreflective subcategory of a topological category* **A**. *Each upper JEJ-morphism is* **J***-quotient with a $\mathbf{JEJ}^{\leq I}$-codomain. In particular, an upper JEJ-morphism with codomain in* **J** *is* **J***-quotient with a $\mathbf{JE}^{\leq I}$-codomain. Conversely, a* **J***-quotient map with a $\mathbf{JEJ}^{\leq I}$-codomain is an upper JEJ-morphism.*

Proof. Let $f : X \to Y$ be an upper JEJ-morphism. Then $\overrightarrow{f} X \geq Y \geq JEJ\left(\overrightarrow{f} X\right)$. Therefore, $Y \geq JEJY$ and, since $E \geq I$ and J is idempotent $Y \geq J\left(\overrightarrow{f} X\right)$. Hence f is **J**-quotient with a $\mathbf{JEJ}^{\leq I}$-codomain. Conversely, if $f : X \to Y$ is such that $Y \geq JEJY$ and $\overrightarrow{f} X \geq Y \geq J\left(\overrightarrow{f} X\right)$, then $Y \geq JEJ\left(\overrightarrow{f} X\right)$. □

Corollary 52. *Let* **J** *be a reflective and* **E** *a coreflective subcategory of a topological category* **A**. *Let $f : X \to Y$ where Y is a* **J***-object. Then f is an upper JEJ-morphism if and only if f is* **J***-quotient and Y is a $\mathbf{JE}^{\leq I}$-object.*

For instance, in **Conv**, a map f with a pseudotopological range is biquotient with bisequential range if and only if it is an upper $S\mathrm{First}S$-morphism; biquotient with a bi-k range if and only if it is an upper SQ_KS-morphism, etc. If the range is paratopological, then f is countably biquotient with strongly Fréchet range if and only if it is an upper $P_\omega\mathrm{First}P_\omega$-morphism; countably biquotient with a strongly k' range if and only if it is an upper $P_\omega KP_\omega$-morphism, etc. If the range is topological, then f is quotient with sequential range if and only if it is an upper $T\mathrm{First}T$-morphism; quotient with k-range if and only if it is an upper TKT-morphism; quotient with quasi-k range if and only if it is an upper TQT-morphism, etc.

If **J** is a given subcategory of **A**, then a question arises for what functors F of **A** do **J**-quotient maps and upper F-morphisms coincide.

Proposition 53. *Let* **J** *be a subcategory of a topological category* **A** *and let* F *be a (concrete endo-) functor of* **A**. *The following are equivalent.*

(1) *Each upper* F*-map is* **J***-final;*
(2) $\mathbf{J} \subset \mathbf{F}^{\geq I}$.

Proof. Let X be a **J**-object. The identity carried map $i_{X,FX} : X \to FX$ is an upper F-map. By assumption, it is also a **J**-final map. Hence, the identity carried map $i_{FX,X}$ is a morphism because the composition $i_X = i_{FX,X} \circ i_{X,FX}$ is a morphism. Thus, $FX \geq X$ and $\mathbf{J} \subset \mathbf{F}^{\geq I}$.

Conversely, assume that $\mathbf{J} \subset \mathbf{F}^{\geq I}$ and let $f : X \to Y$ be an upper F-map. Consider a **J**-object Z and a map $g : Y \to Z$ such that $g \circ f : X \to Z$ is a morphism. In other words, $\left(\overrightarrow{g \circ f}\right) X \geq Z$. By isotony of F, $F(\overrightarrow{g \circ f}X) \geq FZ$. As F is a functor, $\overrightarrow{g}(F\overrightarrow{f}X) \geq F(\overrightarrow{g \circ f}X) \geq FZ$. By assumption $Y \geq F\overrightarrow{f}X$ and $FZ \geq Z$ so that $\overrightarrow{g}Y \geq Z$. Thus g is a morphism and f is **J**-final. □

Corollary 54. *Each upper* F*-map is* $\mathbf{F}^{\geq I}$*-final.*

As observed in Theorem 49, if $MF \geq F$, then each upper M-map with domain in $\mathbf{F}^{\geq I}$ is an upper F-map, hence a $\mathbf{F}^{\geq I}$-final map. In case M is a reflector J and $F = JE$ where E is a coreflector, we get

Corollary 55. *Let* **J** *be a reflective subcategory and let* **E** *be a coreflective subcategory of a topological category* **A**. *Assume that* X *is a* $\mathbf{JE}^{\leq I}$*-object. Then* $f : X \to Y$ *is* **J***-quotient if and only if for every* $\mathbf{JE}^{\geq I}$*-object (equivalently every* **J***-object)* Z*, a map* $g : Y \to Z$ *is a morphism provided that* $g \circ f$ *is a morphism.*

If f is an upper J-map with the domain in $\mathbf{JE}^{\leq I}$ then it is a $\mathbf{JE}^{\geq I}$-final map. Once again, the duality between properties which are characteristic of $\mathbf{F}^{\leq I}$ and $\mathbf{F}^{\geq I}$-objects appears in the characterization above of **J**-quotient maps: If a **J**-quotient map has a domain in $\mathbf{JE}^{\leq I}$, the universal property defining **J**-quotient is not only true for maps g with the codomain in $Ob(\mathbf{J})$ but more generally for maps with the codomain in $\mathbf{JE}^{\geq I}$.

Proposition 56. *Let* $F : \mathbf{A} \to \mathbf{J}$ *be a functor of* **A** *valued in* **J**. *The following are equivalent:*

(1) *Upper* F*-maps and* **J***-final maps coincide;*
(2) *Upper* F*-morphisms and* **J***-quotients coincide;*
(3) F *is contractive and* $\mathbf{J} \subset \mathbf{F}^{\geq I}$.

Proof. $1 \Longrightarrow 2$ follow from the definitions.

$2 \Longrightarrow 3$. Each identity map i_X is **J**-quotient. Thus it is an upper F-morphism, so that $X \geq FX$ and F is contractive. Hence, in the proof of Proposition 53, $i_{Z,FZ}$ is not only an upper F-map but an upper F-morphism. Its **J**-quotientness leads to $\mathbf{J} \subset \mathbf{F}^{\geq I}$.

$3 \Longrightarrow 1$. By Proposition 53, each upper F-map is **J**-final. Consider a **J**-final map $f : X \to Y$. Notice that $F\left(\overrightarrow{f}X\right)$ is a **J**-object because $F : \mathbf{A} \to \mathbf{J}$. The identity carried map $i_{Y,F\overrightarrow{f}X}$ is a morphism because the composite $i_{Y,F\left(\overrightarrow{f}X\right)} \circ f : X \to F\left(\overrightarrow{f}X\right)$ is a morphism. Indeed, $\overrightarrow{f}X \geq F\left(\overrightarrow{f}X\right)$ because F is contractive. Thus $Y \geq F\left(\overrightarrow{f}X\right)$. □

We gather without proofs the dual notions and results.

If **E** is a subcategory of **A**, a source $(f_i : |A| \to |B_i|)_{i \in I}$ is called **E**-*initial* if a map $g : |C| \to |A|$ with domain in **E** is an **A**-morphism provided that $f_i \circ g$ is an **A**-morphism for each i in I. An **E**-initial map need not be an **A**-morphism.

Notice that f is initial in **E** if and only if it is an **E**-initial morphism with domain and codomain in **E**.

Proposition 57. *Let* **E** *be a coreflective subcategory of a topological category* **A**. *The following are equivalent:*

1. $(f_i : A \to B_i)_{i \in I}$ *is* **E**-*initial;*
2. $E(\bigvee_{i \in I} \overleftarrow{f_i} B_i) \geq A$;
3. $(Ef_i : EA \to EB_i)_{i \in I}$ *is initial in* **E**.

In particular, if **E** is a coreflective subcategory of **A**, $f : X \to Y$ is **E**-initial if and only if

$$E(\overleftarrow{f}Y) \geq X.$$

Let F denote a functor of **A**. An **X**-morphism $f : |X| \to |Y|$ is a *lower* F-*map* provided that

$$F(\overleftarrow{f}Y) \geq X.$$

Notice that a lower F-map need not be an **A**-morphism. Thus, we shall distinguish between *lower* F-*maps* and *lower* F-*morphisms.* Analogously, a **X**-source $(f_i : |A| \to |B_i|)_I$ is a *lower* F-*sink* provided that

$$E(\bigvee_I \overleftarrow{f_i} B_i) \geq A.$$

Theorem 58. *Let F and M be two functors of a topological category* **A** *such that $F \geq MF$. Then each lower M-map with the codomain in* $\mathbf{F}^{\geq I}$ *is a lower F-map. In particular, each lower M-morphism with the codomain in* $\mathbf{F}^{\geq I}$ *has a domain in* $\mathbf{F}^{\geq I}$.

Corollary 59. *Let* **J** *be a reflective and* **E** *a coreflective subcategory of a topological category* **A**. *Each* **E**-*initial morphism inversely preserve* $\mathbf{EJ}^{\geq I}$-*objects.*

Proposition 60. *Let* **J** *be a subcategory of a topological category* **A** *and let F be a functor of* **A**. *The following are equivalent.*

1. *each lower F-map is* **E**-*initial;*
2. $\mathbf{E} \subset \mathbf{F}^{\leq I}$.

Corollary 61. *Let* $\mathbf{J}$ *be a reflective subcategory and let* $\mathbf{E}$ *be a coreflective subcategory of a topological category* $\mathbf{A}$*. Assume that* X *is a* $\mathbf{EJ}^{\geq I}$*-object. Then* $f : X \to Y$ *is an* $\mathbf{E}$*-initial morphism if and only if for every* $\mathbf{EJ}^{\leq I}$*-object (equivalently every* $\mathbf{E}$*-object)* Z*, a map* $g : Z \to X$ *is a morphism provided that* $f \circ g$ *is a morphism.*

Proposition 62. *Let* $F : \mathbf{A} \to \mathbf{E}$ *be a concrete functor. The following are equivalent:*

(1) *Lower* F*-maps and* $\mathbf{E}$*-initial maps coincide;*
(2) *Lower* F*-morphisms and* $\mathbf{J}$*-initial morphism coincide;*
(3) F *is expansive and* $\mathbf{J} \subset \mathbf{F}^{\leq I}$*.*

11. Covering maps

Recall that a continuous map $f : X \to Y$ between topological spaces is *sequence-covering* if for every sequence $(y_n)_{n\in\omega}$ convergent to y in Y, there exists a sequence $(x_n)_{n\in\omega}$ in X convergent to x such that $f(x) = y$ and $f(x_n) = y_n$. We gather [36, Theorems 4.1, 4.2, 4.4] of F. Siwiec in the following

Theorem 63. *A topological space* X *is sequential (resp. Fréchet, strongly Fréchet) if and only if every sequence-covering map onto* X *is quotient (resp. hereditarily quotient, countably biquotient).*

Before showing that these three theorems not only extend to arbitrary convergence spaces, but are special cases of a single abstract result, let us consider another group of analogous results.

A continuous map $f : X \to Y$ is *compact-covering* if for every Y-compact set K, there exists a X-compact set C such that $f(C) = K$. Analogously to Theorem 63, the following theorem gathers results of F. Siwiec and V. J. Mancuso [37], A. V. Arhangel'skii [2], E. Michael [26, Lemma 11.2] and F. Siwiec [36] follows from the same abstract result as Theorem 63.

Theorem 64. *A topological space* X *is a* k*-topology (resp.* k'*, strongly* k'*, locally compact) if and only if each compact-covering map onto* X *is quotient (resp. hereditarily quotient, countably biquotient, biquotient).*

We shall see that Theorems 63 and 64 follow both from a single abstract result (Theorem 65).

It was observed [17] that $f : (X, \xi) \to (Y, \tau)$ is sequence-covering if and only if

$$\mathrm{Seq}\tau \geq \overrightarrow{f}(\mathrm{Seq}\xi).$$

This inequality is meaningful for arbitrary convergence spaces (X, ξ) and (Y, τ) and will define a *sequence-covering map* in **Conv**. Analogously [17, Proposition 5.3],

$$K\tau \geq \overrightarrow{f}(K\xi) \Longrightarrow f : X \to Y \text{ is compact-covering } \Longrightarrow K\tau \geq S\overrightarrow{f}(K\xi).$$

Let J be a reflector and let E be a coreflector of $\mathbf{A}$. A map $f : X \to Y$ is called [17] *E-relatively J-map* if Ef is an upper J-map, that is, if

$$EY \geq J\overrightarrow{f}(EX),$$

and *weakly E-relatively J-map* if $f : X \to EY$ is an upper J-map, that is, if

$$EY \geq J\left(\overrightarrow{f}X\right).$$

The following extends [17, Theorem 5.4] from the category **Conv** to an arbitrary topological category.

Theorem 65. *Let* $\mathbf{E}$ *be a coreflective subcategory and let* $\mathbf{J} \subset \mathbf{D}$ *be two reflective subcategories of a topological category* $\mathbf{A}$*. The following are equivalent:*

(1) X *is a* $\mathbf{JE}^{\leq I}$*-object;*
(2) *every* E*-relatively* D*-map onto* X *is an upper* J*-map;*
(3) *every weakly* E*-relatively* D*-map onto* X *is an upper* J*-map.*

Proof. $1 \Longrightarrow 3$. Assume that $f : Z \to X$ is a weakly E-relatively D-map , that is, $EX \geq D\left(\overrightarrow{f} Z\right)$. Hence $JEX \geq JD\left(\overrightarrow{f} Z\right)$. By assumption, $X \geq JEX$ so that $X \geq JD\left(\overrightarrow{f} Z\right)$. Hence, $X \geq J\left(\overrightarrow{f} Z\right)$ because $D \geq J$.

$3 \Longrightarrow 2$ because every E-relatively J-map is a weakly E-relatively J-map.

$2 \Longrightarrow 1$. If $X \ngeq JEX$ then i: $EX \to X$ is a E-relatively D-map which is not a J-map. □

Equivalence between the two first points above recover Theorem 63 when $D = I$, $E = \mathrm{Seq}$ and J runs over T, P, P_ω. If D is either I or S, $E = K$ and J runs over T, P, P_ω, S then Theorem 64 is recovered. Moreover, both theorems are generalized because the class of weakly E-relatively D-maps is essentially broader than that of E-relatively D-maps [17, Example 5.6].

12. Modified continuity

It is well known (e.g., [18]) that a sequentially continuous map between two topological spaces is continuous provided that the domain is a sequential space. We shall see that this classical fact not only extends to **Conv** but is an instance of a general but simple scheme.

Let F be a (concrete endo-) functor of a topological category $\mathbf{A}$. If an $\mathbf{X}$-morphism $f : |A| \to |B|$ is such that $f : FA \to FB$ is an $\mathbf{A}$-morphism, then f is called an *F-morphism*. In this section, we are interested in conditions ensuring that an F-morphism is a morphism (in $\mathbf{A}$). Of course, sequential continuity corresponds to $F = \mathrm{Seq}$.

Proposition 66. *Let* F *be a (concrete endo-) functor of a topological category* $\mathbf{A}$*. If* A *is an* $\mathbf{F}^{\leq I}$*-object and* B *is an* $\mathbf{F}^{\geq I}$*-object, then* $f : A \to B$ *is a morphism whenever it is an* F*-morphism.*

Proof. As f is an F-morphism, $FB \leq \overrightarrow{f}(FA)$. Under our assumptions

$$B \leq FB \leq \overrightarrow{f}(FA) \leq \overrightarrow{f} A,$$

so that $f : A \to B$ is a morphism. □

Each $\mathbf{J}$-object is a $\mathbf{JE}^{\geq I}$-object, and each E-morphism is also a JE-morphism. Hence Proposition 66 particularizes to the following if $F = JE$, where J is a reflector and E a coreflector.

Corollary 67. *Let* $\mathbf{E}$ *and* $\mathbf{J}$ *denote respectively a coreflective and a reflective subcategory of a topological category* $\mathbf{A}$*. Each* E*-morphism with the domain in* $\mathbf{JE}^{\leq I}$ *and with the codomain in* $\mathbf{JE}^{\geq I}$ *(in particular with the codomain in* $\mathbf{J}$*) is a morphism.*

When $E = \mathrm{Seq}$ and J runs over T, P, P_ω, S, we obtain

Corollary 68. (1) *A sequentially continuous map from a sequential convergence to a* $\mathbf{TSeq}^{\geq I}$*-convergence (in particular a topology) is continuous;*
(2) *A sequentially continuous map from a Fréchet convergence to a* $\mathbf{PSeq}^{\geq I}$*-convergence (in particular a pretopology) is continuous;*
(3) *A sequentially continuous map from a strongly Fréchet convergence to a* $\mathbf{P}_\omega\mathbf{Seq}^{\geq I}$*-convergence (in particular a paratopology) is continuous;*
(4) *A sequentially continuous map from a sequentially based pseudotopology to a* $\mathbf{SSeq}^{\geq I}$*-convergence (in particular a pseudotopology) is continuous.*

A convergence space is *sequentially determined* if a countably based filter converges to x whenever each finer sequence does.

One of the main motivations for the introduction of *sequentially determined convergence spaces* by R. Beattie and H.P. Butzmann in [4] is that in general sequential continuity of a map between two convergence spaces does not imply continuity, even if these convergence spaces are first-countable. However

Theorem 69. [4, Theorem 2.10] *If* (X,ξ) *is first-countable and* (Y,τ) *is sequentially determined, then* $f:(X,\xi)\to(Y,\tau)$ *is continuous if and only if it is sequentially continuous.*

Among the large classes of convergence spaces shown to be sequentially determined are all first-countable pretopological spaces, second-countable convergence spaces and web-spaces (see [4] and [3]). It is interesting to note that every first-countable convergence is Fréchet and every first-countable pretopological space (even every pretopological space!) is a $\mathbf{PSeq}^{\geq I}$-convergence. Hence Corollary 68 (2) gives a useful alternative to [4, Theorem 2.10]. However,

Proposition 70. *Every* $\mathbf{PSeq}^{\geq I}$*-convergence is sequentially determined.*

Proof. If $\mathcal{F}$ is countably based, then $\mathcal{F}=\bigwedge_{(x_n)_n\geq\mathcal{F}}(x_n)_n$. If each $(x_n)_n$ finer than $\mathcal{F}$ converges to x for ξ, then $\mathcal{F}=\bigwedge_{(x_n)_n\geq\mathcal{F}}(x_n)_n$ converges for $P\mathrm{Seq}\xi$, hence for ξ because $P\mathrm{Seq}\xi\geq\xi$. □

This fact justifies a closer look, taken in [15], at sequentially determined convergences as well as the range of [4, Theorem 2.10]. Let us just mention that sequentially determined convergences can be characterized as convergences ξ satisfying

$$\xi\leq \mathrm{First}U_{\mathbb{E}}\xi$$

where $U_{\mathbb{E}}\xi$ is a projector, but not a reflector. It becomes a functor in restriction to convergences based in a certain class of filters, whose study is part of [15].

13. Exponential objects

The *continuous convergence* $[\xi,\sigma]$ on the set of continuous maps from a convergence space (X,ξ) to another (Y,σ) is the coarsest convergence making the evaluation map $ev:\xi\times[\xi,\sigma]\to\sigma$ jointly continuous ([18]). Therefore the category **Conv** is cartesian-closed. In particular,

$$[\xi\times\tau,\sigma]=[\tau,[\xi,\sigma]] \tag{23}$$

[18]Explicitely, $f\in\lim_{[\xi,\sigma]}\mathcal{F}$ iff for every $x\in|\xi|$ and every filter $\mathcal{G}$ converging to x for ξ, we have $f(x)\in\lim_\sigma ev(\mathcal{G}\times\mathcal{F})$.

for every convergence spaces ξ, τ, σ, where the equality means that the *exponential map* $\exp : [\xi \times \tau, \sigma] \to [\tau, [\xi, \sigma]]$ defined by $\exp f(y)(x) = f(x, y)$ is a bijection (actually a homeomorphism). A reformulation of the definition of continuous convergence that is more in line with our approach is that $[\xi, \sigma]$ is the coarsest of the convergences τ on $|[\xi, \sigma]|$ satisfying

$$\xi \times \tau \geq \overleftarrow{ev}\,(\sigma)\,. \tag{24}$$

This formalism leads to a simple and elegant proof of the fact, often proved in a more technical way (e.g., [20], [3, Theorem 1.5.5]), that the continuous convergence is pseudotopological whenever σ is. Indeed, it is easy to see that the pseudotopologizer S commutes with arbitrary products ([19]), so that, in particular

$$\xi \times S\tau \geq S(\xi \times \tau) \tag{25}$$

for every pair of convergences ξ, τ. Now, by definition, $\xi \times [\xi, \sigma] \geq \overleftarrow{ev}\,(\sigma)$. Applying the reflector S to this inequality yields

$$\xi \times S[\xi, \sigma] \geq S\,(\xi \times [\xi, \sigma]) \geq S\,(\overleftarrow{ev}\sigma)\,.$$

Since S is a functor, $S\,(\overleftarrow{ev}\sigma) \geq \overleftarrow{ev}\,(S\sigma)\,,$ so that, under the assumption that $\sigma = S\sigma$, we obtain

$$\xi \times S[\xi, \sigma] \geq \overleftarrow{ev}\,(\sigma)\,.$$

But the continuous convergence $[\xi, \sigma]$ is the coarsest convergence that satisfies (24). Hence $[\xi, \sigma] \leq S[\xi, \sigma]$ and $[\xi, \sigma]$ is pseudotopological. To our knowledge, this type of argument was first used in [22], and rediscovered in [**?**]. The same simple proof gives (1) $\Longrightarrow$ (2) in the following particular case of [29, Theorem 3.1], and (2) $\Longrightarrow$ (1) is a simple application of the exponential law (23).

Proposition 71. *Let L be a reflector of* **Conv**. *The following are equivalent:*

(1) *ξ satisfies*
$$\xi \times L\tau \geq L(\xi \times \tau)$$
for every convergence τ;

(2) *$L[\xi, \sigma] \geq [\xi, \sigma]$ for every $\sigma = L\sigma$.*

Proof. (2) $\Longrightarrow$ (1). Because L is a reflector, to show that $\xi \times L\tau \geq L(\xi \times \tau)$ for every τ, it is sufficient to show that every continuous map $f : \xi \times \tau \to \sigma = L\sigma$ is also continuous from $\xi \times L\tau$ to σ. If $f : \xi \times \tau \to \sigma$ is continuous, then $\exp f : \tau \to [\xi, \sigma]$ is also continuous, so that $\exp f : L\tau \to L[\xi, \sigma]$ is as well. But $L[\xi, \sigma] = [\xi, \sigma]$ so that $\exp f : L\tau \to [\xi, \sigma]$ is continuous. In other words, $f : \xi \times L\tau \to \sigma$ is continuous. □

In particular, as already observed by F. Schwarz [35], if **L** is a reflective and finally dense ([20]) subcategory of **Conv**, then $\xi = L\xi$ is exponential in **L** if and only if

$$\xi \times L\tau \geq L(\xi \times \tau) \tag{26}$$

for every convergence space τ. This approach to characterizing exponential objects in various subcategories of **Conv** proved particularly fruitful [13], [29], [34]. In

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[19] The argument is essentially that of the ultrafilter proof of the Tychonoff theorem: Assume $(x_i)_{i \in I} \in \lim_{\prod_{i \in I} S\xi_i} \mathcal{F}$ and let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{F}$. Then for each $i \in I$, the filter $p_i\mathcal{U}$ is an ultrafilter finer than $p_i\mathcal{F}$, so that $x_i \in \lim_{\xi_i} p_i\mathcal{U}$. In other words, $(x_i)_{i \in I} \in \lim_{\prod_{i \in I} \xi_i} \mathcal{U}$ so that $(x_i)_{i \in I} \in \lim_{S\left(\prod_{i \in I} \xi_i\right)} \mathcal{F}$.

particular, functorial inequalities such as (26) can be proved for a general functor $L = \mathrm{Adh}_{\mathbb{J}}$, providing unified characterizations of exponential objects in the categories $\mathbf{T}$, $\mathbf{P}$ and $\mathbf{P}_\omega$ [34].

Also, the well-known connection between exponentiality and quotientness of product maps is particularly easy to interpret in this formalism: if ξ satisfies (26) for every τ and $f : \tau \to \sigma$ is an upper L-map, then

$$\xi \times \sigma \geq \xi \times L\left(\overrightarrow{f}\tau\right) \geq L\left(\xi \times \overrightarrow{f}\tau\right) = L\left(\overrightarrow{i_\xi \times f}\,(\xi \times \tau)\right)$$

so that $i_\xi \times f$ is also an upper L-map. Conversely, if $i_\xi \times f$ is also an upper L-map for every upper L-map, then in particular the product of i_ξ with each identity carried upper L-map $i_{\tau,L\tau} : \tau \to L\tau$ is an upper L-map, which rephrases as (26).

As already observed in [32], the considerations above extend to more general categories. More specifically, consider the case of a *topological construct* $\mathbf{A}$ *that has function spaces* in the sense of [1] ([21]). Given two $\mathbf{A}$-objects A and B, we denote by $[A, B]$ the canonical $\mathbf{A}$-object on $\mathrm{Hom}_{\mathbf{A}}(A, B)$. Then $[A, B]$ is the coarsest $\mathbf{A}$-object on $\mathrm{Hom}_{\mathbf{A}}(A, B)$ making the evaluation $ev : A \times [A, B] \to B$ a morphism. In other words, $[A, B]$ is the coarsest of the $\mathbf{A}$-objects C such that $|C| = \mathrm{Hom}_{\mathbf{A}}(A, B)$ and

$$A \times C \geq \overleftarrow{ev}B. \tag{27}$$

In such a category

$$[A \times B, C] = [B, [A, C]] \tag{28}$$

where the equality stands for isomorphism via the *exponential map* $\exp : [A \times B, C] \to [B, [A, C]]$ defined by $(\exp f)(b)(a) = f(a, b)$. Moreover, each functor $(A \times \bullet)$ preserves final sinks. In particular,

$$\overrightarrow{i_A \times f}(A \times B) = A \times \overrightarrow{f}B. \tag{29}$$

Since the $\mathbf{Set}$-map $g \times f : |A \times B| \to |C \times D|$ factors into $(g \times |i_D|) \circ (|i_A| \times f)$ and in view of (9), we also have

$$\overrightarrow{g \times f}\,(A \times B) = \overrightarrow{g}A \times \overrightarrow{f}B. \tag{30}$$

We provide here generalizations of results from [32] with the hope that the reader will find applications in categories different from $\mathbf{Conv}$.

Theorem 72. *Let F and L be two functors of $\mathbf{A}$. Then*

$$C \times LB \geq F(A \times B)$$

for every $B \geq LB$ if and only if

$$i_{A,C} \times f : A \times B \to C \times D$$

is an upper F-map for every upper L-morphism $f : B \to D$.

Proof. In view of (29), we have $F\left(\overrightarrow{i_A \times f}\,(A \times B)\right) = F(A \times \overrightarrow{f}B)$. Moreover, $\overrightarrow{f}B \geq D \geq L\overrightarrow{f}B$ because f is an upper L-morphism, so that

$$F(A \times \overrightarrow{f}B) \leq C \times L\overrightarrow{f}B \leq C \times D.$$

Hence $i_{A,C} \times f : A \times B \to C \times D$ is an upper F-map.

[21] that is, a topological category over $\mathbf{Set}$, which is additionally concretely cartesian-closed.

Conversely, the identity carried map $i : B \to LB$ is an upper L-morphism whenever $B \geq LB$, so that $i_{A,C} \times i : A \times B \to C \times LB$ is an upper F-map, that is,

$$F(A \times B) \leq C \times LB.$$

□

Theorem 73. *Let L be a functor of* **A** *and F be a reflector of* **A**. *The following are equivalent:*

(1) *A satisfies*

$$A \times LB \geq F(A \times B)$$

for every **A**-*object B;*

(2) *$L[A,R] \geq [A,R]$ for every* **F**-*object R.*

Proof. Assume that $F(A \times B) \leq A \times LB$ for every **A**-object B. In view of (27) , we have

$$\overleftarrow{ev}R \leq A \times [A,R]$$

for every $R \leq FR$. Applying the functor F to this inequality, we obtain

$$F\left(\overleftarrow{ev}R\right) \leq F\left(A \times [A,R]\right).$$

By assumption $F\left(A \times [A,R]\right) \leq A \times L[A,R]$. Moreover, $F\left(\overleftarrow{ev}R\right) \geq \overleftarrow{ev}(FR)$ because F is a functor, and $FR \geq R$, so that

$$\overleftarrow{ev}R \leq A \times L[A,R].$$

By definition $[A,R]$ is the coarsest **A**-object satisfying this property. Hence $[A,R] \leq L[A,R]$.

Conversely, consider the identity carried morphism $i : A \times B \to F(A \times B)$. In view of (28) , the map $\exp i : B \to [A, F(A \times B)]$ is a morphism. By functoriality of L, so is $\exp i : LB \to L[A, F(A \times B)]$. By assumption, $L[A, F(A \times B)] \geq [A, F(A \times B)]$ so that $\exp i : LB \to [A, F(A \times B)]$ is a morphism. Since exp is an isomorphism, $i : A \times LB \to F(A \times B)$ is a morphism. Hence $A \times LB \geq F(A \times B)$. □

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