A RADEMACHER TYPE FORMULA FOR OVERPARTITIONS

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Abstract. Rademacher-type convergent series formulas are presented for the number of overpartitions of n and the number of partitions of n where no odd part is repeated.

1. Background and Statement of Results

1.1. **Partitions.** A *partition* of an integer n is a representation of n as a sum of positive integers, where the order of the summands (called *parts*) is considered irrelevant. It is customary to write the parts in nonincreasing order. For example, there are three partitions of the integer 3, namely 3, $2+1$, and $1+1+1$. Let $p(n)$ denote the number of partitions of n, with the convention that $p(0)=1$, and let $f(x)$ denote the generating function of $p(n)$, i.e. let

$$f(x) := \sum_{n=0}^{\infty} p(n) x^n.$$

Euler [12] was the first to systematically study partitions. He showed that

$$f(x) = \prod_{m=1}^{\infty} \frac{1}{1-x^m}. \tag{1.1}$$

Euler also showed that

$$\frac{1}{f(x)} = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2}, \tag{1.2}$$

and since the exponents appearing on the right side of (1.2) are the pentagonal numbers, Eq. (1.2) is often called "Euler's pentagonal number theorem."

Although Euler's results can all be treated from the point of view of formal power series, the series and infinite products above (and indeed all the series and infinite products mentioned in this paper) converge absolutely when $|x|<1$, which is important for analytic study of these series and products.

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Hardy and Ramanujan were the first to study $p(n)$ analytically and produced an incredibly accurate asymptotic formula [27, p. 85, Eq. (1.74)], namely

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{\lfloor \alpha\sqrt{n}\rfloor} \sqrt{k} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i h n/k} \frac{d}{dn}\left(\frac{\exp\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) + O(n^{-1/4}), \tag{1.3}$$

where

$$\omega(h,k) = \exp\left(\pi i \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \lfloor \frac{hr}{k} \rfloor - \frac{1}{2}\right)\right),$$

α is an arbitrary constant, and here and throughout (h,k) is an abbreviation for $\gcd(h,k)$.

Later Rademacher [45] improved upon (1.3) by finding the following convergent series representation for $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} \omega(h,k) e^{-2\pi i n h/k} \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right). \tag{1.4}$$

Rademacher's method was used extensively by many practitioners, including Grosswald [15, 16], Haberzetle [17], Hagis [18, 19, 20, 21, 22, 23, 24, 25, 26], Hua [30], Iseki [31, 32, 33], Lehner [35], Livingood [36], Niven [44], and Subramanyasastri [51] to study various restricted partitions functions.

Recently, Bringmann and Ono [4] have given exact formulas for the coeffcients of all harmonic Maass forms of weight $\leqq \frac{1}{2}$. The generating functions considered herein are weakly holomorphic modular forms of weight $-\frac{1}{2}$, and thus they are harmonic Maass forms of weight $\leqq \frac{1}{2}$. Accordingly, the results of this present paper could be derived from the general theorem in [4]. However, here we opt to derive the results via classical method of Rademacher.

1.2. **Overpartitions.** Overpartitions were introduced by S. Corteel and J. Lovejoy in [9] and have been studied extensively by them and others including Bringmann, Chen, Fu, Goh, Hirschhorn, Hitczenko, Lascoux, Mahlburg, Robbins, Rødseth, Sellers, Yee, and Zho [3, 6, 7, 8, 9, 10, 11, 14, 28, 29, 37, 38, 39, 40, 41, 42, 43, 49, 50].

An *overpartition* of n is a representation of n as a sum of positive integers with summands in nonincreasing order, where the last occurrence of a given summand may or may not be overlined. Thus the eight overpartitions of 3 are 3, $\bar{3}$, $2+1$, $\bar{2}+1$, $2+\bar{1}$, $\bar{2}+\bar{1}$, $1+1+1$, $1+1+\bar{1}$.

Let $\bar{p}(n)$ denote the number of overpartitions of n and let $\bar{f}(x)$ denote the generating function $\sum_{n=0}^{\infty} \bar{p}(n)x^n$ of $\bar{p}(n)$. Elementary techniques are sufficient to show that

$$\bar{f}(x) = \prod_{m=1}^{\infty} \frac{1+x^m}{1-x^m} = \frac{f(x)^2}{f(x^2)}.$$

Note that

$$\frac{1}{\bar{f}(x)} = \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2}$$

via an identity of Gauss [1, p. 23, Eq. (2.2.12)], so that the reciprocal of the generating function for overpartitions is a series wherein a coefficient is nonzero if and only if the exponent of x is a perfect square, just as the reciprocal of the generating function for partitions is a series wherein a coefficient is nonzero if and only if the exponent of x is a pentagonal number.

Hardy and Ramanujan, writing more than 80 years before the coining of the term "overpartition," stated [27, p. 109–110] that the function which we are calling $\bar{p}(n)$ "has no very simple arithmetical interpretation; but the series is none the less, as the direct reciprocal of a simple ϑ-funciton, of particular interest." They went on to state that

(1.5)
$$\bar{p}(n) = \frac{1}{4\pi}\frac{d}{dn}\left(\frac{e^{\pi\sqrt{n}}}{\sqrt{n}}\right) + \frac{\sqrt{3}}{2\pi}\cos\left(\frac{2}{3}n\pi - \frac{1}{6}\pi\right)\frac{d}{dn}\left(e^{\pi\sqrt{n}/3}\right) + \cdots + O(n^{-1/4}).$$

In fact, (1.5) can be improved to the following Rademacher-type convergent series.

Theorem 1.1.
$$\bar{p}(n) = \frac{1}{2\pi}\sum_{\substack{k\geqq 1\\ 2\nmid k}}\sqrt{k}\sum_{\substack{0\leqq h<k\\ (h,k)=1}}\frac{\omega(h,k)^2}{\omega(2h,k)}e^{-2\pi i n h/k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{n}}{k}\right)}{\sqrt{n}}\right).$$

The above formula arguably retains much of the elegance of Rademacher's series (1.4). Theorem 1.1 will be proved in section 2.

1.3. **Partitions where no odd part is repeated.** Let $pod(n)$ denote the number of partitions of n where no odd part appears more than once. Let $g(x)$ denote the generating function of $pod(n)$, so we have
$$g(x) = \sum_{n=0}^{\infty} pod(n)x^n = \prod_{m=1}^{\infty}\frac{1+x^{2j-1}}{1-x^{2j}} = \frac{f(x)f(x^4)}{f(x^2)}.$$

Via another identity of Gauss [1, p. 23, Eq. (2.2.13)], it turns out that
$$\frac{1}{g(x)} = \sum_{n=0}^{\infty}(-x)^{n(n+1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n x^{2n^2-n},$$
so in this case the reciprocal of the generating function under consideration has nonzero coefficients at the exponents which are triangular (or equivalently, hexagonal) numbers.

The analogous Rademacher-type formula for $pod(n)$ is as follows.

Theorem 1.2.
$$pod(n) = \frac{2}{\pi}\sum_{k\geqq 1}\sqrt{k\left(1-(-1)^k+\lfloor\frac{(k,4)}{4}\rfloor\right)}$$
$$\times \sum_{\substack{0\leqq h<k\\ (h,k)=1}}\frac{\omega(h,k)\ \omega\left(\frac{4h}{(k,4)},\frac{k}{(k,4)}\right)}{\omega\left(\frac{2h}{(k,2)},\frac{k}{(k,2)}\right)}e^{-2\pi i n h/k}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{(k,4)(8n-1)}}{4k}\right)}{\sqrt{8n-1}}\right).$$

A proof of Theorem 1.2 will be sketched in section 3.

2. A Proof of Theorem 1.1

The method of proof is based on Rademacher's proof of (1.4) in [46] with the necessary modifications. Additional details of Rademacher's proof of (1.4) are provided in [47], [48, Ch. 14] and [2, Ch. 5].

Of fundamental importance is the path of integration to be used. In [46], Rademacher improved upon his original proof of (1.4) given in [45], by altering his path of integration from a carefully chosen circle to a more complicated path based on Ford circles, which in turn led to considerable simplifications later in the proof. Here, we shall use a slight variation on Rademacher's "Ford circle path," cf. Chan [5] and Kane [34].

2.1. **Farey fractions.** The sequence $\mathcal{F}_N$ of *proper Farey fractions of order* N is the set of all h/k with $(h,k)=1$ and $0 \leqq h/k < 1$, arranged in increasing order. Thus, e.g., $\mathcal{F}_4 = \left\{ \frac{0}{1}, \frac14, \frac13, \frac12, \frac23, \frac34 \right\}$.

For a given N, let h_p, h_s, k_p, and k_s be such that $\frac{h_p}{k_p}$ is the immediate predecessor of $\frac{h}{k}$ and $\frac{h_s}{k_s}$ is the immediate successor of $\frac{h}{k}$ in $\mathcal{F}_N$. It will be convenient to view each $\mathcal{F}_N$ cyclically, i.e. to view $\frac01$ as the immediate successor of $\frac{N-1}{N}$.

2.2. **Ford circles and the Rademacher path.** Let h and k be integers with $(h,k)=1$ and $0 \leqq h < k$. The *Ford circle* [13] $C(h,k)$ is the circle in $\mathbb{C}$ of radius $\frac{1}{2k^2}$ centered at the point

$$\frac{h}{k} + \frac{1}{2k^2} i.$$

The *upper arc* $\gamma(h,k)$ *of the Ford circle* $C(h,k)$ is those points of $C(h,k)$ from the initial point

$$\alpha_I(h,k) := \frac{h}{k} - \frac{k_p}{k(k^2+k_p^2)} + \frac{1}{k^2+k_p^2} i \tag{2.1}$$

to the terminal point

$$\alpha_T(h,k) := \frac{h}{k} + \frac{k_s}{k(k^2+k_s^2)} + \frac{1}{k^2+k_s^2} i, \tag{2.2}$$

traversed in the clockwise direction.

Note that we have

$$\alpha_I(0,1) = \alpha_T(N-1,N).$$

Every Ford circle is in the upper half plane. For $\frac{h_1}{k_1}, \frac{h_2}{k_2} \in \mathcal{F}_N$, $C(h_1,k_1)$ and $C(h_2,k_2)$ are either tangent or do not intersect.

The *Rademacher path* $P(N)$ of order N is the path in the upper half of the τ-plane from i to $i+1$ consisting of

$$\bigcup_{\frac{h}{k} \in \mathcal{F}_N} \gamma(h,k) \tag{2.3}$$

traversed left to right and clockwise. In particular, we consider the left half of the Ford circle $C(0,1)$ and the corresponding upper arc $\gamma(0,1)$ to be translated to the right by 1 unit. This is legal given then periodicity of the function which is to be integrated over $P(N)$.

However, since here we are interested in integrating a different function from that of Rademacher, we will define the *modified Rademacher path* $\bar{P}(N)$ of order N to be the path from i to $i+1$ consisting of

$$\bigcup_{\substack{\frac{h}{k}\in\mathcal{F}_N \\ 2\nmid k}} \gamma(h,k) \ \cup \bigcup_{\substack{\frac{h}{k}\in\mathcal{F}_N \\ 2\mid k}} \bar{\gamma}(h,k), \tag{2.4}$$

where $\bar{\gamma}(h,k)$ is the line segment connecting $\alpha_I(h,k)$ to $\alpha_T(h,k)$.

2.3. **Set up the integral.** Let n be fixed.

Since

$$\bar{f}(x) = \sum_{n=0}^{\infty} \bar{p}(n) x^n,$$

Cauchy's integral formula implies that

$$\bar{p}(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\bar{f}(x)}{x^{n+1}}\, dx, \tag{2.5}$$

where $\mathcal{C}$ is any simply closed contour enclosing the origin and inside the unit circle. We introduce the change of variable

$$x = e^{2\pi i\tau}$$

so that the unit disk $|x| \leqq 1$ in the x-plane maps to the infinitely tall, unit-wide strip in the τ-plane where $0 \leqq \Re\tau \leqq 1$ and $\Im\tau \geqq 0$. The contour $\mathcal{C}$ is then taken to be the preimage of $\bar{P}(N)$ under the map $x \mapsto e^{2\pi i\tau}$.

Better yet, let us replace x with $e^{2\pi i\tau}$ in (2.5) to express the integration in the τ-plane:

$$\begin{aligned}
\bar{p}(n) &= \int_{\bar{P}(N)} \bar{f}(e^{2\pi i\tau}) e^{-2\pi i n\tau} d\tau \\
&= \sum_{\substack{\frac{h}{k}\in\mathcal{F}_N \\ 2\nmid k}} \int_{\gamma(h,k)} \bar{f}(e^{2\pi i\tau}) e^{-2\pi i n\tau} d\tau + \sum_{\substack{\frac{h}{k}\in\mathcal{F}_N \\ 2\mid k}} \int_{\bar{\gamma}(h,k)} \bar{f}(e^{2\pi i\tau}) e^{-2\pi i n\tau} d\tau \\
&= \sum_{\substack{k=1 \\ 2\nmid k}}^{N} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} \int_{\gamma(h,k)} \bar{f}(e^{2\pi i\tau}) e^{-2\pi i n\tau} d\tau + \sum_{\substack{k=1 \\ 2\mid k}}^{N} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} \int_{\bar{\gamma}(h,k)} \bar{f}(e^{2\pi i\tau}) e^{-2\pi i n\tau} d\tau.
\end{aligned}$$

2.4. **Another change of variable.** Next, we change variables again, taking

$$\tau = \frac{iz+h}{k}, \tag{2.6}$$

so that

$$z = -ik\left(\tau - \frac{h}{k}\right). \tag{2.7}$$

Thus $C(h,k)$ (in the τ-plane) maps to the clockwise-oriented circle $K_k^{(-)}$ (in the z-plane) centered at $1/2k$ with radius $1/2k$.

So we now have

$$\bar{p}(n) = i\sum_{\substack{k=1\\ 2\nmid k}}^{N} k^{-1} \sum_{\substack{0\leqq h<k\\ (h,k)=1}} e^{-2\pi inh/k} \int_{\substack{z_I(h,k)\\ \text{arc}}}^{z_T(h,k)} e^{2n\pi z/k}\, \bar{f}(e^{2\pi ih/k - 2\pi z/k})\, dz$$
$$+ i\sum_{\substack{k=1\\ 2\mid k}}^{N} k^{-1} \sum_{\substack{0\leqq h<k\\ (h,k)=1}} e^{-2\pi inh/k} \int_{\substack{z_I(h,k)\\ \text{seg}}}^{z_T(h,k)} e^{2n\pi z/k}\, \bar{f}(e^{2\pi ih/k - 2\pi z/k})\, dz, \tag{2.8}$$

where $z_I(h,k)$ (resp. $z_T(h,k)$) is the image of $\alpha_I(h,k)$ (see (2.1)) (resp. $\alpha_T(h,k)$ [see (2.2)]) under the transformation (2.7).

So the transformation (2.7) maps the upper arc $\gamma(h,k)$ of $C(h,k)$ in the τ-plane to the arc on $K_k^{(-)}$ which initiates at

$$z_I(h,k) = \frac{k}{k^2+k_p^2} + \frac{k_p}{k^2+k_p^2} i \tag{2.9}$$

and terminates at

$$z_T(h,k) = \frac{k}{k^2+k_s^2} - \frac{k_s}{k^2+k_s^2} i. \tag{2.10}$$

Similarly, (2.7) maps the directed segment $\bar{\gamma}(h,k)$ to the directed segment from $z_I(h,k)$ to $z_T(h,k)$.

2.5. **Exploiting a modular transformation.** From the theory of modular forms, we have the transformation formula [27, p. 93, Lemma 4.31]

$$f(e^{2\pi ih/k-2\pi z/k}) = \omega(h,k) e^{\pi(z^{-1}-z)/12k} \sqrt{z} f(e^{2\pi i(iz^{-1}+H)/k}), \tag{2.11}$$

where $\sqrt{z}$ is the principal branch, $(h,k)=1$, and H is a solution to the congruence

$$hH \equiv -1 \pmod{k}.$$

From (2.11), we deduce the analogous transformation for $\bar{f}(x) = [f(x)]^2/f(x^2)$. It will be necessary to consider two cases. When k is even, $k/2$ is integer, so we can obtain $f(x^2)$ from $f(x)$ by replacing k by $k/2$ in $f(e^{2\pi ih/k-2\pi z/k})$. On the other hand, when k is odd, we instead replace h by $2h$ and z by $2z$ in $f(e^{2\pi ih/k-2\pi z/k})$. Thus,

$$\bar{f}(e^{2\pi ih/k-2\pi z/k}) = \begin{cases} \frac{\omega(h,k)^2}{\omega(h,\frac{k}{2})} \sqrt{z} \bar{f}(e^{2\pi i(H_1+i/z)/k}), & \text{if } 2 \mid k, \\ \frac{\omega(h,k)^2}{\omega(2h,k)} \sqrt{\frac{z}{2}} \exp\left(\frac{\pi}{8kz}\right) \frac{\left[f(e^{2\pi i(H_2+i/z)/k})\right]^2}{f(e^{\pi i(H_2+i/z)/k})}, & \text{if } 2 \nmid k, \end{cases} \tag{2.12}$$

where H_j is a solution to the congruence $jhH_j \equiv -1 \pmod{k}$.

Apply (2.12) to (2.8) to obtain

(2.13)

$$\bar{p}(n) = i\sum_{\substack{k=1\\2|k}}^{N} k^{-1} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(h,\frac{k}{2})} \int_{\substack{z_I(h,k)\\ \text{seg}}}^{z_T(h,k)} e^{2n\pi z/k}\sqrt{z}\bar{f}(e^{2\pi i(H_1+i/z)/k})\,dz$$
$$+ \frac{i}{\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{N} k^{-1} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)}$$
$$\times \int_{\substack{z_I(h,k)\\ \text{arc}}}^{z_T(h,k)} \sqrt{z}\exp\left(\frac{2\pi nz}{k}+\frac{\pi}{8kz}\right)\frac{\left[f(e^{2\pi i(H_2+i/z)/k})\right]^2}{f(e^{\pi i(H_2+i/z)/k})}\,dz.$$

2.6. **Normalization.** Next, introduce a normalization $\zeta = z/k$. (This is not *strictly* necessary, but it will allow us in the sequel to quote various useful results directly from the literature.)

(2.14)

$$\bar{p}(n) = i\sum_{\substack{k=1\\2|k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(h,\frac{k}{2})} \int_{\substack{\zeta_I(h,k)\\ \text{seg}}}^{\zeta_T(h,k)} e^{2n\pi\zeta/k^2}\sqrt{\zeta}\bar{f}(e^{2\pi iH_1/k-2\pi/\zeta})\,d\zeta$$
$$+ \frac{i}{\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)}$$
$$\times \int_{\substack{\zeta_I(h,k)\\ \text{arc}}}^{\zeta_T(h,k)} \sqrt{\zeta}\exp\left(\frac{2\pi n\zeta}{k^2}+\frac{\pi}{8\zeta}\right)\frac{\left[f(e^{2\pi iH_2/k-2\pi/\zeta})\right]^2}{f(e^{\pi iH_2/k-\pi/\zeta})}\,d\zeta,$$

where

$$\zeta_I(h,k) = \frac{k^2}{k^2+k_p^2} + \frac{kk_p}{k^2+k_p^2}i \tag{2.15}$$

and

$$\zeta_T(h,k) = \frac{k^2}{k^2+k_s^2} - \frac{kk_s}{k^2+k_s^2}i. \tag{2.16}$$

Let us now rewrite (2.14) as

$$\bar{p}(n) = i\sum_{\substack{k=1\\2|k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(h,\frac{k}{2})}(\mathcal{I}_1+\mathcal{I}_2) \tag{2.17}$$
$$+ \frac{i}{\sqrt{2}}\sum_{\substack{k=1\\2\nmid k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)}(\mathcal{I}_3+\mathcal{I}_4),$$

where

$$\mathcal{I}_1 := \int_{\substack{\zeta_I(h,k)\\ \text{seg}}}^{\zeta_T(h,k)} e^{2n\pi\zeta/k^2}\sqrt{\zeta}\ d\zeta,$$

$$\mathcal{I}_2 := \int_{\substack{\zeta_I(h,k)\\ \text{seg}}}^{\zeta_T(h,k)} e^{2n\pi\zeta/k^2}\sqrt{\zeta}\left\{-1+\bar{f}(e^{2\pi i H_1/k-2\pi/\zeta})\right\}\ d\zeta,$$

$$\mathcal{I}_3 := \int_{\substack{\zeta_I(h,k)\\ \text{arc}}}^{\zeta_T(h,k)} \sqrt{\zeta}\exp\left(\frac{2\pi n\zeta}{k^2}+\frac{\pi}{8\zeta}\right)\left\{-1+\frac{\left[f(e^{2\pi i H_2/k-2\pi/\zeta})\right]^2}{f(e^{\pi i H_2/k-\pi/\zeta})}\right\}\ d\zeta,$$

and

$$\mathcal{I}_4 := \int_{\substack{\zeta_I(h,k)\\ \text{arc}}}^{\zeta_T(h,k)} \sqrt{\zeta}\exp\left(\frac{2\pi n\zeta}{k^2}+\frac{\pi}{8\zeta}\right)\ d\zeta.$$

2.7. **Estimation.** It will turn out that as $N \to \infty$, only $\mathcal{I}_4$ ultimately makes a contribution. Note that all the integrations in the ζ-plane occur on arcs and chords of the circle K of radius $\frac{1}{2}$ centered at the point $\frac{1}{2}$. So, inside and on K, $0 < \Re\zeta \leqq 1$ and $\Re\frac{1}{z} \geqq 1$.

2.7.1. *Estimation of* $\mathcal{I}_1$. By, [2, p. 104, Thm. 5.9], the length of the path of integration does not exceed $2\sqrt{2}k/N$, and on the segment connecting $\zeta_I(h,k)$ to $\zeta_T(h,k)$, $|\zeta| < \sqrt{2}k/N$. Thus, the absolute value of the integrand,

$$\left|e^{2n\pi\zeta/k^2}\sqrt{\zeta}\right| = |\zeta|^{1/2}\exp\left(\frac{2n\pi\Re\zeta}{k^2}\right) \leqq |\zeta|^{1/2}\exp(2\pi n).$$

Thus,

$$|\mathcal{I}_1| \leqq \frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}k}{N}\right)^{1/2} e^{2\pi n} \leqq ck^{3/2}N^{-3/2},$$

for a constant c (recalling that n is fixed).

2.7.2. *Estimation of* $\mathcal{I}_2$. We have the absolute value of the integrand,

$$\begin{aligned}
&\left|\sqrt{\zeta}e^{2n\pi\zeta/k^2}\left\{-1+\bar{f}(e^{2\pi i H_1/k-2\pi/\zeta})\right\}\right|\\
=&|\zeta|^{1/2}\exp\left(\frac{2\pi n\Re\zeta}{k}\right)\sum_{m=1}^{\infty}\bar{p}(m)\exp\left(\frac{2\pi i H_1 m}{k^2}\right)\exp\left(\frac{-2\pi m}{\zeta}\right)\\
\leqq&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\bar{p}(m)\exp\left(-2\pi m\Re\frac{1}{\zeta}\right)\\
<&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\bar{p}(m)\exp\left(-2\pi\left(m-\frac{1}{24}\right)\Re\frac{1}{\zeta}\right)\\
\leqq&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\bar{p}(m)\exp\left(-\frac{\pi}{12}(24m-1)\right)\\
<&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\bar{p}(24m-1)y^{24m-1}, \qquad \text{(where } y = e^{-\pi/12}\text{)}\\
=&c'|\zeta|^{1/2},
\end{aligned}$$

for a constant c'. So,

$$|\mathcal{I}_2| \leqq \frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}k}{N}\right)^{1/2} c' < Ck^{3/2}N^{-3/2}$$

for a constant C.

2.7.3. *Estimation of I_3.* Let $p^*(x)$ be defined by

$$\sum_{n=0}^{\infty} p^*(n)x^n = \frac{[f(x^2)]^2}{f(x)} = 1-x+x^2-2x^3+3x^4-4x^5+5x^6-7x^7+10x^3-13x^9+\cdots .$$

The regularity of the integrand allows us to alter the path of integration from the arc connecting $\zeta_I(h,k)$ and $\zeta_T(h,k)$ to the directed segment.

With this in mind, we estimate the absolute value of the integrand,

$$\begin{aligned}
&\left|\sqrt{\zeta}\exp\left(\frac{2\pi n\zeta}{k^2}+\frac{\pi}{8\zeta}\right)\sum_{m=1}^{\infty} p^*(m)\exp\left(\frac{\pi i H_2 m}{k}-\frac{\pi m}{\zeta}\right)\right| \\
=&\left|\sqrt{\zeta}\exp\left(\frac{2\pi n\zeta}{k^2}\right)\sum_{m=1}^{\infty} p^*(m)\exp\left(\frac{\pi i H_2 m}{k}\right)\exp\left(\frac{\pi}{\zeta}\left(\frac{1}{8}-m\right)\right)\right| \\
\leqq&|\zeta|^{1/2}\exp\left(\frac{2\pi n\Re\zeta}{k^2}\right)\sum_{m=1}^{\infty}|p^*(m)|\exp\left(\frac{\pi i H_2 m}{k}\right)\exp\left(\pi\Re\frac{1}{\zeta}\left(\frac{1}{8}-m\right)\right) \\
\leqq&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|p^*(m)|\exp\left(-\frac{\pi}{8}(8m-1)\right) \\
<&|\zeta|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}|p^*(8m-1)|y^{8m-1} \quad (\text{where } y=e^{-\pi/8}) \\
=&c''|\zeta|^{1/2}
\end{aligned}$$

for a constant c''. So,

$$|\mathcal{I}_3| \leqq \frac{2\sqrt{2}k}{N}\left(\frac{\sqrt{2}k}{N}\right)^{1/2} c'' < C'k^{3/2}N^{-3/2}$$

for a constant C'.

2.7.4. *Combining the estimates.*

$$\begin{aligned}
\left| i\sum_{\substack{k=1\\2|k}}^{N} k^{-5/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}} e^{-2\pi inh/k}\frac{\omega(h,k)^2}{\omega(h,\frac{k}{2})}(\mathcal{I}_1+\mathcal{I}_2)\right| &< \sum_{k=1}^{N}\sum_{0\leqq h<k}(c+C)k^{-1}N^{-3/2} \\
&\leqq (c+C)N^{-3/2}\sum_{k=1}^{N}1 \\
&= O(N^{-1/2}).
\end{aligned}$$

Also,

$$\left| \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)} \mathcal{I}_3 \right| < \sum_{k=1}^{N} \sum_{0\leqq h<k} \frac{C'}{\sqrt{2}} k^{-1} N^{-3/2}$$

$$\leqq \frac{C'}{\sqrt{2}} N^{-3/2} \sum_{k=1}^{N} 1$$

$$= O(N^{-1/2}).$$

Thus, we may revise (2.14) to

$$\bar{p}(n) = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)}\, \mathcal{I}_4 + O(N^{-1/2}). \tag{2.18}$$

2.8. **Evaluation of $\mathcal{I}_4$.** Write $\mathcal{I}_4$ as

$$\mathcal{I}_4 = \int_{K^{(-)}} \sqrt{\zeta} \exp\left(\frac{2\pi n\zeta}{k^2} + \frac{\pi}{8\zeta} \right)\, d\zeta - \mathcal{I}_5 - \mathcal{I}_6,$$

where

$$\mathcal{I}_5 := \int_0^{\zeta_I(h,k)} \sqrt{\zeta} \exp\left(\frac{2\pi n\zeta}{k^2} + \frac{\pi}{8\zeta} \right)\, d\zeta$$

and

$$\mathcal{I}_6 := \int_{\zeta_T(h,k)}^{0} \sqrt{\zeta} \exp\left(\frac{2\pi n\zeta}{k^2} + \frac{\pi}{8\zeta} \right)\, d\zeta.$$

2.8.1. *Estimation of $\mathcal{I}_5$ and $\mathcal{I}_6$.* We note that the length of the arc of integration in $\mathcal{I}_5$ is less than $\frac{\pi k}{\sqrt{2}N}$, and on this arc $|z| < \sqrt{2}k/N$. [48, p. 272]. Also, $\Re\frac{1}{\zeta} = 1$ on K [48, p. 271, Eq. (120.2)]. Further, $0 < \Re\zeta < 2k^2/N^2$ [48, p. 271, Eq. (119.6)]. The absolute value of the integrand is thus

$$|\zeta|^{1/2} \exp\left(\frac{2\pi n\Re\zeta}{k^2} + \frac{\pi}{8}\Re\frac{1}{\zeta} \right) < 2^{1/4} k^{1/2} N^{-1/2} \exp(4\pi nk^2N^{-2} + \pi/8)$$

so that

$$\begin{aligned} |\mathcal{I}_5| &< \pi k 2^{-1/2} N^{-1} 2^{1/4} k^{1/2} N^{-1/2} \exp(4\pi nk^2N^{-2} + \pi/8) \\ &= 2^{-1/4}\pi k^{3/2} \exp\left(4\pi nk^2N^{-2} + \frac{\pi}{8} \right) N^{-3/2} \\ &= O(k^{3/2}N^{-3/2}e^{4\pi k^2N^{-2}}). \end{aligned}$$

By the same reasoning, $|\mathcal{I}_6| = O(k^{3/2}N^{-3/2}e^{4\pi k^2N^{-2}})$.

We may therefore revise (2.18) to

(2.19)

$$\bar{p}(n) = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{N} k^{-5/2} \sum_{\substack{0\leqq h<k \\ (h,k)=1}} e^{-2\pi inh/k} \frac{\omega(h,k)^2}{\omega(2h,k)} \int_{K^{(-)}} \sqrt{\zeta} \exp\left(\frac{2\pi n\zeta}{k^2} + \frac{\pi}{8\zeta} \right)\, d\zeta$$

$$+ O(N^{-1/2}),$$

and upon letting N tend to infinity, obtain

(2.20)

$$\bar{p}(n) = \frac{i}{\sqrt{2}} \sum_{\substack{k=1 \\ 2\nmid k}}^{\infty} k^{-5/2} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2}{\omega(2h,k)} \int_{K^{(-)}} \sqrt{\zeta} \exp\left(\frac{2\pi n \zeta}{k^2} + \frac{\pi}{8\zeta} \right) \, d\zeta.$$

2.9. **The final form.** We may now introduce the change of variable

$$\zeta = \frac{\pi}{8t}$$

which allows the integral to be evaluated in terms of $I_{3/2}$, the Bessel function of the first kind of order $3/2$ with purely imaginary argument [53, p. 372, §17.7] when we bear in mind that a "bent" path of integration is allowable according to the remark preceding Eq. (8) on p. 177 of [52]. The final form of the formula is then obtained by using the fact that Bessel functions of half-odd integer order can be expressed in terms of elementary functions.

We therefore have

$$\begin{aligned} \bar{p}(n) &= \frac{\pi^{3/2}}{32i} \sum_{\substack{k=1 \\ k\nmid 2}}^{\infty} k^{-5/2} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2}{\omega(2h,k)} \int_{\pi/8 - i\infty}^{\pi/8 + i\infty} t^{-5/2} \exp\left(t + \frac{n\pi^2}{4k^2 t} \right) \, dt \\ &= \pi 2^{-5/2} n^{-3/4} \sum_{\substack{k=1 \\ k\nmid 2}}^{\infty} k^{-1} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} e^{-2\pi i n h/k} \frac{\omega(h,k)^2}{\omega(2h,k)} I_{3/2}\left(\frac{\pi\sqrt{n}}{k} \right) \\ &= \frac{1}{2\pi} \sum_{\substack{k \geqq 1 \\ 2\nmid k}} \sqrt{k} \sum_{\substack{0 \leqq h < k \\ (h,k)=1}} \frac{\omega(h,k)^2}{\omega(2h,k)} e^{-2\pi i n h/k} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi\sqrt{n}}{k}\right)}{\sqrt{n}} \right), \end{aligned}$$

which is Theorem 1.1. □

3. Proof of Theorem 1.2

The proof of 1.2 differs from that of Theorem 1.1 on in minor computational details. It will therefore be sufficient to move quickly through the steps, highlighting these differences, rather than present the complete proof.

$$\begin{aligned} pod(n) = \sum_{\substack{0 \leqq h < k \leqq N \\ 4|k; (h,k)=1}} \int_{\gamma(h,k)} g(e^{2\pi i \tau}) e^{-2\pi i n \tau} \, d\tau + \sum_{\substack{0 \leqq h < k \leqq N \\ 2\nmid k; (h,k)=1}} \int_{\gamma(h,k)} \\ + \sum_{\substack{0 \leqq h < k \leqq N \\ k \equiv 2 \pmod 4; (h,k)=1}} \int_{\bar{\gamma}(h,k)}, \end{aligned}$$

where the integrand of the last two integrals is the same as that of the first. Notice that we alter the path of integration from the original Rademacher path $P(N)$ only when $k \equiv 2 \pmod 4$.

After setting $z=-ik(\tau-h/k)$, use the following modular transformation:

$$g(e^{2\pi ih/k-2\pi z/k}) \tag{3.1}$$
$$=\begin{cases} \frac{\omega(h,k)\omega(h,\frac{k}{4})}{\omega(h,\frac{k}{2})}\sqrt{z}\exp\left(\frac{\pi}{4kz}-\frac{\pi z}{4k}\right)g(e^{2\pi i(H_1+i/z)/k}), & \text{if } k\equiv 0 \pmod 4,\\ \frac{\omega(h,k)\omega(2h,\frac{k}{2})}{\omega(h,\frac{k}{2})}\sqrt{2z}\exp\left(\frac{-\pi z}{4k}\right)\frac{\left[f(e^{2\pi i(H_2+i/z)/k})\right]^2}{f(e^{4\pi i(H_2+i/z)/k})}, & \text{if } k\equiv 2 \pmod 4,\\ \frac{\omega(h,k)\omega(4h,k)}{\omega(2h,k)}\sqrt{2z}\exp\left(\frac{\pi}{16kz}-\frac{\pi z}{4k}\right)g(e^{\pi i(H_4+i/z)/(2k)}), & \text{if } k\equiv \pm 1 \pmod 4,\end{cases}$$

where H_j is a solution to the congruence $jhH_j\equiv -1 \pmod k$.

Normalize with the transformation $\zeta=z/k$. The sum over k with $k\equiv 2 \pmod 4$ is $O(N^{-1/2})$. For the other k, split the integral as before into two pieces, one with the factor $(-1+g(x))$ and the other with the factor 1. Both of the sums with the factor $(-1+g(x))$ are $O(N^{-1/2})$. For the integral in the summation over k where $4\mid k$, use the substitution $\zeta=\pi/(4t)$. For the integral in the summation over k odd, use the substitution $\zeta=\pi/(16t)$. We obtain

$$pod(n)=\frac{\pi}{(8n-1)^{3/4}}\sum_{k\geqq 1}k^{-1}\sum_{\substack{0\leqq h<k\\(h,k)=1}}e^{-2\pi inh/k}\omega(h,k)$$
$$\times\left\{\frac{\omega(h,\frac{k}{4})}{2\omega(h,\frac{k}{2})}I_{3/2}\left(\frac{\pi\sqrt{8n-1}}{2k}\right)\chi(4\mid k)+\frac{\omega(4h,k)}{\omega(2h,k)}I_{3/2}\left(\frac{\pi\sqrt{8n-1}}{4k}\right)\chi(2\nmid k)\right\}.$$

And finally,

$$pod(n)=\frac{2}{\pi}\sum_{k\geqq 1}k^{1/2}\sum_{\substack{0\leqq h<k\\(h,k)=1}}e^{-2\pi inh/k}\omega(h,k)$$
$$\times\left\{\frac{\omega(h,\frac{k}{4})}{\omega(h,\frac{k}{2})}\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi\sqrt{8n-1}}{2k}\right)}{\sqrt{8n-1}}\right)\chi(4\mid k)\right.$$
$$\left.+\frac{\sqrt{2}\omega(4h,k)}{\omega(2h,k)}\left(\frac{\sinh\left(\frac{\pi\sqrt{8n-1}}{4k}\right)}{\sqrt{8n-1}}\right)\chi(2\nmid k)\right\},$$

which is equivalent to Theorem 1.2. □

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