# ABSTRACT COREFLECTIVELY MODIFIED DUALITY AND APPLICATIONS TO CONVERGENCE-APPROACH SPACES

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Abstract. The general mechanism of coreflectively modified duality recently developed for convergences is shown to be of purely categorical nature. Applications in the category **CAP** of convergence-approach spaces are presented as a first example of fruitful use of the method in an other category than that of convergences. Many of the main tools of convergence theory are extended to convergence-approach spaces. In particular, explicit descriptions of some of the fundamental reflectors and coreflectors available in **Conv** are extended to **CAP**, leading to new concepts and to a better understanding of well-known notions in **CAP**. In this way, general results on product of various classes of quotient maps in **CAP** are obtained together with new simple proofs of characterizations of exponential objects and cartesian-closed hull of **PRAP**.

## 1. Introduction

Coreflectively modified continuous duality was introduced and developed in [12] and [34] in the category **Conv** of convergence spaces (with continuous maps as morphisms) in view of a unification of topological product theorems. The method turned out to be particularly efficient to the effect that a small kernel of structural results allowed to unify and generalize about twenty classical product theorems of Michael, Arhangel'skii, Cohen, Bagley and Weddington, Tanaka, among others. Moreover, it led to a dozen of new theorems on stability under product of various topological properties and various types of quotientness as well as on relationship between a given topology and the corresponding upper Kuratowski convergence on its closed sets (see [12], [32], [34] for details). For example, my solution of the problem of Y. Tanaka [41] of characterizing the topologies which product with every metrizable space is sequential that I obtained in [33] is an instance of these new results.

*Date*: December 13, 2002.

The proofs of results on coreflectively modified duality of [34] made explicit use of the particular context of the category of convergences **Conv**. It turns out that new purely categorical proofs can be given, and that the scope of this method can be broadened to a large class of cartesian-closed topological categories.

I give an example of applications of coreflectively modified duality to other categories than **Conv** to the category **CAP** of convergence-approach spaces. Basic definitions concerning both **Conv** and **CAP** are postponed to Section 5, because they are not necessary in the first categorical part except as illustrating examples. The category **AP** of approach spaces was introduced by R. Lowen in [25] in order to unify topological and metric concepts. Roughly speaking, an approach space can be described by a distance $\delta$ between points and sets. The metric approach spaces are those for which $\delta(x, A) = \bigwedge_{y\in A} \delta(x, \{y\})$, while ([1]) the topological ones are those for which $\delta$ takes only values 0 and $\infty$. In the latter case $x \in \operatorname{cl} A$ if and only if $\delta(x, A) = 0$. Hence **AP** contains both the categories $pq^\infty$-**Met** of infinite pseudo-quasi metric spaces and **Top** of topological spaces. For example, the theory of approach spaces allowed to unify total boundedness and compactness via measure of compactness [27], usual topological connectedness and (metric) Cantor-connectedness [28] and to obtain a canonical distance (in **AP**) on arbitrary products of metric spaces. Moreover some important objects such as $\beta\mathbb{N}$, spaces of measures, function spaces and, in particular, hyperspaces, spaces of random variables admit natural **AP**-structures. I refer to [26] for terminology and details of the theory of approach spaces and to [1], [35] for categorical notions. As closures of **AP** with respect to certain categorical constructions, E. Lowen-Colebunders and R. Lowen introduced the bigger categories **PRAP**, **PSAP** and **CAP** of pre-approach spaces, pseudo-approach spaces and convergence-approach spaces. **PRAP** contains the category **PrTop** of pretopologies both bireflectively and bicoreflectively and the category **PrMet** of premetric spaces only bicoreflectively. Analogously, the category **PsTop** of pseudotopologies is a bireflective and bicoreflective subcategory of **PSAP** as well as **Conv** in **CAP**. Hence, the results obtained in **CAP** are generalization of some of my results of [34] in **Conv** based on extension to **CAP** of convergence-theoretic notions and methods. In particular, explicit descriptions of some of the fundamental reflectors and coreflectors available in **Conv** are extended to

[1] $\bigwedge$ and $\bigvee$ stand for infimum and supremum respectively.

**CAP**, leading to new concepts and to a better understanding of well-known notions in **CAP** (for example para-approach spaces that fill a gap in the Figure **??** : paratopological spaces, introduced by S. Dolecki in [10], play an important role in applications of convergence theory to topology and it was natural to look for an analogue in **CAP**). For instance, I give a new description of the reflector on **PRAP** and a new proof characterizing exponential objects in **PRAP** that highlights the analogy between the situations in **PrTop** and in **PRAP**. Analogously the characterization of **PSAP** as the cartesian-closed hull of **PRAP** and that of **PsTop** as the cartesian-closed hull of **PrTop** follow from the same formal result. Moreover I obtain several results on product of various classes of quotient maps in **CAP**. diag.lpfig:diag1

## 2. Categorical preliminaries

2.1. **Toponomes.** If $\xi$ is a $\mathcal{C}$-object, I denote by $|\xi|$ the underlying set. In the sequel $\mathcal{C}$ is a *toponome* in the sense of [35], i.e., a concrete topological category for which :

For every pair $(\xi, \sigma)$ of $\mathcal{C}$-objects there exists a canonical $\mathcal{C}$-structure $[\xi, \sigma]$ on the set $\mathrm{Hom}(\xi, \sigma)$ of $\mathcal{C}$-morphism between $\xi$ and $\sigma$ such that

$$[\xi \times \tau, \sigma] = [\tau, [\xi, \sigma]], \tag{2.1}$$

for every $\mathcal{C}$-objects $\xi$, $\tau$, $\sigma$, where the equality stands for a $\mathcal{C}$-isomorphism via the transposition map $^t$ defined by $^t f(y)(x) = f(x, y)$ (see [35] for details). If $g \in |[\tau, [\xi, \sigma]]|$ I denote by $\widehat{g}$ the (unique) map for which $^t(\widehat{g}) = g$. Notice that

$$\widehat{g} = w \circ (\mathrm{Id}_\xi \times g),$$

where $w : |\xi \times \mathrm{Hom}(\xi, \sigma)| \to |\sigma|$ is the evaluation map.

Since $\mathcal{C}$ is topological, initial and final structures always exist. Products, coproducts, subspaces are defined in terms of initial and final structure. I say that $\xi$ *is finer than* $\theta$ (in symbols $\xi \geq \theta$) if $\mathrm{Id} : \xi \to \theta$ is a $\mathcal{C}$-morphism, where Id stands for the identity of the underlynig set $|\xi| = |\theta|$. If $f : |\xi| \to |\tau|$, I denote by $f\xi$ the final $\mathcal{C}$-structure induced by $f$ and $\xi$, and I denote by $f^- \tau$ the initial $\mathcal{C}$-structure induced by $f$ and $\tau$.

Obviously $[\xi, \sigma]$ is the coarsest $\mathcal{C}$-structure on $\mathrm{Hom}(\xi, \sigma)$ among the $\alpha$ that verify

$$\xi \times \alpha \geq w^- \sigma. \tag{2.2}$$

If $f : \xi \to \tau$, I denote by $f^* : [\tau, \sigma] \to [\xi, \sigma]$ the $\mathcal{C}$-map defined by $f^*(h) = h \circ f$ and by $f_* : [\theta, \xi] \to [\theta, \tau]$ the $\mathcal{C}$-map defined by $f_*(h) =$

$f \circ h$. In a toponome [35, Proposition 18.i]

$$[\bigwedge_i f_i\xi_i, \sigma] = \bigvee_i (f_i^*)^-([\xi_i, \sigma]), \tag{2.3}$$

for every $\sigma$, every family $(\xi_i)_i$ of $\mathcal{C}$-objects and every surjective family $(f_i)_i$.

2.2. **Reflectors and coreflectors.** A map $J : \mathcal{C} \to \mathcal{C}$ with $|J\xi| = |\xi|$ for every $\mathcal{C}$-object $\xi$ is a *bireflector* if it is isotone (i.e., $\xi \geq \theta \Longrightarrow J\xi \geq J\theta$), contractive (i.e., $\xi \geq J\xi$), idempotent (i.e., $J(J\xi) = J\xi$) and if for each $\mathcal{C}$-morphism $f : \xi \to \tau$,

$$f(J\xi) \geq J(f\xi) \tag{2.4}$$

what amounts to $J(f^-\tau) \geq f^-(J\tau)$. The class of $\mathcal{C}$-objects fixed by $J$ is denoted by $\operatorname{fix} J$ and is a *bireflective subcategory* of $\mathcal{C}$. The $J$-reflection $J\xi$ of $\xi$ is the finest structure of $\operatorname{fix} J$ that is coarser than $\xi$.

Dually, a *bicoreflector* $E$ is isotone, expansive (i.e., $E\xi \geq \xi$) and idempotent. Moreover

$$f(E\xi) \geq E(f\xi), \tag{2.5}$$

for each $\mathcal{C}$-morphism $f : \xi \to \tau$. The latter amounts to $E(f^-\tau) \geq f^-(E\tau)$. The $E$-coreflection $E\xi$ of $\xi$ is the coarsest structure of $\operatorname{fix} E$ that is finer than $\xi$.

In the sequel, all (co)reflectors are supposed to be bi(co)reflectors, i.e., a $\mathcal{C}$-object and its (co)reflection have the same underlying set.

2.3. ***J*-quotients and *JE*-structures.** The following concept is inspired by the situation in **Conv**. Indeed, it is known [11], [10] that numerous classes of convergences (and in particular classes of topologies) $\tau$ can be characterized by the inequalities of the type

$$\tau \geq JE\tau, \tag{2.6}$$

where $J$ is a reflector and $E$ is a coreflector. A convergence fulfilling (2.6) is called a *JE-convergence.*

For example ([2]), if $E$ is the first-countable modifier, then, by choosing $J$ to be respectively the topologizer, pretopologizer, paratopologizer, pseudotopologizer and identity, (2.6) characterizes sequential, Frchet, strongly Frchet, bisequential and first-countable convergences; if $E$ is the compact localizer then (2.6) characterizes $k, k'$, strongly $k'$, locally compact and, once again locally compact convergences respectively. See [10] for details.

[2]See Section 5 for details and for basic terminology of convergence theory and of theory of approach spaces.

More generally, I call *$JE$-structure* a $\mathcal{C}$-object satisfying (2.6) where $J$ and $E$ are respectively a reflector and a coreflector in $\mathcal{C}$.

2.4. **Classes of quotients.** Analogously, it is known [10] that almost open, biquotient, countably biquotient, hereditarily quotient and quotient maps $f:\xi\to\tau$ can be characterized in **Conv** as surjections that fulfill

$$\tau \geq J(f\xi), \tag{2.7}$$

where $J$ is respectively the identity, pseudotopologizer, paratopologizer, pretopologizer and topologizer. A continuous surjection $f$ that fulfills (2.7) is called *$J$-quotient*. More generally, given a reflector $J$ in $\mathcal{C}$, I call $J$-quotient (in $\mathcal{C}$) a surjective $\mathcal{C}$-morphism satisfying (2.7). Notice that a $J$-quotient map between elements of $\operatorname{fix} J$ is precisely a quotient map in the subcategory $\operatorname{fix} J$ of $\mathcal{C}$.

**Proposition 2.1.** *If $J$ is a reflector and $E$ is a coreflector, and if $\xi$ is a $JE$-structure, then $f\xi$ is also a $JE$-structure. Moreover, each $J$-quotient image of a $JE$-structure is a $JE$-structure.*

The proof is formally the same as in **Conv** [10, Theorem 4.2] and is a direct consequence of (2.4) and (2.5).

Analogously, the following generalizes [34, Proposition 2.3] proved in **Conv** and its proof (once again easily derived from (2.4), (2.5) and (2.7)) is formally the same.

**Proposition 2.2.** *Let $J$ be a reflector and let $E$ be a coreflector in a toponome $\mathcal{C}$. If $f:\xi\to\tau$ is a surjective $\mathcal{C}$-morphism with $\xi\geq JE\xi$, then $f$ is $J$-quotient if and only if for each $\theta\leq JE\theta$ (equivalently $\theta=J\theta$) and each map $g:\tau\to\theta$, $g$ is a $\mathcal{C}$-morphism provided that $g\circ f$ is a $\mathcal{C}$-morphism.*

## 3. General mechanism of duality

If $\xi$ and $\sigma$ are $\mathcal{C}$-objects, $J$ is a reflector and $E$ is a coreflector, then the modifier $\mathrm{Epi}^{\sigma}_{JE}$ ([3]) is defined by

$$\mathrm{Epi}^{\sigma}_{JE}(\xi)=i^{-}([JE[\xi,\sigma],\sigma]) \tag{3.1}$$

where $i:|\xi|\to|[[\xi,\sigma],\sigma]|$ is defined by $i(x)(f)=f(x)$. If $L$ is a reflector, then

$$\mathrm{Epi}^{L}_{JE}(\xi)=\bigvee_{\sigma=L\sigma}\mathrm{Epi}^{\sigma}_{JE}(\xi). \tag{3.2}$$

[3] In [34], I used $A^{\sigma}_{JE}$ and $A^{L}_{JE}$ instead.

The following extends [34, Theorem 3.1] ( that was proved for $\mathcal{C} =$ **Conv**).

**Theorem 3.1.** *Let $\mathcal{C}$ be a toponome. Let $\xi$ and $\theta$ be two $\mathcal{C}$-structures on the same set such that $\xi \geq \theta$. $J$ and $L$ are reflectors and $E$ is a coreflector. The following are equivalent:*

(1) *For every $\tau \geq JE\tau$ in $\mathcal{C}$*

$$\theta \times J\tau \geq L(\xi \times \tau); \tag{3.3}$$

(2) (3.3) *holds for every $\tau = E\tau$;*

(3) $\mathrm{Id}_{\xi,\theta} \times f$ *is $L$-quotient for every $J$-quotient map $f$ with $JE$-domain;*

(4) *For every $\tau = E\tau$ and every $\sigma = L\sigma$,*

$$\mathrm{Hom}(\theta \times J\tau, \sigma) = \mathrm{Hom}(\xi \times \tau, \sigma); \tag{3.4}$$

(5) (3.4) *holds for every $\tau = E\tau$ and every $\sigma$ in an initially dense subclass of the category* $\mathrm{fix}\, L$*;*

(6) *For every $\sigma = L\sigma$,*

$$JE[\xi, \sigma] \geq [\theta, \sigma]; \tag{3.5}$$

(7) (3.5) *holds for every $\sigma$ in an initially dense subclass of* $\mathrm{fix}\, L$*;*

(8) $\theta \geq \mathrm{Epi}^L_{JE}\xi$.

A subclass $\mathcal{D}$ of $\mathcal{C}$-objects is said to be *initially dense* in $\mathcal{C}$ if each $\mathcal{C}$-object $\xi$ is an initial image of elements of $\mathcal{D}$. In other words, there exists an initial source (in $\mathcal{C}$) $(f_i : \xi \to \sigma_i)_{i \in I}$ with codomains $\sigma_i$ in $\mathcal{D}$: $\xi = \bigvee_{i \in I} f_i^- \sigma_i$.

Dually a subclass $\mathcal{D}$ of $\mathcal{C}$-objects is *finally dense* in $\mathcal{C}$ if for every $\mathcal{C}$-object $\xi$ there exists a final sink $(f_i : \sigma_i \to \xi)_{i \in I}$ with domains in $\mathcal{D}$: $\xi = \bigwedge_{i \in I} f_i \sigma_i$. If moreover each $f_i$ can be chosen to underly an identity map, then $\mathcal{D}$ is said to be *rigidly finally dense* ([4]).

If $\mathcal{D}$ is an initially dense subcategory of $\mathrm{fix}\, L$, then, in view of Theorem 3.1,

$$\mathrm{Epi}^L_{JE}\xi = \bigvee_{\sigma \in \mathcal{D}} \mathrm{Epi}^\sigma_{JE}\xi; \tag{3.6}$$

and, by definition of initial density,

$$L\xi = \bigvee_{\sigma \in \mathcal{D}} \bigvee_{f \in \mathrm{Hom}(\xi, \sigma)} f^- \sigma. \tag{3.7}$$

[4] I used *concretely finally dense* instead in [34].

Since there is a unique $\mathcal{C}$-structure on a singleton, this (unique) structure is fixed by every reflector and every coreflector. Thus, if $\theta$ verifies (3.3) for every $\tau = E\tau$, it does in particular if $\tau$ is a singleton. Consequently $\theta \geq L\xi$. Since $L$ is a reflector, $\mathrm{Hom}(\xi, \sigma) = \mathrm{Hom}(L\xi, \sigma)$ for every $\sigma = L\sigma$, so that $\mathrm{Hom}(\xi, \sigma) = \mathrm{Hom}(\theta, \sigma)$, because $\xi \geq \theta \geq L\xi$.

Notice that the proofs of $1 \implies 2 \implies 3 \implies 4 \iff 5 \implies 6 \iff 7$ are formally the same as in [34]. They are added only by the sake of completness.

*Proof.* $1 \implies 2$ because $\tau = E\tau$ implies $\tau \geq JE\tau$.

$2 \implies 3$: Consider $f : \xi_1 \to \tau_1 \geq J(f\xi_1)$ with $\xi_1 \geq JE\xi_1$. Then $f\xi_1 \geq JE(f\xi_1)$ by Proposition 2.1. Applying 2 with $\tau = E(f\xi_1)$ we get $\theta \times JE(f\xi_1) \geq L(\xi \times Ef\xi_1) \geq L(\mathrm{Id}_{\xi,\theta} \times f)(\xi \times \xi_1)$. Since $f$ is $J$-quotient, $\tau_1 \geq JE(f\xi_1)$ so that $\theta \times \tau_1 \geq L(\mathrm{Id}_{\xi,\theta} \times f)(\xi \times \xi_1)$. In view of (2.7), $\mathrm{Id}_{\xi,\theta} \times f$ is $L$-quotient.

$3 \implies 4$: $\xi \times \tau \geq \theta \times J\tau$ because $\xi \geq \theta$, so that $\mathrm{Hom}(\theta \times J\tau, \sigma) \subset \mathrm{Hom}(\xi \times \tau, \sigma)$. Consider $g \in \mathrm{Hom}(\xi \times \tau, \sigma)$. Let $\overline{g}$ denote the map $g$ considered from $\theta \times J\tau$ to $\sigma$. By definition, $g = \overline{g} \circ (\mathrm{Id}_{\xi,\theta} \times \mathrm{Id}_{\tau,J\tau})$. The map $\mathrm{Id}_{\tau,J\tau}$ is $J$-quotient with $JE$-domain, so that, by 3, $\mathrm{Id}_{\xi,\theta} \times \mathrm{Id}_{\tau,J\tau}$ is $L$-quotient. In view of Proposition 2.2, $\overline{g}$ is a morphism because $g$ is a morphism. Thus, $\mathrm{Hom}(\xi \times \tau, \sigma) \subset \mathrm{Hom}(\theta \times J\tau, \sigma)$.

$4 \iff 5$ is obvious in view of the definition of initial density.

$4 \implies 6$: For each $\sigma = L\sigma$, let $\tau = E[\xi, \sigma]$. By 4, $\mathrm{Hom}(\xi \times E[\xi, \sigma], \sigma) \subset \mathrm{Hom}(\theta \times JE[\xi, \sigma], \sigma)$. Since the evaluation $w$ is a morphism from $\xi \times E[\xi, \sigma]$ to $\sigma$, it is a morphism from $\theta \times JE[\xi, \sigma]$ to $\sigma$. Hence, $JE[\xi, \sigma] \geq [\theta, \sigma]$, by definition (2.2) of $[\theta, \sigma]$.

$6 \iff 7$ follows from the equivalence between 4 and 5.

$6 \implies 8$: Let $\sigma$ belong to $\mathrm{fix}\, L$. Since $JE[\xi, \sigma] \geq [\theta, \sigma]$,

$$[[\theta, \sigma], \sigma] \geq [JE[\xi, \sigma], \sigma],$$

in restriction to $\mathrm{Hom}(JE[\xi, \sigma], \sigma) \supset i(|\xi|)$. Thus, $\theta \geq \mathrm{Epi}^{\sigma}\theta \geq \mathrm{Epi}^{\sigma}_{JE}\xi$. Hence $\theta \geq \mathrm{Epi}^{L}_{JE}\xi$.

$8 \implies 4$: Let $f : \xi \times \tau \to \sigma = L\sigma$ be a morphism. Then ${}^{t}f : \tau \to [\xi, \sigma]$ is a morphism. By (2.4) and (2.5), ${}^{t}f : JE\tau \to JE[\xi, \sigma]$ is also a morphism. Hence, since $JE\tau = J\tau$,

$$({}^{t}f)^{*} : [JE[\xi, \sigma], \sigma] \to [J\tau, \sigma]$$

is a morphism. By 8, the map $i : |\xi| \to |\mathrm{Hom}([\xi, \sigma], \sigma)|$ defined by $i(x)(f) = f(x)$ is an element of $\mathrm{Hom}(\theta, [JE[\xi, \sigma], \sigma])$. Hence, $({}^{t}f)^{*} \circ i : \theta \to [J\tau, \sigma]$ is a morphism so that, in view of (2.1),

$$\widehat{({}^{t}f)^{*} \circ i} : \theta \times J\tau \to \sigma$$

is a morphism. I conclude by the observation that $\widehat{({}^t f)^* \circ i}$ and $f$ have the same underlying map.

$4 \Longrightarrow 1$: because $L\alpha = \bigvee_{\sigma = L\sigma} \bigvee_{f \in \mathrm{Hom}(\alpha,\sigma)} f^- \sigma$. □

The last result should be compared with the following combination of corollaries of [36, Theorem 3.1] and [40, Theorem 3.3]. F. Schwarz's results cited above apply in the more general context of initially structured categories and epireflective subcategories. I only give a reformulation in case of a toponome $\mathcal{C}$.

**Theorem 3.2.** *Let $\mathcal{C}$ be a toponome, let $L$ be an epireflector and let $\mathcal{D}$ be an initially dense subclass of* $\operatorname{fix} L$. *If $\xi = L\xi$, the following are equivalent :*

1. $\xi$ *is exponential in* $\operatorname{fix} L$*;*
2. $\xi \times L\tau \geq L(\xi \times \tau)$ *for every $\mathcal{C}$-object $\tau$;*
3. $\xi \times -$ *preserves coproduct and quotient in* $\operatorname{fix} L$*;*
4. $[\xi, \sigma] = L[\xi, \sigma]$ *for every $\sigma = L\sigma$;*
5. $[\xi, \sigma] = L[\xi, \sigma]$ *for every $\sigma \in \mathcal{D}$.*

In case $J = L$, $E$ is the identity and $\theta = \xi = L\xi$, Theorem 3.1 recovers Theorem 3.2 (only for a bireflector $L$). A $\mathcal{C}$-object $\xi = L\xi$ verifying one (hence all) of th conditions of Theorem 3.2 is said to be *exponential* in $\operatorname{fix} L$. In [36, Notes added in proof, page 253], F. Schwarz adresses the problem of characterizing objects $\xi$ for which $[\xi, \sigma] = L[\xi, \sigma]$ for every $\sigma = L\sigma$, dropping the condition that $\xi$ belongs to $\operatorname{fix} L$ and notices that few is known about this question. I call such objects *$\mathcal{C}$-quasi-exponential* in $\operatorname{fix} L$ and I usually drop $\mathcal{C}$ when it is clear from the context. Theorem 3.1 characterizes quasi-exponential objects when $E$ is particularized to the identity, the reflectors $J$ and $L$ coincide and $\theta = \xi$. In particular, quasi-exponential objects in **Top** were characterized in [12], while quasi-exponential objects in **PrTop** and **ParaTop** were characterized in [34]. On the other hand, the use of two different reflectors $J$ and $L$ and of a coreflector $E$ other than the identity appears to be of fundamental interest in applications. This is precisely this flexibility that allowed to apply the general scheme in [12], [34] and [33] to a large class of topological problem, including preservation under product of sequentiality, Frchetness and $k$-ness (among others), a unified treatement of product theorems for quotient maps and new links between a topology and the upper Kuratowski convergence on its closed sets.

## 4. $\mathrm{Epi}^L_{JE}$-MODIFIERS

In case $J = I$, I write $\mathrm{Epi}^L_E$ (respectively $\mathrm{Epi}^\sigma_E$) instead of $\mathrm{Epi}^L_{IE}$ (resp. $\mathrm{Epi}^\sigma_{IE}$). If moreover $E = I$, I simply write $\mathrm{Epi}^L$ (resp. $\mathrm{Epi}^\sigma$).

**Proposition 4.1.** *Let $J, L$ be two reflectors and let $E$ be a coreflector. Then*

$$f(\mathrm{Epi}^L_{JE}\xi) \geq \mathrm{Epi}^L_{JE}(f\xi),$$

*for every map $f : |\xi| \to Y$.*

*Proof.* In view of the definition (3.2) of $\mathrm{Epi}^L_{JE}$, it suffices to prove

$$f(\mathrm{Epi}^\sigma_{JE}\xi) \geq \mathrm{Epi}^\sigma_{JE}(f\xi),$$

for every $\sigma = L\sigma$. As $f : \xi \to f\xi$ is a $\mathcal{C}$-morphism, $f^* \in \mathrm{Hom}([f\xi, \sigma], [\xi, \sigma])$. Moreover, $f^* \in \mathrm{Hom}(JE[f\xi, \sigma], JE[\xi, \sigma])$, because $J$ is a reflector and $E$ is a coreflector. Thus $f^{**} : [JE[\xi, \sigma], \sigma] \to [JE[f\xi, \sigma], \sigma]$ is also a $\mathcal{C}$-morphism. Since $i_1 : \mathrm{Epi}^\sigma_{JE}\xi \to [JE[\xi, \sigma], \sigma]$ defined by $i_1(x)(f) = f(x)$ is a $\mathcal{C}$-morphism and $i_2 : \mathrm{Epi}^\sigma_{JE} f\xi \to JE[f\xi, \sigma]$ defined analogously, is also a $\mathcal{C}$-morphism, the following diagram commutes and $i_2 \circ f = f^{**} \circ i_1$ is a $\mathcal{C}$-morphism. 0.8mm *[h]comm1.lp

Thus $f \in \mathrm{Hom}(\mathrm{Epi}^\sigma_{JE}\xi, \mathrm{Epi}^\sigma_{JE} f\xi)$, because $i_2$ is initial. □

On the other hand, $\mathrm{Epi}^L_{JE}$ is isotone because for each $\sigma$, $\mathrm{Epi}^\sigma_{JE}$ is isotone. However, $\mathrm{Epi}^\sigma_{JE}\xi$ is not always comparable to $\xi$. The only observations of interest seem to be

$$\mathrm{Epi}^L_{JE}\xi = \mathrm{Epi}^L_{JE}(\mathrm{Epi}^L_E\xi) \geq \mathrm{Epi}^L_E\xi, \tag{4.1}$$

$$\mathrm{Epi}^L(\mathrm{Epi}^L_{JE}\xi) = \mathrm{Epi}^L_{JE}\xi, \tag{4.2}$$

for every $\xi$. In contrast, when $J = I$, the way of behavior of $\mathrm{Epi}^L_E$ is better understood. Since $i : \xi \to [E[\xi, \sigma], \sigma]$ is a morphism, $\xi \geq \mathrm{Epi}^\sigma_E\xi$ for every $\sigma$, so that $\xi \geq \mathrm{Epi}^L_E\xi$. On the other hand, $\mathrm{Epi}^L_E$ is idempotent because for every $\sigma$,

$$E[\mathrm{Epi}^L_E\xi, \sigma] = E[\xi, \sigma]. \tag{4.3}$$

Indeed, $\xi \geq \mathrm{Epi}^L_E\xi$ so that $[\mathrm{Epi}^L_E\xi, \sigma] \geq [\xi, \sigma]$, and in view of Theorem 3.1, $E[\xi, \sigma] \geq [\mathrm{Epi}^L_E\xi, \sigma]$. Since $\xi \geq \mathrm{Epi}^L_E\xi \geq L\xi$ and since $L$ is a reflector, $\mathrm{Hom}(\xi, \sigma) = \mathrm{Hom}(\mathrm{Epi}^L_E\xi, \sigma)$ for every $\sigma = L\sigma$. Hence $\mathrm{Epi}^L_E$ is idempotent, contractive and isotone. In view of Proposition 4.1,

**Proposition 4.2.** $\mathrm{Epi}^L_E$ *is a (bi)reflector.*

Such reflectors will play a key role in the sequel. Indeed, by Theorem 3.1, $\mathrm{Epi}^L_E\xi \times \tau \geq L(\xi \times \tau)$ for every $\tau = E\tau$, so that

$$L(\xi \times \tau) = L(\mathrm{Epi}^L_E\xi \times \tau). \tag{4.4}$$

4.1. **Commutation of $\mathrm{Epi}_E^L$ with product.**

**Theorem 4.3.** *Let $E$ and $B$ be two finitely productive coreflectors in a toponome $\mathcal{C}$ such that $E \geq B$. Then*

$$\mathrm{Epi}^L\ E\ \mathrm{Epi}_B^L \xi \times \mathrm{Epi}_E^L \tau \geq \mathrm{Epi}_E^L(\xi \times \tau), \tag{4.5}$$

*for every $\tau = B\tau$.*

Before proving this theorem, let me observe that in case $B = E = I$, Theorem 4.3 amounts to

$$\mathrm{Epi}^L \xi \times \mathrm{Epi}^L \tau \geq \mathrm{Epi}^L(\xi \times \tau). \tag{4.6}$$

Hence, by Theorem 3.1, the category $\mathrm{fix}\,\mathrm{Epi}^L$ is *cartesian closed*, i.e., every $\mathrm{Epi}^L$-object is exponential (in the category $\mathrm{fix}\,\mathrm{Epi}^L$). Moreover, in view of [14, Theorem 3.9] (see also [15]) and of (2.3), it is the cartesian closed topological hull of the category $\mathrm{fix}\,L$, provided that $\mathrm{fix}\,L$ is finally dense in $\mathrm{fix}\,\mathrm{Epi}^L$. For example, if $\mathcal{C} = \mathbf{Conv}$, *atomic* (i.e., all but one point are isolated) topologies are rigidly finally dense in **Conv**. Moreover F. Schwarz proved [36, Proposition 4.4] that each epireflective subcategory of **Conv** contains atomic topologies if and only if it contains a finite non-indiscrete space. Thus the above remark applies to a very large class of bireflectors $L$ in **Conv**. Analogously, if $\mathcal{C} =$ **CAP**, the subclass $\{\lambda_{\mathcal{F},f} : \mathcal{F} \in \varphi X, f \in \mathbb{P}^X, X \in \mathbf{Sets}\ \}$ of **AP** is rigidly finally dense in **CAP** [21, Theorem 3.2], with

$$\lambda_{\mathcal{F},f}\mathcal{G}(x) = \begin{cases} f(x) & \text{if } \mathcal{G} \neq (x) \text{ and } \mathcal{G} \geq \mathcal{F} \wedge (x) \\ \infty & \text{if } \mathcal{G} \ngeq \mathcal{F} \wedge (x) \\ 0 & \text{if } \mathcal{G} = (x). \end{cases}$$

I call such spaces *atomic approach spaces.*

**Corollary 4.4.** *The cartesian closed hull of the category* $\mathrm{fix}\,L$ *is the category* $\mathrm{fix}\,\mathrm{Epi}^L$ *provided* $\mathrm{fix}\,L$ *is finally dense in* $\mathrm{fix}\,\mathrm{Epi}^L$.

Hence, Theorem 3.1 allows to describe both exponential objects in reflective subcategories of $\mathcal{C}$ and cartesian closed hulls of such subcategories.

Notice that, by definition of $\mathrm{Epi}_E^L$,

$$\mathrm{Epi}^L \geq \mathrm{Epi}_B^L \geq \mathrm{Epi}_E^L \geq L = \mathrm{Epi}_{\mathrm{Dis}}^L, \tag{4.7}$$

whenever $E$ and $B$ are two coreflectors such that $E \geq B$.

*Proof of Theorem 4.3.* It suffices to prove

$$\mathrm{Epi}_E^\sigma \xi \times E\tau \geq \mathrm{Epi}_E^\sigma(\xi \times \tau), \tag{4.8}$$

for each $\sigma = L\sigma$. Indeed,

$$\mathrm{Epi}_E^L \xi \times E\tau \geq \mathrm{Epi}_E^L(\xi \times \tau), \tag{4.9}$$

follows directly from (4.8) so that it proves that

$$\mathrm{Epi}_B^L \xi \times \tau \geq \mathrm{Epi}_B^L(\xi \times \tau),$$

for every $\tau = B\tau$. Moreover, $\mathrm{Epi}_B^L \geq \mathrm{Epi}_E^L$ because $E \geq B$, so that $\mathrm{Epi}_E^L(E\ \mathrm{Epi}_B^L \xi \times \tau) \geq \mathrm{Epi}_E^L(\xi \times \tau)$. In view of (4.9),

$$E\ \mathrm{Epi}_B^L \xi \times \mathrm{Epi}_E^L \tau \geq \mathrm{Epi}_E^L(\xi \times \tau). \tag{4.10}$$

On the other hand, $\mathrm{Epi}^L$ commutes with finite products ([5]) so that applying $\mathrm{Epi}^L$ to (4.10) we get (4.5).

The evaluation $w_1 : \tau \times [\tau, [\xi, \sigma]] \to [\xi, \sigma]$ is a morphism so that by (2.5), $w_1 \in \mathrm{Hom}\,(E(\tau \times [\tau, [\xi, \sigma]]), [\xi, \sigma])$. Since $E$ is finitely productive, $w_1 \in \mathrm{Hom}\,(E\tau \times E([\tau, [\xi, \sigma]]), [\xi, \sigma])$, so that, by (2.2),

$$E[\tau, [\xi, \sigma]] \geq [E\tau, E[\xi, \sigma]].$$

Moreover, $E[\xi, \sigma] \geq [\mathrm{Epi}_E^\sigma \xi, \sigma]$ by Theorem 3.1, so that

$$E[\tau, [\xi, \sigma]] \geq [E\tau, [\mathrm{Epi}_E^\sigma \xi, \sigma]].$$

Applying the isomorphism (2.1), we get

$$E[\xi \times \tau, \sigma] \geq [\mathrm{Epi}_E^\sigma \xi \times E\tau, \sigma].$$

Once again by Theorem 3.1, the above inequality amounts to

$$\mathrm{Epi}_E^\sigma \xi \times E\tau \geq \mathrm{Epi}_E^\sigma(\xi \times \tau).$$

□

The two following theorems summarize the situations (when $J = I$ in (3.3)) concerning the preservation of $LE$-properties under product on one hand, and concerning product of quotient maps on the other hand. They generalize [34, Theorems 4.6 and 4.7] from **Conv** to an arbitrary toponome $\mathcal{C}$. The proofs are formally the same.

**Theorem 4.5.** *Let $E$ be a finitely productive coreflector and let $L$ be a reflector in a toponome $\mathcal{C}$. The following are equivalent:*

1. *$\alpha \times \tau$ is a $LE$-structure for every $\tau$ in a rigidly finally dense subclass of the* fix $E$*;*
2. *$\alpha \times \tau$ is a $LE$-structure for every $\tau = E\tau$;*
3. *$\alpha \times \tau$ is a $\mathrm{Epi}_E^L E$-structure for every $\tau \geq \mathrm{Epi}^L E\tau$;*
4. *$\alpha \geq \mathrm{Epi}_E^L E\alpha$.*

[5] Apply (4.9) two times with $E = I$.

**Theorem 4.6.** *Let $E$ be a finitely productive coreflector and let $L$ be a reflector in a toponome $\mathcal{C}$. Let $f:\xi_1 \to \tau_1$ be a surjective morphism. Then the following are equivalent:*

1. *$f$ is $\mathrm{Epi}_E^L$-quotient;*
2. *$f\times \mathrm{Id}_\tau$ is $L$-quotient for every $\tau$ in a rigidly finally dense subclass of $\operatorname{fix} E$;*
3. *$f\times \mathrm{Id}_\tau$ is $L$-quotient for every $\tau = E\tau$;*
4. *$f\times g$ is $\mathrm{Epi}_E^L$-quotient for every $\mathrm{Epi}^L$-quotient map $g$ with $\mathrm{Epi}^L E$-range ([6]).*

Recall that, for example, atomic topologies are rigidly finally dense in **Conv**, while metrizable atomic topologies are rigidly finally dense in first countable convergences. On the other hand, atomic approach spaces are rigidly finally dense in **CAP**.

Notice that in case $E$ is the identity and $\operatorname{fix} L$ is finally dense in $\mathcal{C}$, then, in view of Corollary 4.4, Theorem 4.6 states that a map is product-stable in $\operatorname{fix} L$ in the sense of Schwarz [39] ([7]) if and only if it is quotient in the cartesian closed hull of $\operatorname{fix} L$ [39, Theorem 3].

In view of Theorem 3.1, internal characterizations of $\theta$ for which

$$\theta \times J\tau \geq L(\xi\times\tau),$$

for every $\tau = E\tau$ (for various reflectors $L$ and $J$ and coreflector $E$) provides a large collection of applications. The challenging problem is to provide internal characterizations of $\mathrm{Epi}_{JE}^L\xi$ and, as a particulary interesting first step (see Theorems 4.5 and 4.6), of $\mathrm{Epi}_E^L$ whose structural behavior is better than $\mathrm{Epi}_{JE}^L$.

## 5. Elementary convergence-theoretic notions extended to CAP

5.1. **Convergences.** Recall that a *convergence* $\xi$ on a set $X$ is a relation between $X$ and the set $\varphi X$ of filters on $X$

$$x \in \lim_\xi \mathcal{F}$$

that fulfills :

$$(CONV1) \qquad \forall x \in X,\ x\in \lim_\xi (x);$$

$$(CONV2) \qquad \mathcal{G}\geq\mathcal{F} \Longrightarrow \lim_\xi \mathcal{G} \supset \lim_\xi \mathcal{F};$$

$$(CONV3) \quad \forall \mathcal{F},\mathcal{G}\in\varphi X,\ \lim_\xi(\mathcal{F}\wedge\mathcal{G}) = \lim_\xi \mathcal{F} \bigcap \lim_\xi \mathcal{G};$$

[6]Notice that $\mathrm{Epi}_E^L$ being a reflector while $E$ is a coreflector, the range of a $\mathrm{Epi}_E^L$-quotient map is a $\mathrm{Epi}_E^L E$-structure whenever the domain is a $\mathrm{Epi}_E^L E$-structure by Proposition 2.1.

[7]that is, $f\times \mathrm{Id}_\tau$ is $L$-quotient for every $\tau = L\tau$.

where $(x)$ denotes the principal ultrafilter generated by $x$. A convergence $\xi$ is *finer* than a convergence $\vartheta$ ($\xi \geq \vartheta$) whenever $\lim_\xi \mathcal{F} \subset \lim_\vartheta \mathcal{F}$ for every filter $\mathcal{F}$. A map $f : \xi \to \tau$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$. The category **Conv** with convergences as objects and continuous maps as morphisms is a toponome. Indeed, the canonical Hom-structure in **Conv** on the set $\mathrm{Hom}(\xi, \sigma)$ of continuous maps from the convergence $\xi$ to the convergence $\sigma$, is the *continuous convergence* $[\xi, \sigma]$. A filter $\mathcal{F}$ converges to a continuous function $f : \xi \to \sigma$ for $[\xi, \sigma]$ if and only if $f(x) \in \lim_\sigma w(\mathcal{F} \times \mathcal{G})$ for every $x \in |\xi|$ and every filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$. See [4] for details. Thus, I use in **Conv** the same notations as in a toponome $\mathcal{C}$, in particular for initial and final convergences.

The *adherence* of a filter $\mathcal{F}$ is the union of the limits of all filters that are finer than $\mathcal{F}$ :

$$\mathrm{adh}_\xi \mathcal{F} = \bigcup_{\mathcal{G} \geq \mathcal{F}} \lim_\xi \mathcal{G}. \tag{5.1}$$

The *adherence* $\mathrm{adh}_\xi A$ of a subset $A$ of $X$ is the adherence of the principal filter of $A$. A set $V$ is a $\xi$-*vicinity* of $x$ whenever $x \notin \mathrm{adh}_\xi V^c$. We denote $\mathcal{V}_\xi(x)$ the set of all the vicinities of $x$. A subset $A$ of $X$ is $\xi$-*closed* whenever for every filter $\mathcal{F}$ with $A \in \mathcal{F}$, one has $\lim_\xi \mathcal{F} \subset A$. A set is $\xi$-*open* if its complement is $\xi$-closed. The *closure* $\mathrm{cl}_\xi A$ is the least closed set that includes $A$. A set $V$ is a *neighborhood* of $x$ if and only if $x \notin \mathrm{cl}_\xi V^c$. The set of all the neighborhoods is denoted by $\mathcal{N}_\xi(x)$.

A convergence $\xi$ is a *topology* if $x \in \lim_\xi \mathcal{F}$ amounts to $\mathcal{F} \geq \mathcal{N}_\xi(x)$; a *pretopology* if $x \in \lim_\xi \mathcal{F}$ amounts to $\mathcal{F} \geq \mathcal{V}_\xi(x)$; a *pseudotopology* if and only if

$$\lim_\xi \mathcal{F} = \bigcap_{\mathcal{U} \in \beta(\mathcal{F})} \lim_\xi \mathcal{U},$$

where $\beta(\mathcal{F})$ denotes the set of all the ultrafilters finer than $\mathcal{F}$. Analogously, $\beta(X)$ denotes the set of ultrafilters on $X$.

All these classes are closed for arbitrary suprema in the complete lattice of convergences. Moreover, the initial convergence of a topology (resp., pretopology, pseudotopology) is a topology (resp., pretopology, pseudotopology). Hence the above classes (together with continuous maps) are bireflective subcategories of **Conv**. In the case of topologies, pretopologies and pseudotopologies, the reflectors are denoted by $T, P$ and $S$ respectively, and are called the *topologizer, pretopologizer and pseudotopologizer*. As mentioned before, various classes of quotient maps classically used in topology can be characterized via these reflectors, e.g., quotient maps are $T$-quotients, hereditary quotient means $P$-quotient and biquotient amounts to $S$-quotient.

A subset $A$ of $|\xi|$ is *compact* (for $\xi$) if $\lim_\xi \mathcal{U} \bigcap A \neq \varnothing$ for every $\mathcal{U} \in \beta(A)$. A convergence is *locally compact* if every convergent filter contains a compact set. The class of locally compact convergences is coreflective in **Conv** and the associated coreflector called *compact localizer* is denoted by $K$. A convergence is *based in a class of filters* $\mathfrak{J}$ or $\mathfrak{J}$-*based* if there exists a filter $\mathcal{G}$ of $\mathfrak{J}$ coarser than $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{G}$ for every $\mathcal{F}$ for which $x \in \lim_\xi \mathcal{F}$. The classes of convergences based in countably based filters and in principal filters are coreflective in **Conv** and the associated coreflectors called *first-countable modifier* and *finitely generated modifier* are denoted by First and Fin respectively. As said before, numerous topological properties are characterized via (2.6), where $J$ is one of the above mentioned reflectors and $E$ one of the above mentioned coreflectors (see [10]).

5.2. **Convergence-approach spaces.** Following [20] and [21], I call *convergence-approach* on $X$ a map $\lambda : \varphi X \to [0,\infty]^X$ which fulfills the properties:

$$\begin{array}{ll} (CAL1) & \forall x \in X,\ \lambda(x)(x) = 0; \\ (CAL2) & \mathcal{G} \geq \mathcal{F} \Longrightarrow \lambda(\mathcal{F}) \geq \lambda(\mathcal{G}); \\ (CAL3) & \forall \mathcal{F}, \mathcal{G} \in \varphi X,\ \lambda(\mathcal{F} \wedge \mathcal{G}) = \lambda(\mathcal{F}) \bigvee \lambda(\mathcal{G}). \end{array}$$

A map $f : (X,\xi) \to (Y,\tau)$ between two convergence-approach spaces is a *contraction* if

$$\tau(f(\mathcal{F}))\ (f(\cdot)) \leq \xi(\mathcal{F})(\cdot),$$

for every $\mathcal{F} \in \varphi X$. The category with convergence-approach spaces as objects and contractions as morphisms is a toponome denoted **CAP** [20]. Each convergence $\xi$ can be considered as a convergence-approach by stating

$$\xi(\mathcal{F})(x) = \begin{cases} 0 & \text{if } x \in \lim_\xi \mathcal{F} \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, **Conv** (together with continuous maps) is included both reflectively and coreflectively in **CAP**. Indeed, if $\lambda$ is a convergence-approach, then its **Conv**-coreflection is $c(\lambda)$ defined by $x \in \lim_{c(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) = 0$, while its **Conv**-reflection is $r(\lambda)$ defined by $x \in \lim_{r(\lambda)} \mathcal{F}$ if and only if $\lambda(\mathcal{F})(x) < \infty$. In **CAP** the canonical Hom-structure is described for example in [21]. If $\xi$ and $\sigma$ are now two convergence-approach spaces, the limit $\lambda$ on the set $\mathrm{Hom}(\xi,\sigma)$ of contractions from $\xi$ to $\sigma$ is given by

$$\lambda(\mathcal{F})(f) = \bigwedge \{\alpha : \underset{\mathcal{G} \in \varphi(|\xi|)}{\forall} \sigma(w(\mathcal{G} \times \mathcal{F})) \circ f(\cdot) \leq \xi(\mathcal{G})(\cdot) \bigvee \alpha\},$$

and is called *continuous convergence-approach*. Since $\lambda$ coincide with the continuous convergence in case $\xi$ and $\sigma$ are convergences, I extend to **CAP** the notation of **Conv** and use $[\xi, \sigma]$ instead of $\lambda$. Of course both have the properties of the Hom-structure in a toponome.

A convergence-approach $\lambda$ is a *pseudo-approach space* [21] if

$$(PSAP) \quad \forall \mathcal{F} \in \varphi X,\ \lambda(\mathcal{F}) = \bigvee_{\mathcal{U} \in \beta(\mathcal{F})} \lambda(\mathcal{U});$$

and it is a *pre-approach space* [20] if $(CAL3)$ is strengthened to

$$(PRAP) \quad \lambda(\bigwedge_{j \in J} \mathcal{F}_j) = \bigvee_{j \in J} \lambda(\mathcal{F}_j), \text{ for any family } (\mathcal{F}_j)_{j \in J} \text{ of filters.}$$

The category **PSAP** of pseudo-approach spaces contains the category **PsTop** of pseudotopologies and the category **PRAP** of pre-approach spaces contains the category **PrTop** of pretopologies both reflectively and coreflectively (via the restrictions of $c$ and $r$).

An *approach space* is a pre-approach space fulfilling

$$(AP) \quad \text{for any } \mathcal{F} \in \varphi X \text{ and any selection } (\mathcal{F}_x)_{x \in X} \text{ of filters,}$$
$$\lambda(\bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} \mathcal{F}_x)(\cdot) \leq \lambda(\mathcal{F})(\cdot) + \bigvee_{x \in X} \lambda(\mathcal{F}_x)(x).$$

The category **Top** of topological spaces (with continuous maps) is a reflective and coreflective (via the restrictions of $r$ and $c$) subcategory of the category **AP** of approach spaces [25] (see Figure **??**). There are several other equivalent descriptions of **AP** and **PRAP** (see [26] and [22] for details). An approach space of particular interest is $\mathbb{P}$, defined on $[0, \infty]$ by

$$\mathbb{P}(\mathcal{F})(x) = (x - \bigvee_{F \in \mathcal{F}} \bigwedge F) \vee 0.$$

The space $\mathbb{P}$ is initially dense in **AP** [21, Theorem 3.7]. From $\mathbb{P}$ we derive the pre-approach space $\mathbb{P}^{\triangle}$ on $[0, \infty] \bigcup \{x_0\}$ by stating

$$\mathbb{P}^{\triangle}(\mathcal{F})(y) = \begin{cases} \mathbb{P}(\mathcal{F}_{|\ [0,\infty]})(y) & \text{if } y \in [0, \infty] \\ 0 & \text{if } y = x_0. \end{cases}$$

The space $\mathbb{P}^{\triangle}$ is initially dense in **PRAP** [21, Theorem 4.1].

5.3. **Reflectors and coreflectors.** Two families of subsets $\mathcal{A}$ and $\mathcal{B}$ *mesh* $(\mathcal{A} \# \mathcal{B})$ if $A \bigcap B \neq \emptyset$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. If $\{A\} \# \mathcal{B}$, I denote $A \# \mathcal{B}$ or $A \in \mathcal{B}^{\#}$.

If a class of filters $\mathfrak{J}(\xi)$ (possibly depending of the convergence $\xi$) fulfills

**Condition 5.1.**

$$\xi \geq \theta \Longrightarrow \mathfrak{J}(\xi) \supset \mathfrak{J}(\theta);$$
$$\mathfrak{J}(\mathrm{Adh}_{\mathfrak{J}}\xi) = \mathfrak{J}(\xi);$$
$$f : \xi \to \tau \text{ and } \mathcal{H} \in \mathfrak{J}(\tau) \Longrightarrow f^{-}\mathcal{H} \in \mathfrak{J}(\xi);$$

then the map $\mathrm{Adh}_{\mathfrak{J}}$ defined by

$$\lim_{\xi} \mathcal{F} = \bigcap_{\mathfrak{J} \ni \mathcal{H} \# \mathcal{F}} \mathrm{adh}_{\xi} \mathcal{H} \tag{5.2}$$

is a reflector in **Conv**. The notion of adherence (5.1) available in **Conv** has already been extended to **CAP** [20] via

$$(\mathrm{adh}_{\xi} \mathcal{H})(x) = \bigwedge_{\mathcal{U} \in \beta(\mathcal{H})} \xi(\mathcal{U})(x) = \bigwedge_{\mathcal{G} \# \mathcal{H}} \xi(\mathcal{G})(x).$$

Analogously, if the class of filters $\mathfrak{J}(\cdot)$ verifies

**Condition 5.2.**

$$\xi \geq \theta \Longrightarrow \mathfrak{J}(\xi) \supset \mathfrak{J}(\theta);$$
$$\mathfrak{J}(\mathrm{Base}_{\mathfrak{J}}\xi) = \mathfrak{J}(\xi);$$
$$f : \xi \to \tau \text{ and } \mathcal{H} \in \mathfrak{J}(\xi) \Longrightarrow f\mathcal{H} \in \mathfrak{J}(\tau);$$

then the map $\mathrm{Base}_{\mathfrak{J}}$ defined by

$$x \in \lim_{\xi} \mathcal{F} \iff \underset{\mathfrak{J} \ni \mathcal{H} \leq \mathcal{F}}{\exists} x \in \lim_{\xi} \mathcal{H}. \tag{5.3}$$

is a coreflector in **Conv**.

I extend these definitions to **CAP** via

$$\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F})(x) = \bigvee_{\mathfrak{J} \ni \mathcal{H} \# \mathcal{F}} \mathrm{adh}_{\xi} \mathcal{H}(x) = \bigvee_{\mathfrak{J} \ni \mathcal{H} \# \mathcal{F}} \bigwedge_{\mathcal{U} \in \beta(\mathcal{H})} \xi(\mathcal{U})(x) \tag{5.4}$$

and

$$\mathrm{Base}_{\mathfrak{J}}\xi(\mathcal{F})(x) = \bigwedge_{\mathfrak{J} \ni \mathcal{H} \leq \mathcal{F}} \xi(\mathcal{H})(x). \tag{5.5}$$

Notice that, as in **Conv**, for each $\mathcal{H} \in \mathfrak{J}(\xi) = \mathfrak{J}(\mathrm{Adh}_{\mathfrak{J}}\xi)$,

$$\mathrm{adh}_{\xi} \mathcal{H}(\cdot) = \mathrm{adh}_{\mathrm{Adh}_{\mathfrak{J}}\xi} \mathcal{H}(\cdot). \tag{5.6}$$

**Theorem 5.3.** *If $\mathfrak{J}$ verifies conditions 5.1 then* $\mathrm{Adh}_{\mathfrak{J}}$ *is a bireflector in* **CAP**. *If $\mathfrak{J}$ verifies conditions 5.2 then* $\mathrm{Base}_{\mathfrak{J}}$ *is a bicoreflector in* **CAP**.

*Proof.* $\mathrm{Adh}_{\mathfrak{J}}$ is isotone. If $\xi \geq \theta$, $\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F})(\cdot) = \bigvee_{\mathfrak{J}(\xi)\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_{\xi}\,\mathcal{H}(\cdot) \geq \bigvee_{\mathfrak{J}(\theta)\ni\mathcal{H}\#\mathcal{F}} \mathrm{adh}_{\theta}\,\mathcal{H}(\cdot)$ because $\mathfrak{J}(\xi) \supset \mathfrak{J}(\theta)$ and $\mathrm{adh}_{\xi}\,\mathcal{H}(\cdot) \geq \mathrm{adh}_{\theta}\,\mathcal{H}(\cdot)$. $\mathrm{Adh}_{\mathfrak{J}}$ is contractive because $\xi(\mathcal{F})(\cdot) \geq \mathrm{adh}_{\xi}\,\mathcal{H}(\cdot)$ for every $\mathcal{H}\#\mathcal{F}$.

$\mathrm{Adh}_{\mathfrak{J}}$ is idempotent because of (5.6).

Hence, $\mathrm{Adh}_{\mathfrak{J}}\xi$ is the greatest convergence approach fixed by $\mathrm{Adh}_{\mathfrak{J}}$ and coarser than $\xi$.

Moreover, $\mathrm{Adh}_{\mathfrak{J}}(f^{-}\tau) \geq f^{-}(\mathrm{Adh}_{\mathfrak{J}}\tau)$ for every contraction $f : \xi \to \tau$. Indeed,$\mathrm{Adh}_{\mathfrak{J}}(f^{-}\tau)(\mathcal{F})(x) = \bigvee_{\mathfrak{J}(f^{-}\tau)\ni\mathcal{H}\#\mathcal{F}} \bigwedge_{\mathcal{W}\in\beta(\mathcal{H})} f^{-}\tau(\mathcal{W})(x)$.

Since $f^{-}\mathcal{H} \in \mathfrak{J}(f^{-}\tau)$ and $f^{-}\mathcal{H}\#\mathcal{F}$ for each $\mathcal{H} \in \mathfrak{J}(\tau)$ such that $\mathcal{H}\#f(\mathcal{F})$,

$$\mathrm{Adh}_{\mathfrak{J}}(f^{-}\tau)(\mathcal{F})(x) \geq \bigvee_{\mathfrak{J}(\tau)\ni\mathcal{H}\#f(\mathcal{F})} \bigwedge_{\mathcal{W}\in\beta(f^{-}\mathcal{H})} f^{-}\tau(\mathcal{W})(x)$$

$$= \bigvee_{\mathfrak{J}(\tau)\ni\mathcal{H}\#f(\mathcal{F})} \bigwedge_{f(\mathcal{W})\in\beta(\mathcal{H})} \tau(f(\mathcal{W}))(f(x)) = f^{-}(\mathrm{Adh}_{\mathfrak{J}}\tau)(\mathcal{F})(x).$$

Thus $\mathrm{Adh}_{\mathfrak{J}}$ is a bireflector. The proof is similar for $\mathrm{Base}_{\mathfrak{J}}$. □

In the sequel, the class $\mathfrak{J}$ is always supposed to fulfill the right conditions to ensure that $\mathrm{Adh}_{\mathfrak{J}}$ is a reflector and $\mathrm{Base}_{\mathfrak{J}}$ a coreflector. $\mathrm{Adh}_{\mathfrak{J}}$ and $\mathrm{Base}_{\mathfrak{J}}$ really extend the existing reflectors and coreflectors in **Conv**.

**Lemma 5.4.** *If* $\xi \in |CAP|$ *and* $\xi(\cdot) = \mathrm{Adh}_{\mathfrak{J}}\xi(\cdot)$, *then* $r(\xi) = \mathrm{Adh}_{\mathfrak{J}}r(\xi)$ *and* $c(\xi) = \mathrm{Adh}_{\mathfrak{J}}c(\xi)$ *in* **Conv**. *If* $\xi \in |CAP|$ *and* $\xi(\cdot) = \mathrm{Base}_{\mathfrak{J}}\xi(\cdot)$, *then* $r(\xi) = \mathrm{Base}_{\mathfrak{J}}r(\xi)$ *and* $c(\xi) = \mathrm{Base}_{\mathfrak{J}}c(\xi)$ *in* **Conv**.

A class $\mathfrak{J}$ of filters is said to be *composable* if it contains principal filters and if the filter $\mathcal{HG}$ generated by $\{HG : H \in \mathcal{H}, G \in \mathcal{G}\}$ ([8]), is a (possibly degenerate) $\mathfrak{J}$-filter on $Y$ whenever $\mathcal{H}$ is a $\mathfrak{J}$-filter on $X \times Y$ and $\mathcal{G}$ a $\mathfrak{J}$-filter on $X$. In particular, if $\mathfrak{J}$ is composable and if $\mathcal{F}$ and $\mathcal{G}$ are $\mathfrak{J}$-filters such that $\mathcal{F}\#\mathcal{G}$ then $\mathcal{F} \vee \mathcal{G}$ is a $\mathfrak{J}$-filter. For example, the classes of principal filters and of countably based filters are composable, while that of filters generated by sequences is not. In **Conv**, [34, Lemma 2.1] states that $\mathrm{Base}_{\mathfrak{J}}$ is finitely productive if $\mathfrak{J}$ is composable. This can be extended to **CAP** with the observation that

$$\bigwedge_{a\in A}(x \bigvee a) = x \bigvee (\bigwedge_{a\in A} a). \tag{5.7}$$

**Lemma 5.5.** *If* $\mathfrak{J}$ *is a composable class of filters, the coreflector* $\mathrm{Base}_{\mathfrak{J}}$ *is finitely productive in* **CAP**.

[8] $HG = \{y : \exists_{x\in G}(x, y) \in H\}$

The three typical cases for the class $\mathfrak{J}$ are the classes of all filters, of countably based filters and of principal filters. In these cases, I will use the following notations, extending to **CAP** the notations in **Conv**.

| $\mathfrak{J}$ | $\mathrm{Base}_{\mathfrak{J}}$ | $\mathrm{Adh}_{\mathfrak{J}}$ |
|---|---|---|
| all filters | $I$ | $S$ |
| countably based filters | First | $P_\omega$ |
| principal filters | Fin | $P$ |

The following proposition [13, Exemple 3] refines the well known case of ultrafilters [26, Proposition 1.8.29].

**Proposition 5.6.** *Let $\mathcal{B}$ be an isotone family of subsets of $X$ not containing the empty set. Then for every $f : X \to \overline{\mathbb{R}}$,*

$$\lim_{\mathcal{B}} f = \bigvee_{B\in\mathcal{B}} \bigwedge_{x\in B} f(x) = \bigwedge_{A\#\mathcal{B}} \bigvee_{x\in A} f(x);$$

*and*

$$\lim_{\mathcal{B}^{\#}} f = \bigvee_{A\#\mathcal{B}} \bigwedge_{x\in A} f(x) = \bigwedge_{B\in\mathcal{B}} \bigvee_{x\in B} f(x).$$

**Theorem 5.7.** *If $\mathfrak{J}$ is the class of all filters,* $\mathrm{Adh}_{\mathfrak{J}}$ *is the reflector on* **PSAP**. *If $\mathfrak{J}$ is the class of principal filters then* $\mathrm{Adh}_{\mathfrak{J}}$ *is the reflector on* **PRAP**.

*Proof.* The case of all filters is clear.

Let $P$ denote the reflector $\mathrm{Adh}_{\mathfrak{J}}$ in case $\mathfrak{J}$ is the class of principal filters. If $\xi = P\xi$ then $\xi$ is a pre-approach space. Indeed, $\xi(\bigwedge_{\alpha\in\mathcal{A}} \mathcal{F}_\alpha)(\cdot) = \bigvee_{A\#\bigwedge \mathcal{F}_\alpha} \bigwedge_{\mathcal{W}\in\beta(A)} \xi(\mathcal{W})(\cdot)$. Since $A\#\bigwedge_{\alpha\in\mathcal{A}} \mathcal{F}_\alpha \iff \exists_{\alpha_0\in\mathcal{A}} A\#\mathcal{F}_{\alpha_0}$, I conclude that

$$\xi(\bigwedge_{\alpha\in\mathcal{A}} \mathcal{F}_\alpha)(\cdot) = \bigvee_{\alpha\in\mathcal{A}} \bigvee_{A\#\mathcal{F}_\alpha} \bigwedge_{\mathcal{W}\in\beta(A)} \xi(\mathcal{W})(\cdot).$$

Hence $\xi(\bigwedge_{\alpha\in\mathcal{A}} \mathcal{F}_\alpha)(\cdot) = \bigvee_{\alpha\in\mathcal{A}} \xi(\mathcal{F}_\alpha)(\cdot)$.

Moreover, if $\xi$ is a pre-approach space then $\xi = P\xi$. Indeed, given an ultrafilter $\mathcal{U}$,

$$P\xi(\mathcal{U})(x) = \bigvee_{U\in\mathcal{U}} \bigwedge_{\mathcal{W}\in\beta(U)} \xi(\mathcal{W})(x) = \bigvee_{V\in\mathcal{N}_{\beta(|\xi|)}(\{\mathcal{U}\})} \bigwedge_{\mathcal{W}\in V} \xi(\mathcal{W})(x),$$

using the stone transformation $\beta$ from $|\xi|$ to the set $\beta(|\xi|)$ of its ultrafilters. Thus, by Proposition 5.6 applied in $\beta(|\xi|)$,

$$P\xi(\mathcal{U})(x) = \bigwedge_{V\#\mathcal{N}_{\beta(|\xi|)}(\{\mathcal{U}\})} \bigvee_{\mathcal{W}\in V} \xi(\mathcal{W})(x).$$

For every $V\#\mathcal{N}_{\beta(|\xi|)}(\{\mathcal{U}\})$ and every $U \in \mathcal{U}$, there exists an ultrafilter $\mathcal{W}_U \in V$ such that $U \in \mathcal{W}_U$. Thus, $\bigvee_{\mathcal{W}\in V} \xi(\mathcal{W})(x) \geq \bigvee_{U\in\mathcal{U}} \xi(\mathcal{W}_U)(x)$. Since $\xi$ is a pre-approach space, $\bigvee_{U\in\mathcal{U}} \xi(\mathcal{W}_U)(x) = \xi(\bigwedge_{U\in\mathcal{U}} \mathcal{W}_U)(x)$. From $\mathcal{U} \geq \bigwedge_{U\in\mathcal{U}} \mathcal{W}_U$, I conclude that $P\xi(\mathcal{U})(x) \geq \xi(\mathcal{U})(x)$. The result for arbitrary filter follows immediately. $\square$

In view of Figure **??** and of Lemma 5.4, it is natural to call *para-approach convergences* the convergence-approach spaces fixed by $\mathrm{Adh}_{\mathfrak{J}}$ when $\mathfrak{J}$ is the class of countably based filters. I denote **ParaAP** the category of para-approach convergences.

## 6. Commutation-theorems for the reflectors $\mathrm{Adh}_{\mathfrak{J}}$

In this section, I show that for $\mathrm{Adh}_{\mathfrak{J}}$-reflectors associated with a composable class of filters $\mathfrak{J}$, the situation in **Conv** extends to **CAP** without new pathology. Indeed, the following [34, Theorem 7.1] extends from **Conv** to **CAP** without formal change.

**Theorem 6.1.** *Let $\mathfrak{J}$ be a composable class of filters. Then*

$$\mathrm{Adh}_{\mathfrak{J}} = \mathrm{Epi}_{\mathfrak{J}}^{P}.$$

*Proof in* **CAP**. In view of Theorem 3.1, it suffices to show that

$$[\forall \tau = \mathrm{Base}_{\mathfrak{J}}\tau,\ \theta \times \tau \geq P(\xi \times \tau)] \iff \theta \geq \mathrm{Adh}_{\mathfrak{J}}\xi.$$

If $\theta \ngeq \mathrm{Adh}_{\mathfrak{J}}\xi$ then there exists a filter $\mathcal{F}_0$ and a point $x_0$ in $|\xi|$ such that $\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F}_0)(x_0) > \theta(\mathcal{F}_0)(x_0)$. Thus, there exists a $\mathfrak{J}$-filter $\mathcal{H}_0\#\mathcal{F}_0$ such that $\mathrm{adh}_\xi\,\mathcal{H}_0(x_0) > \theta(\mathcal{F}_0)(x_0)$. Let $\tau$ denote the convergence-approach on a copy of $|\xi|$ defined by $\tau(y)(y) = 0$ for every $y \in |\xi|$, $\tau(\mathcal{G})(x_0) = 0$ whenever $\mathcal{G} \geq \mathcal{H}_0 \wedge (x_0)$ and $\tau(\mathcal{F})(x) = \infty$ otherwise. Notice that $\tau$ is $\mathfrak{J}$-based ($\mathcal{H}_0$ and fixed ultrafilters are $\mathfrak{J}$-filters). However, $\theta \times \tau \ngeq P(\xi \times \tau)$. Indeed,

$$P(\xi\times\tau)(\mathcal{F}_0\vee\mathcal{H}_0\times\mathcal{F}_0\vee\mathcal{H}_0)(x_0,x_0) = \bigvee_{A\#(\mathcal{F}_0\vee\mathcal{H}_0)^{\otimes 2}} \bigwedge_{\mathcal{U}\in\beta(A)} (\xi\times\tau)(\mathcal{U})(x_0,x_0).$$

Hence,

$$P(\xi \times \tau)(\mathcal{F}_0 \vee \mathcal{H}_0 \times \mathcal{F}_0 \vee \mathcal{H}_0)(x_0, x_0) \geq \bigwedge_{\mathcal{U}\in\beta(A_0)} (\xi \times \tau)(\mathcal{U})(x_0, x_0),$$

where $A_0 = \{(x, x) : x \neq x_0\}$. Since $\mathcal{U} \in \beta(A_0)$, there exists a filter $\mathcal{G}$ on $|\xi|$ such that $\mathcal{U} = \mathcal{G} \times \mathcal{G}$. If $\mathcal{G} = (y)$ for $y \neq x_0$, $\tau(y)(x_0) = \infty$.

If $\mathcal{G} \ngeq \mathcal{H}_0$ then $\tau(\mathcal{G})(x_0) = \infty$. If $\mathcal{G} \geq \mathcal{H}_0$ then $\tau(\mathcal{G})(x_0) = 0$ but $\xi(\mathcal{G})(x_0) > \theta(\mathcal{F}_0)(x_0)$. Consequentely,

$$\bigwedge_{\mathcal{U}\in\beta(A_0)} (\xi \times \tau)(\mathcal{U})(x_0, x_0) > \theta(\mathcal{F}_0)(x_0),$$

so that

$$P(\xi\times\tau)(\mathcal{F}_0\vee\mathcal{H}_0\times\mathcal{F}_0\vee\mathcal{H}_0)(x_0,x_0) > \theta(\mathcal{F}_0\vee\mathcal{H}_0)(x_0)\bigvee\tau(\mathcal{F}_0\vee\mathcal{H}_0)(x_0).$$

Conversely, let $\tau$ be a $\mathfrak{J}$-based convergence approach space. In view of (5.7), it suffices to show that for every $\mathfrak{J}$-filter $\mathcal{G}$ (and every filter $\mathcal{F}$, every $(x, y) \in |\xi \times \tau|$)

$$\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F})(x)\bigvee\tau(\mathcal{G})(y) \geq P(\xi \times \tau)(\mathcal{F} \times \mathcal{G})(x, y).$$

By composability, the filter $A\mathcal{G}$ is a $\mathfrak{J}$-filter for every $A\#\mathcal{F} \times \mathcal{G}$. Moreover $A\mathcal{G}\#\mathcal{F}$. Thus $\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F})(x) \geq \mathrm{adh}_\xi A\mathcal{G}(x)$. Since

$$\bigvee_{A\#\mathcal{F}\times\mathcal{G}} \mathrm{adh}_\xi A\mathcal{G}(x)\bigvee\tau(\mathcal{G})(y) \geq \bigvee_{A\#\mathcal{F}\times\mathcal{G}} \mathrm{adh}_{\xi\times\tau} A(x, y)$$

I conclude that

$$\mathrm{Adh}_{\mathfrak{J}}\xi(\mathcal{F})(x)\bigvee\tau(\mathcal{G})(y) \geq P(\xi \times \tau)(\mathcal{F} \times \mathcal{G})(x, y).$$

□

In particular $S = \mathrm{Epi}^P$, so that in view of Corollary 4.4,

**Corollary 6.2.** $\mathrm{fix}\, S$ *is the cartesian-closed hull of* $\mathrm{fix}\, P$*.*

Hence **PsTop** is the cartesian-closed hull of **PrTop** [5, Thorme II.4.1] and **PSAP** is the cartesian-closed hull of **PRAP** [21, Corollary 5.10].

As **PrTop** in **Conv**, **PRAP** is simple in **CAP**. Indeed, the pre-approach space $\mathbb{P}^\triangle$ is initially dense in **PRAP** [21]. Hence,

$$\mathrm{Adh}_{\mathfrak{J}} = \mathrm{Epi}_{\mathfrak{J}}^{\mathbb{P}^\triangle}. \tag{6.1}$$

Thus Theorem 6.1 recovers (for $\mathfrak{J}$ the class of all filters) [21, Theorem 5.9]. In view of Theorem 6.1 and Lemma 5.5, Theorem 4.3 applies with $E$ and $B$ two coreflectors on convergence-approach spaces based in composable classes of filters to the effect that

**Corollary 6.3.** *Let* $\mathfrak{D}$ *and* $\mathfrak{J}$ *be two composable classes of filters such that* $\mathfrak{D} \subset \mathfrak{J}$*. Then*

$$S\mathrm{Base}_{\mathfrak{D}}\mathrm{Adh}_{\mathfrak{J}}\xi \times \mathrm{Adh}_{\mathfrak{D}}\tau \geq \mathrm{Adh}_{\mathfrak{D}}(\xi \times \tau), \tag{6.2}$$

*for every* $\tau = \mathrm{Base}_{\mathfrak{J}}\tau$*.*

Moreover the commutation theorem [34, Proposition 7.3] for $\mathrm{Adh}_{\mathfrak{J}}$-reflectors also extends from **Conv** to **CAP** without formal change.

**Proposition 6.4.** *Let $\mathfrak{D}$ and $\mathfrak{J}$ be two composable classes of filters. If*

$$\theta \times \mathrm{Adh}_{\mathfrak{D}}\tau \geq P(\xi \times \tau) \tag{6.3}$$

*for every $\tau = \mathrm{Base}_{\mathfrak{J}}\tau$, then*

$$\theta \geq S\mathrm{Base}_{\mathfrak{D}}\mathrm{Adh}_{\mathfrak{J}}\xi.$$

*Proof in* **CAP**. Assume that $\theta \ngeq S\mathrm{Base}_{\mathfrak{D}}\mathrm{Adh}_{\mathfrak{J}}\xi$. Then there exists $x_0 \in |\xi|$ and a filter $\mathcal{F}_0$ on $|\xi|$ such that

$$\bigvee_{\mathcal{U}\in\beta(\mathcal{F}_0)} \bigwedge_{\mathfrak{D}\ni\mathcal{L}\leq\mathcal{U}} \bigvee_{\mathfrak{J}\ni\mathcal{H}\#\mathcal{L}} \mathrm{adh}_\xi\,\mathcal{H}(x_0) > \theta(\mathcal{F}_0)(x_0).$$

Hence, there exists $\mathcal{U}_0 \in \beta(\mathcal{F}_0)$ such that for every $\mathfrak{D}$-filter $\mathcal{L}$ coarser than $\mathcal{U}_0$ there exists a $\mathfrak{J}$-filter $\mathcal{H}_{\mathcal{L}}$ meshing $\mathcal{L}$ such that $\mathrm{adh}_\xi\,\mathcal{H}_{\mathcal{L}}(x_0) > \theta(\mathcal{F}_0)(x_0)$. Let $\tau$ be the convergence approach on $|\xi|$ defined as follows. $\tau(y)(y) = 0$ for every $y \in |\xi|$, $\tau(\mathcal{G})(x_0) = 0$ if and only if $\mathcal{G} \geq \mathcal{H}_{\mathcal{L}} \wedge (x_0)$ for some $\mathfrak{D}$-filter $\mathcal{L} \leq \mathcal{U}_0$, and $\tau(\mathcal{M})(x) = \infty$ otherwise. Notice that $\tau$ is $\mathfrak{J}$-based. Since $\mathcal{U}_0 \times \mathcal{U}_0$ meshes the diagonal $\Delta$ of $|\xi| \times |\xi|$,

$$P(\xi \times \tau)(\mathcal{U}_0 \times \mathcal{U}_0)(x_0, x_0) \geq \bigwedge_{\mathcal{W}\in\beta(\Delta\setminus(x_0,x_0))} (\xi \times \tau)(\mathcal{W})(x_0, x_0).$$

Each of these ultrafilters $\mathcal{W}$ are of the form $\mathcal{G} \times \mathcal{G}$ for some ultrafilter $\mathcal{G}$ on $|\xi|$. If $\mathcal{G} = (y)$ for some $y \neq x_0$ or if $\mathcal{G} \ngeq \mathcal{H}_{\mathcal{L}}$ for every $\mathfrak{D}$-filter $\mathcal{L} \leq \mathcal{U}_0$ then $\tau(\mathcal{G})(x_0) = \infty$. If $\mathcal{G} \geq \mathcal{H}_{\mathcal{L}}$ for some $\mathfrak{D}$-filter $\mathcal{L} \leq \mathcal{U}_0$, then $\tau(\mathcal{G})(x_0) = 0$ but $\xi(\mathcal{G})(x_0) > \theta(\mathcal{F}_0)(x_0)$. Hence,

$$P(\xi \times \tau)(\mathcal{U}_0 \times \mathcal{U}_0)(x_0, x_0) > \theta(\mathcal{F}_0)(x_0) \geq \theta(\mathcal{U}_0)(x_0).$$

On the other hand, $\mathrm{Adh}_{\mathfrak{D}}\tau(\mathcal{U}_0)(x_0) = 0$ because $0 = \tau(\mathcal{H}_{\mathcal{L}})(x_0) \geq \mathrm{adh}_\tau\,\mathcal{L}(x_0)$ for every $\mathfrak{D}$-filter $\mathcal{L}$ meshing $\mathcal{U}_0$. Thus,

$$P(\xi \times \tau)(\mathcal{U}_0 \times \mathcal{U}_0)(x_0, x_0) > \theta(\mathcal{U}_0)(x_0) \bigvee \mathrm{Adh}_{\mathfrak{D}}\tau(\mathcal{U}_0)(x_0).$$

□

Consequently, [34, Theorem 7.5] also extends to **CAP**, only replacing the initially dense pretopology $\Lambda$ by the initially dense pre-approach space $\mathbb{P}^\triangle$.

**Theorem 6.5.** *Let $\mathfrak{D}$ and $\mathfrak{J}$ be two composable classes of filters such that $\mathfrak{D} \subset \mathfrak{J}$. The following are equivalent*

(1) $\theta \times \mathrm{Adh}_{\mathfrak{D}}\tau \geq \mathrm{Adh}_{\mathfrak{D}}(\xi \times \tau)$, *for every* $\tau \geq \mathrm{Adh}_{\mathfrak{D}}\mathrm{Base}_{\mathfrak{J}}\tau$*;*
(2) $\theta \times \mathrm{Adh}_{\mathfrak{D}}\tau \geq P(\xi \times \tau)$, *for every* $\tau = \mathrm{Base}_{\mathfrak{J}}\tau$*;*
(3) $\mathrm{Id}_{\xi,\theta} \times f$ *is* $\mathrm{Adh}_{\mathfrak{D}}$*-quotient for every* $\mathrm{Adh}_{\mathfrak{D}}$*-quotient map $f$ with* $\mathrm{Adh}_{\mathfrak{D}}\mathrm{Base}_{\mathfrak{J}}$*-domain;*
(4) $\mathrm{Adh}_{\mathfrak{D}}\mathrm{Base}_{\mathfrak{J}}[\xi, \sigma] \geq [\theta, \sigma]$ *for every* $\sigma = \mathrm{Adh}_{\mathfrak{D}}\sigma$*;*
(5) $\mathrm{Adh}_{\mathfrak{D}}\mathrm{Base}_{\mathfrak{J}}[\xi, \mathbb{P}^\triangle] \geq [\theta, \mathbb{P}^\triangle]$*;*

(6) $\theta \geq S\mathrm{Base}_{\mathfrak{D}}\mathrm{Adh}_{\mathfrak{J}}\xi$.

## 7. Exponential objects

Theorem 6.5 particularizes for the class $\mathfrak{D}$ of principal filters and the class $\mathfrak{J}$ of all filters to a complete characterization of quasi-exponential objects in **PRAP**. Since $\mathrm{Fin}\,\xi = P(\mathrm{Fin}\,\xi)$ whenever $\xi = P\xi$, we obtain the following corollary that make light on the analogy between exponential objects in **PrTop** and in **PRAP**.

**Corollary 7.1.** *$\xi$ is exponential in* $\mathrm{fix}\,P$ *if and only if* $\xi = \mathrm{Fin}\,\xi$.

Corollary 7.1 is true both in **Conv** and **CAP** ([9]) so that in both **PrTop** and **PRAP**, exponential objects are exactly the structures based in principal filters. The characterization of exponential objects in **PRAP** obtained by E. Lowen, R. Lowen and C. Verbeeck [22, Theorem 3.7] was given in a different language : exponential objects in **PRAP** are the pre-metric pre-approach spaces, i.e., the pre-approach structure $\xi$ is determined by a map $d : |\xi| \times |\xi| \to [0, \infty]$ which is zero on the diagonal in the following way

$$\xi(\mathcal{F})(x) = \bigwedge_{F\in\mathcal{F}} \bigvee_{y\in F} d(x, y).$$

To each $\xi$ in **CAP** we can associate a pre-metric $d_\xi$ defined by $d_\xi(x, y) = \xi(y)(x)$. Conversely, a pre-approach $\xi_d$ can be associated to each pre-metric $d$ via $\xi_d(\mathcal{F})(x) = \bigwedge_{F\in\mathcal{F}} \bigvee_{y\in F} d(x, y)$. If $\xi = \mathrm{Fin}\,\xi$ then $\xi(\mathcal{F})(x) = \bigwedge_{F\in\mathcal{F}} \xi(F)(x) = \bigwedge_{F\in\mathcal{F}} \xi(\bigwedge_{y\in F} y)(x)$. If moreover $\xi = P\xi$ then $\xi(\bigwedge_{y\in F} y)(x) = \bigvee_{y\in F} \xi(y)(x)$ so that $\xi$ is a pre-metric. Conversely, a pre-metric convergence-approach space is obviously a pre-approach space fixed by Fin. Hence Corollary 7.1 unifies [22, Theorem 3.7] and [23].

On the other hand, Theorem 6.5 applies with the class $\mathfrak{D}$ of countably based filters and the class $\mathfrak{J}$ of all filters to the effect that

**Corollary 7.2.** *$\xi$ is quasi-exponential in* $\mathrm{fix}\,P_\omega$ *if and only if* $\xi \geq S\,\mathrm{First}\,S\xi$. *In particular, $\xi$ is exponential in* $\mathrm{fix}\,P_\omega$ *if and only if it is bisequential.*

Once again this holds both in the **Conv** and the **CAP** context.

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[9]actually each of the above results proved in **CAP** admits its counterpart in **Conv** as a corollary.

## 8. Product of quotient maps and topological corollaries

The following gathers the topological corollaries of Theorem 6.5 (via Theorems 4.5 and 4.6) in terms of product of quotient maps (see [34] for details). The parenthesis mark an equivalent condition. With the convention that the topological terminology used in **Conv** is extended to **CAP** (see section 5 and [10]), all these results are still valid in **CAP**, as immediate corollaries of Theorem 6.5.

<table>
<tr><th>for every $g$</th><th>$f \times g$ is</th><th>iff $f$ is</th></tr>
<tr><td>hereditarily quotient</td><td rowspan="3">hereditarily quotient</td><td>biquotient with finitely generated range</td></tr>
<tr><td>biquotient with finitely generated range</td><td>hereditarily quotient</td></tr>
<tr><td>hereditarily quotient with Frchet domain</td><td>countably biquotient with finitely generated range</td></tr>
<tr><td>countably biquotient</td><td rowspan="2">hereditarily quotient (countably biquotient)</td><td>biquotient with bisequential range</td></tr>
<tr><td>countably biquotient with strongly Frchet domain</td><td>countably biquotient with bisequential range</td></tr>
<tr><td>biquotient with bisequential range</td><td>countably biquotient</td><td>countably biquotient</td></tr>
<tr><td>biquotient (identity)</td><td>hereditarily quotient (biquotient)</td><td>biquotient</td></tr>
</table>

Analagously, the convention that the terminology of **Conv** extends to **CAP** allows to generalize the following from **Conv** to **CAP**, once again by applying Theorem 6.5 with various specializations for $\mathfrak{D}$ and $\mathfrak{J}$.

**Theorem 8.1.** *The following are equivalent:*

(1) $\xi$ *is strongly Frchet;*
(2) $\mathrm{adh}_{\xi}\,\mathcal{H} \geq \mathrm{adh}_{\mathrm{First}\,\xi}\,\mathcal{H}$ *for each countably based* $\mathcal{H}$*;*
(3) $\xi \times \tau$ *is Frchet for each first-countable* $\tau$*;*
(4) $\xi \times \tau$ *is strongly Frchet for each bisequential* $\tau$*.*

This last theorem extends [34, Theorem 7.10] whose topological counterparts was a combination of well-known results of E. Michael: [30, Proposition 4.D.4] and [30, Proposition 4.D.5] ([10]).

[10]In [30], E. Michael uses the term *countably bisequential* for strongly Frchet.

On the other hand, Theorem 6.5 applies with the class $\mathfrak{D}$ of principal filters and the class $\mathfrak{J}$ of countably based filters to the effect that

**Corollary 8.2.** *Let $\xi = P_\omega \xi$. The following are equivalent:*

(1) $\mathrm{Id}_\xi \times f$ *is hereditarily quotient for every hereditarily quotient map $f$ (equivalently with Frchet domain);*
(2) $\xi \times \tau$ *is Frchet for every Frchet $\tau$;*
(3) $\xi = \mathrm{Fin}\,\xi$*;*
(4) $[\xi, \sigma] = P[\xi, \sigma]$ *for every $\sigma = P\sigma$;*
(5) $P\,\mathrm{First}([\xi, \mathbb{P}^{\triangle}]) \geq [\xi, \mathbb{P}^{\triangle}]$.

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