An introduction to convergence spaces

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Foreword

These notes were written for a series of seminars given at Georgia Southern University during the Fall of 2005. They are intended to be both accessible to students with little experience of topology, and to make the case to mathematicians familiar with topological ideas that topology has enough shortcomings to justify wanting to go beyond. Most results are left as exercises usually with an approriate number of intermediate questions to make it reasonably easy. Because of the double intent of these notes, some of the exercises will seem trivial to the more experienced reader. Hence, only the statements that do not seem obvious to you need to be worked through. Many notions and statments are introduced without proper reference. When there is a reference, I indicate it either as the source I have used or as a placee to find details on the notion or on the result rather than as the first reference.

Part I

Convergence in topological spaces

Chapter 1

Convergence in topological spaces. Why filters?

At the beginning of the 20^{th} century, the quest for a proper foundational framework for Analysis was attracting a lot of attention and efforts. Several different structures were proposed to meet the basic needs to develop analysis: rigorous notions of convergence, of continuity and of connectedness. The solution proposed by Felix Hausdorff (in 1914)– what we know today as *a topology* – became quickly the almost exclusive structure used to deal with convergence as well as with "rubber geometry" considerations. We will come back later on possible reasons for the success of topology.

The model for convergence was Weierstrass' s definition of a convergent sequence of real numbers:

$$x_n \to x \iff \forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N} : \forall n \geq k_\varepsilon, |x_n - x| < \varepsilon. \tag{1.1}$$

Maurice Fréchet proposed (1906) a first successful generalization from $\mathbb{R}$ to abstract spaces by introducing the notion of *metric spaces* (^{1}). A natural generalization of (1.1) to this context is

$$x_n \to x \iff \forall \varepsilon > 0, \exists k_\varepsilon \in \mathbb{N} : \forall n \geq k_\varepsilon, d(x_n, x) < \varepsilon.$$

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[1] Recall that a metric d on a set X is a map $d : X \times X \to [0, \infty)$ satisfying $d(x, x) = 0$ for every $x \in X$, $d(x, y) = d(y, x)$ for every x, y in X and

$$d(x, y) \leq d(x, z) + d(z, y)$$

for every x, y, z in X. A set X endowed with a metric d is called a *metric space* (X, d).

Hence, metric spaces allow to consider convergence of sequences not only in the reals but in abstract spaces, in a geometrically intuitive way. However, metric spaces rapidly appear to be too narrow of a notion for the purpose of Analysis, since basic concepts of convergence in Analysis such as pointwise convergence of a sequence of functions fail to be the notion of convergence for a metric.

Example 1 *Let* A *be an infinite subset of* $\mathbb{R}$. *Let* f_n *and* f *be functions from* A *to* $\mathbb{R}$. *The sequence* $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f, *in symbols* $f_n \underset{p}{\rightarrow} f$, *if* $f_n(x) \rightarrow f(x)$ *in* $\mathbb{R}$ *for every* $x \in A$. *Assume that pointwise convergence on the set* $\mathcal{B}(A,\mathbb{R})$ *of bounded functions from* A *to* $\mathbb{R}$ *is the convergence of sequences for a metric* d *on* $\mathcal{B}(A,\mathbb{R})$. *For each sequence* $(x_n)_{n\in\mathbb{N}}$ *in* A *satisfying* $x_n \neq x_m$ *whenever* $n \neq m$, *the sequence* $(1_{\{x_n\}})_{n\in\mathbb{N}}$ *where* $1_{\{x_n\}}(x) = 0$ *if* $x \neq x_n$ *and* $1_{\{x_n\}}(x_n) = 1$ *converges pointwise to the zero function* $\overline{0}$. *By our assumption,* $d(1_{\{x_n\}}, \overline{0}) \underset{n\to\infty}{\rightarrow} 0$. *Therefore, for a fixed* $n \in \mathbb{N}$, *the set* $A_n = \{x \in A : d(1_{\{x\}}, \overline{0}) \geq \frac{1}{n}\}$ *is finite. Thus* $A = \bigcup_{n\in\mathbb{N}} A_n$ *is countable. Hence, if* A *is an uncountable subset of* $\mathbb{R}$, *then pointwise convergence of sequences of* $\mathcal{B}(A,\mathbb{R})$ *is NOT the convergence of sequences for a metric.*

A *topology* τ *on* X is a family of subsets of X, called *open*, that contains X and $\varnothing$ and which is stable by finite intersection and arbitrary union. Convergence of sequences is decribed via the notion of *neighborhood.* A subset V of X is a neighborhood of $x \in X$ if there exists an open set O such that $x \in O \subset V$. Let $\mathcal{N}(x)$ be the family of neighborhoods of x.

Exercise 2 *Verify that by declaring a subset* O *of* $\mathbb{R}$ *open if for every* $x \in O$ *there exists* $\varepsilon > 0$ *such that* $\{y : |x-y| < \varepsilon\} \subset O$, *we define a topology on* $\mathbb{R}$, *called* the natural topology of $\mathbb{R}$.

Exercise 3 *Verify that for that topology, a sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* x *in the sense of* (1.1) *if and only if*

$$\forall V \in \mathcal{N}(x), \exists k_V : \{x_n : n \geq k_V\} \subset V.$$

Hence, a natural extension of Weierstrass' s definition of a convergent sequence of real numbers to the general framework of a topological space (X,τ) is

$$x_n \rightarrow x \Longleftrightarrow \forall V \in \mathcal{N}(x), \exists k_V : \{x_n : n \geq k_V\} \subset V. \tag{1.2}$$

The limitation of metric spaces illustrated in Example 1 is overcome by using a topology.

Exercise 4 *Let* $\mathcal{F}(X,\mathbb{R})$ *denote the set of functions from* (X,τ) *to* $\mathbb{R}$ *(with its natural topology). Declare* $U \subset \mathcal{F}(X,\mathbb{R})$ *open for* the topology of pointwise convergence *if for every* $f \in U$ *there exists finitely many points* $x_1...x_n$ *in* X *and* $\varepsilon > 0$ *such that* $|g(x_i) - f(x_i)| < \varepsilon$ *for every* $i \in \{1...n\}$ *and every* $g \in U$*.*

1. *Show that this defines a topology on* $\mathcal{F}(X,\mathbb{R})$*. Denote the resulting topological space by* $\mathcal{F}_p(X,\mathbb{R})$*.*
2. *Show that a sequence* $(f_n)_{n\in\mathbb{N}}$ *of* $\mathcal{F}(X,\mathbb{R})$ *converges pointwise to* $f \in \mathcal{F}(X,\mathbb{R})$ *if and only if* $f_n \to f$ *in* $\mathcal{F}_p(X,\mathbb{R})$*.*

The complement of an open set is called *closed.* Since X is closed (and open) and since an arbitrary intersection of closed sets is closed, there exists, for every $A \subset X$, the smallest closed set containing A, called *closure of* A and denoted $\mathrm{cl}\, A$.

Exercise 5 *Show that*

$$x \in \mathrm{cl}\, A \Longleftrightarrow \exists x_n \in A : x_n \to x, \tag{1.3}$$

for the natural topology of $\mathbb{R}$*.*

Therefore, we would like to interpret the closure of a set A as, roughly speaking, the set of points that can be "reached as limit of something on A”, ideally via (1.3).

As in Calculus, let us call *continuous* a map $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0 : |x - y| < \delta_\varepsilon \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

Exercise 6 *Let* $f : \mathbb{R} \to \mathbb{R}$*, where* $\mathbb{R}$ *carries its natural topology. Show that the following are equivalent:*

1. f *is continuous;*
2. *for every* $x \in \mathbb{R}$*,*
$$x_n \to x \Longrightarrow f(x_n) \to f(x);$$

3. $f^{-1}(O)$ *(*[2]*) is open for every open subset* O *of* $\mathbb{R}$;

4. $f^{-1}(C)$ *is closed for every closed subset* C *of* $\mathbb{R}$;

5. $f(\operatorname{cl} A) \subset \operatorname{cl}(f(A))$ *for every* $A \subset \mathbb{R}$.

In view of the equivalence between 1 and 2 above, we would like to interpret continuity as preservation of convergence. However, as we will see in the next section, while the properties 3, 4 and 5 in Exercise 6 are equivalent for a map $f : (X, \tau) \to (Y, \sigma)$ between two topological spaces, 2 is in general not an equivalent property. Therefore, a map $f : (X, \tau) \to (Y, \sigma)$ between two topological spaces is called *continuous* if the pre-image of every open (closed) subset of Y is an open (closed) subset of X, equivalently if $f(\operatorname{cl}_\tau A) \subset \operatorname{cl}_\sigma(f(A))$ for every subset A of X.

1.1 Inadequacy of sequences

The fact that sequences are not enough for the purpose of Analysis is already clear from the definition of the Riemann integral from Riemann sums: we take a limit on continuously many subdivisions of the interval with continuously many choices of the point at which the function is evaluated in each interval of the subdivison. The definition of the integral is via a limit, but this is not the limit of a sequence. Hence, to give a rigorous definition, there is an obvious need for a theory of convergence for objects more general than sequences.

Moreover, the desired properties (1.3) in Exercise 5 and ($1 \iff 2$) in Exercise 6 do not hold for general topological spaces.

Example 7 (no non-trivial convergent sequence) *Let* X *be an uncountable set. Declare a subset* $A \subset X$ *open if its complement is countable. Verify that this defines a topology* τ *on* X, *called* cocountable topology on X. *Notice that no non constant sequence is convergent for* τ. *Indeed, if we had* $x_n \to x$ *then every open set containing* x *would contain a tail* $\{x_n : n \geq k\}$ *of* $(x_n)_{n\in\mathbb{N}}$ *for some* k. *We can assume that* $(x_n)_{n\in\mathbb{N}}$ *takes infinitely many values* $\{x_{n_p} : p \in \mathbb{N}\}$ *with* $x_{n_{p_1}} \neq x_{n_{p_2}}$ *if* $p_1 \neq p_2$. *But* $\{x_{n_p} \neq x : p \in \mathbb{N}\}^c$ *is an open set containing* x, *but containing no tail of* $(x_n)_{n\in\mathbb{N}}$.

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[2] Recall that if $f : X \to Y$ and $A \subset Y$ then $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Example 8 (Sequences do not suffice to describe the topology of pointwise convergenc
[3] Let $C_p([0,1])$ *denote the set of continuous functions from* $[0,1]$ *to* $\mathbb{R}$ *(*[3]*) endowed with the topology of pointwise convergence, as defined in Exercise 4. Let* $\{r_n : n \in \mathbb{N}\}$ *be a dense subset of* $[0,1]$ *and let* $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ *be a base of* $[0,1]$ *such that the Lebesgue measure of each* U_n *is less than* $\frac{1}{2}$ *and each finite subset of* $[0,1]$ *is contained in one of the* U_n*'s. For each* n*, choose* $f_n \in C_p([0,1])$ *such that* $f_n([0,1]) \subset [0,1]$*,* f_n *is equal to 0 on* $U_n \cup \{r_k : k \leq n\}$ *and big enough on the rest of* $[0,1]$ *to verify* $\int_0^1 f_n dx \geq \frac{1}{2}$*. Let* $\overline{0}$ *denote the zero function. Let* $Z = \{f_n : n \in \mathbb{N}\}$*. We prove that* $\operatorname{cl} Z = Z \cup \{\overline{0}\}$*. Indeed,* $\overline{0} \in \operatorname{cl} Z$ *because any finite subset* F *of* $[0,1]$ *is contained in some* U_n *on which* f_n *is identically zero.*

too much things to explain to a student in this Example?

Now I claim that if $f \in \operatorname{cl} Z \backslash Z$ *then* $f(r_n) = 0$ *for every* n*. Indeed, for every* $n, k > 0$ *there exists* $m_k^n \in \mathbb{N}$ *such that* $\sup_{i \leq n} |f_{m_k^n}(r_i) - f(r_i)| < \frac{1}{k}$*. Moreover* $m_n^n \underset{n}{\to} \infty$*. Otherwise, there exists* m_0 *such that* $f_{m_0}(r_n) = f(r_n)$ *for every* n *and we conclude that* $f \in Z$ *by density of* $\{r_n : n \in \mathbb{N}\}$*. Hence, for a fixed* n*,* $m_k^k \geq n$ *for* k *sufficiently large, so that* $|f_{m_k^k}(r_n) - f(r_n)| = |f(r_n)| < \frac{1}{k}$ *for every* k *sufficiently large. By density of* $\{r_n : n \in \mathbb{N}\}$*, we conclude that* $f = \overline{0}$*.*

Moreover, no sequence on Z *converges to* $\overline{0}$ *because if* $(g_n)_{n \in \mathbb{N}}$ *is a sequence on* Z *then*

$$\int_0^1 g_n dx \geq \frac{1}{2}$$

for every n*, so that by Lebesgue's dominated convergence theorem,* $g_n(x)$ *does not converge to zero for every* x *in a subset of* $[0,1]$ *of positive measure.*

Hence, in general topological spaces, convergence of objects more general than sequences has to be considered.

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[3] where $\mathbb{R}$ carries its natural topology and $[0,1]$ the *topology induced by* $\mathbb{R}$, in which $U \subset [0,1]$ is open iff there exists an open subset V of $\mathbb{R}$ such that $U = V \cap [0,1]$.

Chapter 2

Filter convergence in topological spaces

2.1 Infimum of sequences

In a topological space (X, τ), we have defined convergence of sequences via (1.2). If now $\{(x_n^\alpha)_{n\in\mathbb{N}} : \alpha \in I\}$ is a family of sequences such that $x_n^\alpha \to x$ for every α, we have

$$\forall \alpha \in I, \forall V \in \mathcal{N}(x), \exists k_V^\alpha \in \mathbb{N} : \{x_n^\alpha : n \geq k_V^\alpha\} \subset V.$$

In particular,

$$\forall V \in \mathcal{N}(x), \exists (k_V^\alpha)_{\alpha\in I} \in \mathbb{N}^I : \bigcup_{\alpha\in I}\{x_n^\alpha : n \geq k_V^\alpha\} \subset V. \tag{2.1}$$

It makes sense to say that the family

$$\left\{\bigcup_{\alpha\in I}\{x_n^\alpha : n \geq f(\alpha)\} : f : I \to \mathbb{N}\right\} \tag{2.2}$$

of subsets of X *converges to* x in the sense that every neighborhood of x contains an element of the family. If $\mathcal{A}$ and $\mathcal{B}$ are 2 families of subsets of X, we say that $\mathcal{A}$ *refines* $\mathcal{B}$, in symbols $\mathcal{A} \geq \mathcal{B}$, if

$$\mathcal{A} \geq \mathcal{B} \Longleftrightarrow \forall B \in \mathcal{B}, \exists A \in \mathcal{A} : A \subset B. \tag{2.3}$$

Let us call this family *the infimum of the family of sequences* $\{(x_n^\alpha)_{n\in\mathbb{N}} : \alpha \in I\}$ and denote it by $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\mathbb{N}}$. Then $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\mathbb{N}}$ refines $\mathcal{N}(x)$.

2.2 Filters

A *filter* $\mathcal{F}$ *on* X is a non-empty family of subsets of X satisfying

1. $\varnothing \notin \mathcal{F}$;
2. $A \in \mathcal{F}$ and $B \in \mathcal{F} \Longrightarrow A \cap B \in \mathcal{F}$;
3. $A \subset B$ and $A \in \mathcal{F} \Longrightarrow B \in \mathcal{F}$.

   A family satisfying only conditions 1 and 2 above is called a *filter-base* on X. If $\mathcal{B}$ is a filter-base, then the family

$$\mathcal{B}^{\uparrow} = \{A \subset X : \exists B \in \mathcal{B}, B \subset A\}$$

   is a filter, said to be *generated by* $\mathcal{B}$.

Example 9

- ***neighborhood filter****: the family $\mathcal{N}(x)$ of all neighborhoods of a point x of a topological space (X, τ) is a filter.*
- ***principal filter****: Given $A \subset X$, the family $\{A\}^{\uparrow}$ of supersets of A is a filter on X called* principal filter of A.
- ***elementary filter of a sequence:*** *Given a sequence $(x_n)_{n\in\mathbb{N}}$, the family of its tails*

$$\{\{x_n : n \geq k\} : k \in \mathbb{N}\}$$

  is a filter-base on X. The filter generated by this base is called elementary filter of $(x_n)_{n\in\mathbb{N}}$.
- ***infimum of sequences:*** *The family $\bigwedge_{\alpha\in I} (x_n^{\alpha})_{n\in\mathbb{N}}$ defined by* (2.2) *is also a filter-base on X.*
- ***cofinite filter:*** *Let X be an infinite set. The family*

$$\{A \subset X : \mathrm{card}(X \backslash A) < \omega\}$$

  of subsets of X whose complement is finite is a filter on X.
- ***cocountable filter:*** *Let X be an uncountable set. The family*

$$\{A \subset X : \mathrm{card}(X \backslash A) \leq \omega\}$$

  of subsets of X whose complement is countable is a filter on X.

Subsection 2.3 will show that convergence of filters adequately answers the problem of describing closure and continuity in terms of convergence for general topological spaces. But before we get to that, let us have a closer look at set-theoretic properties of filters.

The set $\mathbb{F}X$ of filters on a set X is partially ordered by inclusion. Notice that because filters are families of subsets closed by supersets, filters $\mathcal{F}$ and $\mathcal{G}$ satisfy

$$\mathcal{F} \subset \mathcal{G} \Longleftrightarrow \mathcal{F} \leq \mathcal{G}$$

in the sense of (2.3). In this case, we say that $\mathcal{G}$ *is finer than* $\mathcal{F}$ or that $\mathcal{F}$ *is coarser than* $\mathcal{G}$.

Example 10 *In view of* (1.2), *a sequence* $(x_n)_{n\in\mathbb{N}}$ *in a topological space* (X,τ) *converges to* x *if and only if its elementary filter is finer than the neighborhood filter of* x. *Hence, the convergence of a sequence only depends on its elementary filter. Therefore, by a convenient abuse of notation, we will from now on identify a sequence with its elementary filter and write*

$$x_n \to x \Longleftrightarrow (x_n)_{n\in\mathbb{N}} \geq \mathcal{N}(x),$$

where the right-hand side of the equivalence means that the elementary filter of $(x_n)_{n\in\mathbb{N}}$ *is finer than* $\mathcal{N}(x)$.

Example 11 *From* (2.1), *we see that if* $x_n^\alpha \to x$ *for every* α, *that is,* $(x_n^\alpha)_{n\in\mathbb{N}} \geq \mathcal{N}(x)$ *for every* α, *then* $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\mathbb{N}} \geq \mathcal{N}(x)$.

Calling $\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\mathbb{N}}$ the *infimum of the sequences* $(x_n^\alpha)_{n\in\mathbb{N}}$ is consistent with the order on filters. Indeed,

Exercise 12

1. *Verify that given two filters* $\mathcal{F}$ *and* $\mathcal{G}$ *on* X, *the filter*

$$\mathcal{F} \wedge \mathcal{G} = \{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}^{\uparrow}$$

is the finest filter, which is coarser than both $\mathcal{F}$ *and* $\mathcal{G}$.

2. *Let* $(\mathcal{F}_\alpha)_{\alpha\in I} \subset \mathbb{F}X$. *Verify that*

$$\bigwedge_{\alpha\in I}\mathcal{F}_\alpha = \left\{\bigcup_{\alpha\in I}F_\alpha : F_\alpha \in \mathcal{F}_\alpha\right\}^{\uparrow}$$

is the finest filter coarser than each $\mathcal{F}_\alpha$.

3. *Verify that* (2.2) *defines a filter-base of the infimum of the elementary filters of the sequences* $(x_n^\alpha)_{\alpha\in I}$.

In contrast, the supremum of two filters $\mathcal{F}$ and $\mathcal{G}$ on X (with respect to the order $\geq$) *does not always exist.* We say that two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh*, in symbols $\mathcal{A}\#\mathcal{B}$, if $A\cap B\neq\varnothing$ whenever $A\in\mathcal{A}$ and $B\in\mathcal{B}$. We also use the notation [7]

$$\mathcal{A}^{\#}=\{H\subset X:\{H\}\#\mathcal{A}\}.$$

Exercise 13 1. *If* $\mathcal{F}\in\mathbb{F}X$ *then*

   (a) $\mathcal{F}\subset\mathcal{F}^{\#}$;

   (b) $A\in\mathcal{F}^{\#}\Longleftrightarrow A^c\notin\mathcal{F}$.

   (c) $\mathcal{F}=\mathcal{F}^{\#\#}$;

2. *If* $(\mathcal{F}_\alpha)_{\alpha\in I}\subset\mathbb{F}X$ *then*

$$\left(\bigwedge_{\alpha\in I}\mathcal{F}_\alpha\right)^{\#}=\bigcup_{\alpha\in I}\mathcal{F}_\alpha^{\#}. \tag{2.4}$$

Exercise 14 *Show that the supremum* $\mathcal{F}\vee\mathcal{G}$ *of two filters* $\mathcal{F}$ *and* $\mathcal{G}$ *on* X *exists if and only if* $\mathcal{F}\#\mathcal{G}$ *and that if it is the case, then*

$$\mathcal{F}\vee\mathcal{G}=\{F\cap G:F\in\mathcal{F},G\in\mathcal{G}\}^{\uparrow}.$$

Example 15 *Identifying subsets of* X *with their principal filters, we will use the following abuse of notation:*
$\{A\cap B\}^{\uparrow}=\{A\}^{\uparrow}\vee\{B\}^{\uparrow}$ *will sometimes be denoted* $A\cap B$ *or* $A\vee B$.
$\{A\cup B\}^{\uparrow}=\{A\}^{\uparrow}\wedge\{B\}^{\uparrow}$ *will be written as* $A\cup B$ *or* $A\wedge B$.

Exercise 16 *Show that for two filters* $\mathcal{F}$ *and* $\mathcal{G}$ *on* X *such that* $\mathcal{F}\#\mathcal{G}$

$$(\mathcal{F}\vee\mathcal{G})^{\#}\subset\mathcal{F}^{\#}\cap\mathcal{G}^{\#}$$

but that the reverse inclusion is not true in general.

Exercise 17 *Let $\mathcal{F} \in \mathbb{F}X$. Show that $\mathcal{F}$ is the principal filter of some subset of X if and only if $\bigcap_{F\in\mathcal{F}} F \in \mathcal{F}$ if and only if $\mathcal{F} = \left\{\bigcap_{F\in\mathcal{F}} F\right\}^{\uparrow}$ is the principal filter of* the kernel $\bigcap_{F\in\mathcal{F}} F$ *of $\mathcal{F}$. Such a filter is called* principal. *We denote by $\mathbb{F}_1$ the class of principal filters.*

In contrast, a filter $\mathcal{F}$ on X is called *free* if $\bigcap_{F\in\mathcal{F}} F = \varnothing$.

Exercise 18 *Let X be an infinite set.*

1. *Show that the cofinite filter of X is the coarsest free filter on X.*
2. *Show that the elementary filter of a free sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X admits the cofinite filter on $\{x_n : n \in \mathbb{N}\}$ as a filter-base.*

If $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$, then either $\bigcap_{F\in\mathcal{F}} F \in \mathcal{F}$ in which case $\mathcal{F}$ is principal, or $\bigcap_{F\in\mathcal{F}} F \notin \mathcal{F}$ which, in view of Exercise 13 (1b), means that $\left(\bigcap_{F\in\mathcal{F}} F\right)^c \# \mathcal{F}$. In this later case, $\mathcal{F} \vee (\cap F)^c$ is a free filter. Hence a filter $\mathcal{F}$ can be viewed as having a *principal part* $\mathcal{F}^\bullet = \left(\bigcap_{F\in\mathcal{F}} F\right)^{\uparrow}$ and a *free part* $\mathcal{F}^\circ = \mathcal{F} \vee \left(\bigcap_{F\in\mathcal{F}} F\right)^c$ each of which might be the degenerate filter.

Exercise 19 *[11] Let $\mathcal{F} \in \mathbb{F}X$. Show that*

$$\mathcal{F} = \mathcal{F}^\bullet \wedge \mathcal{F}^\circ;$$

with the convention that either $\mathcal{F}^\bullet$ or $\mathcal{F}^\circ$ may be the degenerate filter $\{\varnothing\}^{\uparrow} = 2^X$ and that $\mathcal{G} \wedge \{\varnothing\}^{\uparrow} = \mathcal{G}$.

Ultrafilters

A filter $\mathcal{U}$ on X is *an ultrafilter* if it is maximal for $\leq$, that is if there is no strictly finer filter. Let $\mathbb{U}X$ denote the set of ultrafilters on X.

Exercise 20 1. *Show that if $x \in X$, then $\{x\}^{\uparrow} \in \mathbb{U}X$.*

2. *Show that an ultrafilter is either the principal filter of a point – a* fixed ultrafilter *– or a free filter.*

The existence of free ultrafilters depends on the axiom of choice. From now on, we accept the axiom of choice, under which we have:

Proposition 21 *For every $\mathcal{F} \in \mathbb{F}X$, there exists $\mathcal{U} \in \mathbb{U}X$ such that $\mathcal{F} \leq \mathcal{U}$.*

We will write $\mathbb{U}(\mathcal{F})$ for the set of ultrafilters that are finer than the filter $\mathcal{F}$.

Exercise 22

1. *Show that for every $\mathcal{F} \in \mathbb{F}X$*
$$\mathcal{F}^{\#} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \mathcal{U}.$$
2. *Show that for every $\mathcal{F} \in \mathbb{F}X$*
$$\mathcal{F} = \bigwedge_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \mathcal{U}.$$

Exercise 23 *Let $\mathcal{F} \in \mathbb{F}X$. Show that the following are equivalent:*

1. $\mathcal{F} \in \mathbb{U}X$;
2. $\mathbb{U}(\mathcal{F}) = \{\mathcal{F}\}$;
3. $\mathcal{F} = \mathcal{F}^{\#}$;
4. $\forall A \subset X$, $A \in \mathcal{F}$ *or* $A^c \in \mathcal{F}$;
5. $A \cup B \in \mathcal{F} \Longrightarrow A \in \mathcal{F}$ *or* $B \in \mathcal{F}$.

Exercise 24 *Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on X. Show that*

1. *If $\mathcal{F} \# \mathcal{G}$ then*
$$\mathbb{U}(\mathcal{F} \vee \mathcal{G}) = \mathbb{U}(\mathcal{F}) \cap \mathbb{U}(\mathcal{G}).$$
2. $$\mathbb{U}(\mathcal{F} \wedge \mathcal{G}) = \mathbb{U}(\mathcal{F}) \cup \mathbb{U}(\mathcal{G}).$$
3. *Find an example showing that in general*
$$\mathbb{U}\left(\bigwedge_{\alpha \in I} \mathcal{F}_\alpha\right) \not\subseteq \bigcup_{\alpha \in I} \mathbb{U}(\mathcal{F}_\alpha).$$
4. *Show that*
$$\mathbb{U}(\mathcal{F}) \subset \mathbb{U}(\mathcal{G}) \Longleftrightarrow \mathcal{G} \leq \mathcal{F}.$$

2.3 Topological notions in terms of filters

In view of Examples 10 and 11, it is natural to say that a filter $\mathcal{F}$ on a topological space (X,τ) *converges to* $x \in X$ if it is finer than the neighborhood filter of x, that is,

$$\mathcal{F} \to x \Longleftrightarrow \mathcal{F} \geq \mathcal{N}(x). \tag{2.5}$$

We also say that x is *a limit point of* $\mathcal{F}$ and we write $x \in \lim \mathcal{F}$. Notice that $\{x\}^{\uparrow} \to x$ for every $x \in X$.

2.3.1 Atomic topologies and filters

A point x of a topological space is called *isolated* if $\{x\}$ is open, equivalently if $\mathcal{N}(x) = \{x\}^{\uparrow}$, that is, if $\{x\}^{\uparrow}$ is the only filter converging to x. A topology is called *atomic* (or *prime*) if it has at most one non-isolated point. If $\mathcal{F}$ is a filter on Y and $X \subset Y$, then the *trace* $\mathcal{F}_{|X}$ *of* $\mathcal{F}$ *on* X is defined by

$$\mathcal{F}_{|X} = \{F \cap X : F \in \mathcal{F}\}.$$

This is a non-degenerate filter provided that $X \in \mathcal{F}^{\#}$.

If $Y = X \cup \{\infty\}$ is equipped with an atomic topology whose only possibly non-isolated is ∞, then the trace of $\mathcal{N}_Y(\infty)$ on X is a free filter on X ([1]). Conversely, for every free filter $\mathcal{F}$ on X, there exists a unique atomic topology on $X \cup \{\infty\}$ such that the trace on X of the neighborhood filter of ∞ is $\mathcal{F}$. Indeed, this topology is characterized by $\mathcal{N}(\infty) = \{F \cup \{\infty\} : F \in \mathcal{F}\}$.

2.3.2 Open and closed subsets; closure, adherence and limit

Exercise 25 *Let (X,τ) be a topological space and let $A \subset X$.*

1. *Show that A is open if and only if*

$$\mathcal{F} \to x \in A \Longrightarrow A \in \mathcal{F}.$$

2. *Show that A is closed if and only if*

$$\mathcal{F} \# A \Longrightarrow \lim \mathcal{F} \subset A.$$

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[1] This filter on X is degenerate if and only if the topology of Y is discrete, that is, every point of Y is isolated.

Exercise 26 *Let (X,τ) be a topological space, let $x \in X$ and let $A \subset X$. Show that*

$$x \in \operatorname{cl} A \Longleftrightarrow A\#\mathcal{N}(x) \Longleftrightarrow \exists \mathcal{F} \in \mathbb{F}X : A\#\mathcal{F} \text{ and } \mathcal{F} \to x.$$

Exercise 27 *Let (X,τ) be a topological space and let $\mathcal{F} \in \mathbb{F}X$.*

1. *Show that*
$$\bigcap_{F \in \mathcal{F}} \operatorname{cl} F = \bigcup_{\mathcal{G}\#\mathcal{F}} \lim \mathcal{G} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}.$$
The above set is called the adherence of $\mathcal{F}$ *and is denoted* $\operatorname{adh} \mathcal{F}$.

2. *Show that*
$$\operatorname{cl} A = \operatorname{adh}\{A\}^{\uparrow}$$
whenever $A \subset X$.

3. *Let $\operatorname{cl}^{\natural} \mathcal{F}$ denote the filter generated by*
$$\{\operatorname{cl} F : F \in \mathcal{F}\}.$$
Show that
$$\operatorname{adh} \mathcal{F} = \operatorname{adh}\left(\operatorname{cl}^{\natural} \mathcal{F}\right).$$

4. *Show that*
$$\lim \mathcal{F} \subset \operatorname{adh} \mathcal{F}$$
for every $\mathcal{F} \in \mathbb{F}X$.

5. *Show that*
$$\lim \mathcal{U} = \operatorname{adh} \mathcal{U}$$
for every $\mathcal{U} \in \mathbb{U}X$.

Exercise 28 *Let (X,τ) be a topological space and let $\mathcal{F} \in \mathbb{F}X$. Show that*

$$\lim \mathcal{F} = \bigcap_{A\#\mathcal{F}} \operatorname{cl} A = \bigcap_{\mathcal{G}\#\mathcal{F}} \operatorname{adh} \mathcal{G} = \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}.$$

2.3.3 Continuity

Each subset R of a product $X \times Y$ can be interpreted as a (possibly multivalued) map $R : X \rightrightarrows Y$ defined by $R(x) = \{y \in Y : (x,y) \in R\}$. The inverse map $R^{-1} : Y \rightrightarrows X$ is defined by $R^{-1}(y) = \{x \in X : (x,y) \in R\}$. If $\mathcal{F} \in \mathbb{F}X$ then $\{R(F) = \bigcup_{x \in F} R(x) : F \in \mathcal{F}\}$ is a filter-base on Y. We denote by $R(\mathcal{F})$ the filter it generates. If $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$, then

$$\{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}$$

is a filter-base on $X \times Y$ and we denote by $\mathcal{F} \times \mathcal{G}$ the filter it generates.

Exercise 29 *If $R \subset X \times Y$, $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$, show that*

$$R(\mathcal{F}) \# \mathcal{G} \Longleftrightarrow R^{-1}(\mathcal{G}) \# \mathcal{F} \Longleftrightarrow (\mathcal{F} \times \mathcal{G}) \# R.$$

Exercise 30 *Let (X,τ) and (Y,σ) be two topological spaces. Show that the follwing are equivalent:*

1. *$f : X \to Y$ is continuous (i.e., $f^{-1}(O) \in \tau$ whenever $O \in \sigma$);*
2. $$f(\operatorname{cl}_X A) \subset \operatorname{cl}_Y f(A)$$ *for every $A \subset X$;*
3. $$\mathcal{F} \to x \Longrightarrow f(\mathcal{F}) \to f(x)$$ *for every $x \in X$ and every $\mathcal{F} \in \mathbb{F}X$.*

2.3.4 Compactness

A subset K of a topological space (X,τ) is *compact* if every filter that meshes with K has adherent points in K. A family $\mathcal{A}$ of open subsets of X is called *a cover of* $K \subset X$ if $K \subset \bigcup_{A \in \mathcal{A}} A$. We call *subcover (of* K*)* a subfamily of $\mathcal{A}$ that is also a cover of K.

A family $\mathcal{A}$ (of open) subset of X is called *an ideal (of open sets)* if it is stable by finite unions and if $B \in \mathcal{A}$ whenever (B is open and) $B \subset A$ for some $A \in \mathcal{A}$. Notice that a family $\mathcal{A}$ of subsets of X is an ideal if and only if the family $\mathcal{A}_c$ of complements of elements of $\mathcal{A}$ is a filter on X.

Exercise 31 *Let $K \subset (X,\tau)$. Show that the following are equivalent ([2]):*

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[2] Use Exercise 27, in particular (3).

1. *K is compact (i.e., $\mathcal{F}\#K \Longrightarrow \operatorname{adh}\mathcal{F}\#K$);*
2. *$\mathcal{F} = \mathrm{cl}^{\natural}\,\mathcal{F}$ and $\mathcal{F}\#K \Longrightarrow \operatorname{adh}\mathcal{F}\#K$;*
3. *every ultrafilter containing K converges to some point of K;*
4. *every ideal open cover of K has an element that covers K;*
5. *every open cover of K has a finite subcover.*

Exercise 32

1. *Show that if $f : X \to Y$ and $\mathcal{U} \in \mathbb{U}(X)$, then $f(\mathcal{U}) \in \mathbb{U}(Y)$.*
2. *Show that if $f : X \to Y$ and $\mathcal{W} \in \mathbb{U}(Y)$ such that $\mathcal{W}\#f(X)$, then there exits $\mathcal{U} \in \mathbb{U}(X)$ such that $f(\mathcal{U}) = \mathcal{W}$.*
3. *Show that if $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact.*

2.3.5 Product topology and Tychonoff's Theorem

Let X and Y be two sets and let $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ denote the projections from $X \times Y$ onto X and Y. If τ and ξ are topologies on X; and $\tau \subset \xi$, then we say that *τ is coarser than ξ or ξ is finer than τ.*

Exercise 33 *Let (X, τ) and (Y, σ) be two topological spaces.*

1. *Show that the topology on $X \times Y$ with base $\{U \times V : U \in \tau, V \in \sigma\}$ is the coarsest topology on $X \times Y$ making both p_X and p_Y continuous. The topology is called the* product topology *and is denoted $\tau \times \sigma$.*
2. *Show that $\mathcal{M} \in \mathbb{F}(X \times Y)$ converges to (x, y) for $\tau \times \sigma$ if and only if there exists $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$ such that $x \in \lim_{\tau} \mathcal{F}$, $y \in \lim_{\sigma} \mathcal{G}$ and $\mathcal{M} \geq \mathcal{F} \times \mathcal{G}$.*

More generally, let $\{(X_i, \tau_i)\}_{i\in I}$ be a collection of topological spaces and let $\prod_{i\in I} X_i = \left\{(x_i)_{i\in I} : x_i \in X_i\right\}$ .The *Tychonoff product topology* $\prod_{i\in I} \tau_i$ is the coarsest topology on $\prod_{i\in I} X_i$ making each projection $p_i : \Pi X_i \longrightarrow X_i$, continuous. A base for the topology $\prod_{i\in I} \tau_i$ is

$$\left\{ \prod_{i\in I} O_i : O_i \in \tau_i, \text{ where all but finitely many } O_i\text{'s are equal to } X_i \right\}.$$

Let $\mathcal{M} \in \mathbb{F}\left(\prod X_i\right)$. $\mathcal{M}$ converges to $(x_i)_I$ in $\prod_{i\in I} \tau_i$ if and only if for every $i \in I$, there exists $\mathcal{F}_i$ converging to x_i (in τ_i) such that

$$\mathcal{M} \geq \prod_{i\in I} \mathcal{F}_i = \bigvee_{i\in I} p_i^{-1}(\mathcal{F}_i).$$

Theorem 34 *(Tychonoff): Let $(X_i)_I$ be a collection of topological spaces. $\prod_{i\in I} X_i$ is compact if and only if each X_i is compact.*

Exercise 35 *Prove Tychonoff's Theorem using the ultrafilter characterization of compactness.*

Part II

Some of the limitations of topological spaces

Chapter 3

Some non-topological, but "topological-like", phenomena

3.1 Graphs

A directed graph $G = (V, E)$ is formed by a set V of *vertices* and a set of ordered pair of $V \times V$ called *edges.* If $(v_1, v_2) \in E$, we say that there is an *edge from v_1 to v_2*. It is tempting to treat graphs "topologically" by considering that a vertice x *converges to* another vertice y if there is an edge from x to y. In other words, we define the convergence of fixed ultrafilters by $\{x\}^{\uparrow} \to y$ if there is an edge from x to y. But in general, this is not the convergence for a topology.

Exercise 36 *Assume that this convergence is that of a topology τ.*

1. *Show that $N(y) = \{x \in V : x \to y\}$ is the smallest neighborhood of y for τ and that therefore $\{x \in V : x \to y\}$ is open.*
2. *Provide an example showing that we can have $x \in N(y)$ with $N(x) \nsubseteq N(y)$. In other words, $N(y)$ is not open.*

3.2 Convergence almost everywhere

Convergence pointwise is *of topological nature.* Indeed, not only do we have that a sequence of functions converges pointwise if and only if it converges in the topology of pointwise convergence, but the same is true for general fitlers. Specifically,

Exercise 37 *Let* $\mathcal{F} \in \mathbb{F}(C(X,\mathbb{R}))$ *and let* $ev = \langle\cdot|\cdot\rangle : X \times C(X,\mathbb{R}) \to \mathbb{R}$ *be the evaluation map. Show that*

$$f \in \lim_{C_p(X,\mathbb{R})} \mathcal{F} \Longleftrightarrow \forall x \in X, f(x) \in \lim_{\mathbb{R}} \langle x|\mathcal{F}\rangle .$$

In contrast, this is not the case for convergence almost everywhere. Recall that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{F}(X,\mathbb{R})$, where (X,μ) is a measure space *converges almost everywhere* to a function f if $f_n(x) \to f(x)$ for every $x \in X \backslash E$ where $\mu E = 0$. Indeed, there is no topology τ on $\mathcal{F}(X,\mathbb{R})$ for which the convergence of filters is defined by

$$f \in \lim_{\tau} \mathcal{F} \Longleftrightarrow \exists E : \mu E = 0, \forall x \in X \backslash E, f(x) \in \lim_{\mathbb{R}} \langle x|\mathcal{F}\rangle . \tag{3.1}$$

If it was the case, then for instance $\bigwedge_{x\in X} \{1_{\{x\}}\}^{\uparrow} = \{1_{\{x\}} : x \in X\}^{\uparrow}$ would converge for τ to the zero function $\overline{0}$ because each fixed sequence converges almost everywhere to $\overline{0}$. But

$$\left\langle x|\{1_{\{x\}} : x \in X\}^{\uparrow}\right\rangle = \{0,1\}$$

for any $x \in X$. Hence $0 \notin \lim_{\mathbb{R}} \left\langle x|\{1_{\{x\}} : x \in X\}^{\uparrow}\right\rangle$.

Even the convergence almost everywhere *of sequences* is not of topological nature. Indeed, we will produce below an example where the convergence a.e. of sequences fails to have Urysohn's property, which in view of the exercise below, shows that convergence a.e. of sequences does not come from a topology.

Exercise 38 *A convergence of sequences is said to have* Urysohn's property *if* $(x_n)_{n\in\mathbb{N}} \to x$ *whenever* $(x_n)_{n\in\mathbb{N}}$ *has the property that each of its subsequences has a subsequence converging to* x*. Show that the convergence of sequences in a topological space* (X,τ) *has Urysohn's property. [Hint: use Exercise 28].*

Example 39 *Let* $n = k + 2^p$ *with* $0 \leq k < 2^p$*. Define the sequence* $(f_n)_{n\in\mathbb{N}}$ *by*

$$f_n(x) = \begin{cases} 1 \text{ if } x \in \left[\frac{k}{2^p}, \frac{k+1}{2^p}\right] \\ 0 \text{ otherwise} \end{cases} .$$

For any $x \in [0,1]$, *the sequence* $(f_n(x))_{n\in\mathbb{N}}$ *takes the value* 1 *for arbitrarily large values of* n, *so that* $(f_n)_{n\in\mathbb{N}}$ *does* ***not*** *converge almost everywhere to* $\overline{0}$. *However, every subsequence of* $(f_n)_{n\in\mathbb{N}}$ *has a subsequence that converges a.e. to* $\overline{0}$. *Indeed,* $\lim_{n\to\infty} \mu(\{x : f_n(x) = 1\}) = 0$, *so that, given a subsequence* $(f_{n_k})_{k\in\mathbb{N}}$, *there exists for each integer* n_k *an integer* m_k *such that* $\mu\left(\left\{x : f_{n_p}(x) = 1\right\}\right) < 2^{-n_k}$ *for every* $p \geq m_k$. *The sequence* $(f_{n_{m_k}}(x))_{k\in\mathbb{N}}$ *converges to* 0 *for every* $x \notin \bigcap_{k\in\mathbb{N}} \bigcup_{p=k}^{\infty} \{x : f_{n_{m_p}}(x) = 1\}$, *and this set is of measure* 0.

3.3 Continuous convergence of sequences of functions

In 1921, H. Hahn introduced (a particular case of) *continuous convergence* in his book "Theorie der reellen Funktionen" [15]. A sequence $(f_n)_{n\in\mathbb{N}}$ of $C(X,Y)$ *converges continuously to* $f \in C(X,Y)$ if $f_n(x_n) \to f(x)$ in Y, for every convergent sequence $x_n \to x$ in X. Carathéodory suggested in [6] that continuous convergence should replace uniform convergence in Function Theory whenever possible. Continuous convergence is not the convergence of sequences for a topology on $C(X,Y)$ in general, as we will see in the next section.

Chapter 4

Structural limitations

4.1 Function spaces and duality

The power of algebraic duality lies in the exponential rule

$$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}(Y, \mathrm{Hom}(X, Z)) \tag{4.1}$$

where $\mathrm{Hom}(X, Z)$ stands for the set of (say, group or vector space) homomorphisms from X to Z, endowed with the algebraic structure induced by that of Z pointwise (1), and where the equality means that the two sets are in bijection via the *transposition map* ${}^t : \mathrm{Hom}(X \times Y, Z) \to \mathrm{Hom}(Y, \mathrm{Hom}(X, Z))$ defined by ${}^t f(y)(x) = f(x, y)$.

Exercise 40 *Verify that the inverse of the transposition map is* $\widehat{\ } = ev \circ (Id_X \times \cdot) : \mathrm{Hom}(Y, \mathrm{Hom}(X, Z)) \to \mathrm{Hom}(X \times Y, Z)$ *where* $ev = \langle \cdot | \cdot \rangle : X \times \mathrm{Hom}(X, Z) \to Z$ *is the evaluation map defined by* $\langle x | f \rangle = f(x)$ *and* $ev \circ (Id_X \times \cdot)$ *stands from the map associating to each* $f \in \mathrm{Hom}(Y, \mathrm{Hom}(X, Z))$ *the map* $ev \circ (Id_X \times f)$ *(2). In other words, given* $f : Y \to \mathrm{Hom}(X, Z)$*, the map* $\widehat{f}$ *defined by* $\widehat{f}(x, y) = \langle x | f(y) \rangle$ *satisfies* ${}^t\left(\widehat{f}\right) = f$*. On the other hand,* $\widehat{{}^t g} = g$ *whenever* $g : X \times Y \to Z$*.*

In topology, the role of $\mathrm{Hom}(X, Z)$ is played by the set $C(X, Z)$ of continuous maps from X to Z, but there is no canonical topological structure on

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[1] that is, $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$ and $\lambda \cdot g$ is defined by $(\lambda \cdot g)(x) = \lambda \cdot g(x)$.

[2] Note that this is an abuse of notation as the evaluation map is not composed with $(Id_X \times \bullet)$ but with $(Id_X \times f)$, for each f.

$C(X,Z)$, unlike the algebraic structure on $\mathrm{Hom}(X,Z)$. Still, it is natural to ask whether there is a way to canonically defined a topology α on $C(X,Z)$ such that $C(X\times Y,Z)$ and $C(Y,C_\alpha(X,Z))$ are in bijection (via transposition). This is an essential condition to obtain a satisfactory duality theory in the realm of topological spaces [2].

Exercise 41 *Consider the evaluation map* $ev=\langle\cdot|\cdot\rangle : X\times C(X,Z)\to Z$*, and let* α *be a topology on* $C(X,Z)$*.*

1. *Show that* $\langle\cdot|\cdot\rangle : X\times C_\alpha(X,Z)\to Z$ *is (jointly) continuous if and only if*
$$C(X\times Y,Z)\supseteq C(Y,C_\alpha(X,Z))$$
for every topological space Y*, where the inclusion means that* $\widehat{f}:X\times Y\to Z$ *is continuous whenever* $f:Y\to C_\alpha(X,Z)$ *is.*

2. *Show that if*
$$C(X\times Y,Z)\subseteq C(Y,C_\alpha(X,Z))$$
for every topological space Y*, then* α *is coarser than every topology* β *on* $C(X,Z)$ *making* $\langle\cdot|\cdot\rangle : X\times C_\beta(X,Z)\to Z$ *(jointly) continuous, where the inclusion means that* ${}^tf:Y\to C_\alpha(X,Z)$ *is continuous whenever* $f:X\times Y\to Z$ *is.*

Hence, if there exists a topology α on $C(X,Z)$ satisfying

$$C(X\times Y,Z)=C(Y,C_\alpha(X,Z))$$

for every Y, then α is the coarsest topology making the evaluation continuous.

The following result of R. Arens will be proved later, as it is easier to prove using convergence spaces.

Theorem 42 *[1] Let* X *be a completely regular space. If there exists a coarsest topology on* $C(X,[0,1])$ *making the evaluation continuous then* X *is locally compact. In particular, there is* ***no*** *coarsest topology on* $C(\mathbb{Q},[0,1])$ *or on* $C(\mathbb{R}^{\mathbb{N}},[0,1])$*.*

If α makes $\langle\cdot|\cdot\rangle : X\times C_\alpha(X,Z)\to Z$ continuous, then $f(x)\in\lim_Z\langle\mathcal{G}|\mathcal{F}\rangle$ whenever $f\in\lim_\alpha\mathcal{F}$ and $x\in\lim_X\mathcal{G}$. The coarsest *structure* satisfying this condition is the *continuous convergence [8], [5]* for which

$$f\in\lim\nolimits_c\mathcal{F}\Longleftrightarrow f(x)\in\lim\nolimits_Z\langle\mathcal{G}|\mathcal{F}\rangle \text{ whenever } x\in\lim\nolimits_X\mathcal{G}. \qquad \text{(ContConv)}$$

While we will provide explicit examples in the next part, it is not hard to see why this convergence of filters may fail to be that of a topology. Indeed, for a family $(\mathcal{F}_\alpha)_{\alpha\in I}$ of filters that all converge continuously to f_0 and a filter $\mathcal{G} \to x$, we have:

$$\forall \alpha \in I, \forall V \in \mathcal{N}_Z(f_0(x)), \exists F_V^\alpha \in \mathcal{F}_\alpha, \exists G_V^\alpha \in \mathcal{G} : \langle G_V^\alpha | F_V^\alpha \rangle \subset V.$$

But to conclude that $\bigwedge_{\alpha\in I} \mathcal{F}_\alpha$ converges continuously to f_0, we would need, for each $V \in \mathcal{N}_Z(f_0(x))$, to find $G \in \mathcal{G}$ and $F^\alpha \in \mathcal{F}_\alpha$ such that $\left\langle G | \bigcup_{\alpha\in I} F^\alpha \right\rangle \subset V$. The natural candidates are $F^\alpha = F_V^\alpha$ and $G = \bigcap_{\alpha\in I} G_V^\alpha$ but this infinite intersection does not need to be in $\mathcal{G}$ (and may be empty).

4.2 Product of quotient maps

Consider a relation of equivalence $\sim$ on a topological space (X, τ). The *quotient topology* on the quotient set $Y = X/\sim$ is the finest topology on Y making the canonical surjection $q : X \to Y$ (associating to each point its class of equivalence) continuous. In other words, $U \subset Y$ is open in the quotient topology if and only if $q^{-1}(U) \in \tau$. More generally, a surjective map $f : (X, \tau) \to (Y, \sigma)$ is called *quotient* if σ is the finest topology on Y making f continuous (from (X, τ)). Notice that the relation $\sim$ defined by $x \sim y$ if $f(x) = f(y)$ is an equivalence relation on X and that Y can be identified to $X/\sim$ because f is surjective. The map is quotient if and only if $Y = X/\sim$ carries the quotient topology associated to $\sim$.

Exercise 43 *Let $f : (X, \tau) \to (Y, \sigma)$ be a surjection. Show that f is a quotient map if and only if a map $g : (Y, \sigma) \to (Z, \xi)$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \xi)$ is.*

A product of quotient maps may fail to be a quotient map.

Example 44 *[23] Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the canonical surjection, where $\mathbb{R}/\mathbb{Z}$ is endowed with the quotient topology ([3]). I claim that $q \times Id_{\mathbb{Q}} : \mathbb{R} \times \mathbb{Q} \to \mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ is surjective and continuous, but not quotient. Otherwise, it would map*

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[3] Here $\mathbb{R}/\mathbb{Z}$ denotes the quotient $\mathbb{R}/\sim$, where $\sim$ denotes the equivalence relation on $\mathbb{R}$ defined by $x \sim y$ if $x = y$ or $\{x, y\} \subset \mathbb{Z}$.

closed saturated subsets ([4]) *of* $\mathbb{R} \times \mathbb{Q}$ *to closed subsets of* $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$*, but this is not the case. Indeed, let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of irrationals converging to* 0 *in* $\mathbb{R}$ *and for each* n*, let* $(r^n_m)_{m\in\mathbb{N}}$ *be a sequence of rationals converging to* a_n*. The set*

$$A = \left\{ \left(n + \frac{1}{m}, r^n_m \right) : n, m \in \mathbb{N}, m > 1 \right\}$$

is a closed saturated subset of $\mathbb{R} \times \mathbb{Q}$*. But* $(q \times Id_{\mathbb{Q}})(A) = A$ *is not closed in* $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ *because* $(q(0), 0) \notin A$ *but each neighborhood of* $q(0)$ *in* $\mathbb{R}/\mathbb{Z}$ *contains* $[n + \frac{1}{m}] = [\frac{1}{m}]$ *for* m *sufficiently large, and each neighborhood of* 0 *in* $\mathbb{Q}$ *contains and a set of the form* $\{r^n_m : m > k(n), n > p_0\}$ *where* $k : \mathbb{N} \to \mathbb{N}$*. Therefore each neighborhood of* $(q(0), 0)$ *in* $\mathbb{R}/\mathbb{Z} \times \mathbb{Q}$ *meets* A *and* $(q(0), 0) \in \mathrm{cl}_{\mathbb{R}/\mathbb{Z}\times\mathbb{Q}} A$*.*

4.2.1 Quotient maps and function spaces

The example above can be used to provide an example of a function space on which the continuous convergence is not topological. Indeed,

Theorem 45 *If for every topological space* Z*, there is the coarsest topology* α *on* $C(X, Z)$ *making the evaluation continuous, then* $Id_X \times f : X \times Y \to X \times W$ *is quotient for every quotient map* $f : Y \to W$*.*

Proof. In view of Exercise 43, to show that $Id_X \times f : X \times Y \to X \times W$ is quotient, we only need to show that $g : X \times W \to Z$ is continuous whenever $g \circ (Id_X \times f) : X \times Y \to Z$ is. If $g \circ (Id_X \times f)$ is continuous, then by (2), the map ${}^t\left(g \circ (Id_X \times f)\right) : Y \to C_\alpha(X, Z)$ is continuous. On the other hand, ${}^tg : W \to C_\alpha(X, Z)$ is continuous if and only if ${}^tg \circ f : Y \to C_\alpha(X, Z)$ is, because $f : Y \to W$ is quotient. But for every $y \in Y$ and every $x \in X$, we have $\left({}^tg \circ f\right)(y)(x) =^t g(f(y))(x) = g(x, f(y)) = {}^t\left(g \circ (Id_X \times f)\right)(y)(x)$. Hence ${}^tg \circ f =^t \left(g \circ (Id_X \times f)\right)$ is continuous, so that ${}^tg$ is continuous, and in turn, g is continuous. ■

We will see later that the converse is also true.

4.3 Quotients are not hereditary

Topological quotients fail to be hereditary in the following sense: If $f : (X, \tau) \to (Y, \sigma)$ is a quotient map and $A \subset Y$, then the restriction of f to

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[4] If $f : X \to Y$ is a surjection, $A \subset X$ is called *saturated* (for f) if $A = f^{-1}(f(A))$.

$f^{-1}(A)$ is may not be quotient from $f^{-1}(A)$ to A (endowed with the relative topologies induced by τ and σ respectively). This might happen even for finite spaces, as shows the following:

Example 46 *[23] Let* $X = \{0,1,2,3\}$ *endowed with the topology* $\{\varnothing, \{0,2\}, \{1,3\}, \{0,1,2,3\}\}$ *and let* $Y = \{0,1,2\} = X/\{2,3\}$ *endowed with the quotient topology. Let* $f : X \to Y$ *be the associated canonical surjection. Notice that none of the proper subsets of* Y *as an open pre-image, so that the topology of* Y *is anti-discrete. Consider the subset* $A = \{0,1\}$ *of* Y *and the restriction* $\overline{f}$ *of* f

$$\overline{f} : f^{-1}(A) = \{0,1\} \to A = \{0,1\}.$$

Here $f^{-1}(A)$ *carries the topology induced by that of* X *and is therefore discrete, while* A *carries the topology induced by* Y *and is therefore anti-discrete. The map* $\overline{f}$ *is not quotient, because for instance* $\{0\} \subset A$ *is not open in* A *even though* $\overline{f}^{-1}\{0\} = \{0\}$ *is open in* $f^{-1}(A)$*.*

This entails many problems of stability of topological properties by subspace. Indeed, many properties appear naturally as satisfied exactly by the quotient spaces of spaces with a nicer property. For instance a topological space is *sequential* if every sequentially closed subset is closed. Note that Example 8 shows that $C_p([0,1])$ is not sequential. Sequential spaces can be characterized as the quotient images of metrizable spaces. Let S be a sequential space, which is a quotient of the metrizable space M via the canonical map $q : M \to S$. A subspace A of S may fail to be sequential, because the map $q : q^{-1}(A) \to A$ may fail to be quotient.

Part III

A remedy to the structural drawbacks of topological spaces

Chapter 5

Convergence spaces

A general reference is Dolecki's recent survey

$$http://math.u-bourgogne.fr/topo/dolecki/Page/init\_IX.pdf$$

Many of the crucial observations, notational conventions and ideas presented in this chapter have their origin in [9].

A *convergence* ξ on a set X is a relation between X and the set $\mathbb{F}X$ of filters on X denoted

$$x \in \lim_\xi \mathcal{F}$$

whenever $(x, \mathcal{F}) \in \xi$. Moreover, this relation satisfies

$$\mathcal{F} \geq \mathcal{G} \Longrightarrow \lim \mathcal{F} \supseteq \lim \mathcal{G}; \tag{GCONV}$$

$$\forall x \in X, x \in \lim\{x\}^\uparrow \tag{CENTERED}$$

Many authors add to the set of our axioms a third one like

$$\lim(\mathcal{F} \wedge \mathcal{G}) = \lim \mathcal{F} \cap \lim \mathcal{G} \tag{FINITE DEPTH}$$

in which case we call the convergence *of finite depth* or like

$$x \in \lim \mathcal{F} \Longrightarrow x \in \lim(\mathcal{F} \wedge \{x\}^\uparrow) \tag{PT-DEEP}$$

instead, in which case we call the convergence *point-deep.*

Example 47 • *[**Topologies**] A topological space (X, τ) can be identified with a convergence space by setting*

$$x \in \lim \mathcal{F} \Longleftrightarrow \mathcal{F} \geq \mathcal{N}(x).$$

It is clear that τ defines a unique convergence structure (of finite and even arbitrary depth) in that way, and the convergence defined above characterizes the topology τ.

- *[**Convergence almost everywhere**]. Let (X,μ) be a measure space and let $\mathbb{R}^X$ denote the set of maps from X to $\mathbb{R}$ and let $\langle\cdot|\cdot\rangle : X\times\mathbb{R}^X \to \mathbb{R}$ denote the evaluation map The convergence almost everywhere of filters on $\mathbb{R}^X$ defined by*

  $$f \in \lim_\xi \mathcal{F} \iff \exists E \subset X : \mu E = 0 \text{ and } \forall x \in X\backslash E, f(x) \in \lim_{\mathbb{R}} \langle x|\mathcal{F}\rangle$$

  defines a convergence space $(\mathbb{R}^X, \xi)$ (of finite depth).

- *[**Graphs**] Given a directed graph $G = (V,E)$, the relation*

  $$x \in \lim_\xi \mathcal{F} \iff \exists y \in \mathcal{F}^\bullet : (y,x) \in E \tag{Graphs}$$

  defines a convergence space (V,ξ) (of finite and even arbitrary depth) uniquely determined by G and that characterizes G.

- *[**Continuous convergence**] Let X and Y be two topological spaces. The relation $\lim_c$ defined by $(ContConv)$ defines a convergence structure (of finite depth) on the set of $C(X,Y)$ of continuous functions from X to Y. Notice that if $\lim_c$ is defined on the set Y^X of all maps from X to Y, then it satisfies (GCONV) but not (CENTERED). Actually, $f \in \lim_c \{f\}^\uparrow$ if and only if f is continuous.*

5.1 Lattice of convergence structures on a given set

If ξ and σ are two convergences on the same set X, we say that ξ *is finer than* σ or that σ *is coarser than* ξ, in symbols $\xi \geq \sigma$, if $\lim_\xi \mathcal{F} \subseteq \lim_\sigma \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}X$. This defines a partial order on the set $\mathcal{C}X$ of convergence structures on X. $(\mathcal{C}X, \leq)$ is a complete lattice whose inf and sup are defined by

$$\begin{aligned}
\lim_{\bigvee_{i\in I}\xi_i} \mathcal{F} &= \bigcap_{i\in I} \lim_{\xi_i} \mathcal{F};\\
\lim_{\bigwedge_{i\in I}\xi_i} \mathcal{F} &= \bigcup_{i\in I} \lim_{\xi_i} \mathcal{F}.
\end{aligned}$$

The smallest element of $\mathcal{C}X$ is the *antidiscrete topology* o for which $\lim_o \mathcal{F} = X$ for every $\mathcal{F} \in \mathbb{F}X$ and the largest element of $\mathcal{C}X$ is the discrete topology ι for which $x \in \lim_\iota \mathcal{F}$ if and only if $\mathcal{F} = \{x\}^\uparrow$.

5.1.1 topologies

Proposition 48 *The set of topologies on X is a subset of $\mathcal{C}X$ that is closed under supremum.*

Proof. Let $\{\tau_i : i \in I\}$ be a family of topologies on X. The supremum of this family in $\mathcal{C}X$ is defined by $\lim_{\vee_{i\in I}\tau_i} \mathcal{F} = \bigcap\limits_{i\in I} \lim_{\tau_i} \mathcal{F}$, so that $x \in \lim_{\vee_{i\in I}\tau_i} \mathcal{F}$ if $\mathcal{F} \geq \mathcal{N}_{\tau_i}(x)$ for every $i \in I$. The family $\mathcal{N}(x) = \{\cap \mathcal{S} : \mathcal{S} \subset \bigcup\limits_{i\in I} \mathcal{N}_{\tau_i}(x), |\mathcal{S}| < \omega\}$ is a filter-base on X and it is clear that $x \in \lim_{\vee_{i\in I}\tau_i} \mathcal{F} \Longleftrightarrow \mathcal{F} \geq \mathcal{N}(x)$. Remains to see that for every $U \in \mathcal{N}(x)$ there exists $O \in \mathcal{N}(x)$ such that $O \in \mathcal{N}(y)$ for every $y \in O$. If $U \in \mathcal{N}(x)$ then there exist $i_1..i_n$ and $U_1..U_n$ such that $U_j \in \mathcal{N}_{\tau_{i_j}}(x)$ is open in τ_{i_j}, and $\cap_{i=1}^n U_i \subset U$. Let $O = \cap_{i=1}^n U_i$. If $y \in O$ then $y \in U_j \in \mathcal{N}_{\tau_{i_j}}(y)$ for every $j \in \{1..n\}$ so that $O = \cap_{i=1}^n U_i \in \mathcal{N}(y)$. ■

Since the antidiscrete topology is the smallest element of $\mathcal{C}X$, the set of topologies coarser than a given convergence ξ is non empty. It is also closed under supremum. Therefore, for every convergence ξ, there exists the finest of the topologies that are coarser than ξ. We denote this topology by $T\xi$ and call it *topological modification of* ξ.

Exercise 49 *Let (X, ξ) be convergence space.*

1. *Show that O is open in the topology $T\xi$ if and only if $O \in \mathcal{F}$ whenever there exists $x \in \lim_\xi \mathcal{F} \cap O$.*
2. *Show that the "map" T associating to a convergence space (X, ξ) its topological modification $(X, T\xi)$ is monotone (i.e., $\xi \leq \tau \Longrightarrow T\xi \leq T\tau$), contractive (i.e., $T\xi \leq \xi$ for every convergence ξ) and idempotent (i.e., $T(T\xi) = T\xi$ for every convergence ξ).*

However, the set of topologies on a given set is not closed under infimum. Actually,

Proposition 50 *Every point-deep convergence is the infimum of a family of topologies.*

Proof. Let ξ be a point-deep convergence on X.For each $\mathcal{F}$ and x such that $x \in \lim_\xi \mathcal{F}$, define the atomic topology $\tau_{\mathcal{F},x}$ for which x is the only non-isolated point and $\mathcal{N}_{\tau_{\mathcal{F},x}}(x) = \mathcal{F} \wedge \{x\}^\uparrow$. It is easy to verify that

$$\xi = \bigwedge_{(x,\mathcal{F})\in\xi} \tau_{\mathcal{F},x}.$$

■

5.2 Continuity

A map $f : (X, \xi) \to (Y, \tau)$ between two convergence spaces is *continuous at* $x \in X$ if

$$x \in \lim_\xi \mathcal{F} \Longrightarrow f(x) \in \lim_\tau f(\mathcal{F}).$$

The map f is *continuous* if it is continuous at every $x \in X$.

5.2.1 Initial and final convergences

Consider a map $f : X \to (Y, \tau)$, where τ is a convergence. Since $\mathcal{C}X$ is a complete lattice and $f : (X, \iota) \to (Y, \tau)$, there exists the coarsest convergence on X making f continuous (to (Y, τ)). This convergence is called the *initial convergence associated to* f *and* τ and is denoted $f^-\tau$.

Analogously, given a map $f : (X, \xi) \to Y$ where ξ is a convergence, there exists the finest convergence on Y making f continuous, because $f : (X, \xi) \to (Y, o)$ is continuous and $\mathcal{C}Y$ is a complete lattice. This convergence is called *final convergence associated to* f *and* ξ and is denoted $f\xi$. The following is a simple rephrasing of the definitions, but will prove to be a very useful observation.

$$f : (X, \xi) \to (Y, \tau) \text{ is continuous } \Longleftrightarrow \tau \leq f\xi \Longleftrightarrow f^-\tau \leq \xi. \tag{5.2}$$

For the same reasons, given a family $(f_i : X \to (Y_i, \tau_i))_{i\in I}$ of maps from a fixed set X to convergence spaces (Y_i, τ_i), there exists the coarsest convergence on X making each f_i continuous, called *initial convergence associated to* $(f_i : X \to (Y_i, \tau_i))_{i\in I}$. It is clear that this convergence is

$$\bigvee_{i\in I} f_i^- \tau_i.$$

Also, given a family $(f_i : (X_i, \xi_i) \to Y)_{i\in I}$ of maps from convergence spaces (X_i, ξ_i) to a fixed set Y, there exists the finest convergence on Y making each f_i continuous, called *final convergence associated to* $(f_i : (X_i, \xi_i) \to Y)_{i\in I}$. It is clear that this convergence is

$$\bigwedge_{i\in I} f_i\xi_i.$$

Exercise 51 *Let* $f : (X, \xi) \to (Y, \tau)$

1. *Show that* $y \in \lim_{f\xi} \mathcal{F}$ *if and only if there exits* $\mathcal{G}\in\mathbb{F}X$ *such that* $\lim_\xi \mathcal{G}\cap f^{-1}(y) \neq \varnothing$ *and* $\mathcal{F} \geq f(\mathcal{G})$.
2. *Show that* $x \in \lim_{f^-\tau} \mathcal{F}$ *if and only if* $f(x) \in \lim_\tau f(\mathcal{F})$.

Exercise 52

1. *Let* $\leq$ *denote the refinement relation on families of subsets. Show that if* $f : X \to Y$, *then*
   - (a) *Show that*

     $$ff^- \geq Id_Y \text{ and } ff^- = Id_Y \text{ if } f \text{ is surjective.}$$

     In particular, if $\mathcal{G} \in \mathbb{F}Y$, *then* $ff^-\mathcal{G} \geq \mathcal{G}$.
   - (b) *Show that*

     $$f^-f \leq Id_X \text{ and } f^-f = Id_X \text{ if } f \text{ is injective.}$$

     In particular if $\mathcal{F} \in \mathbb{F}X$, *then* $f^-f\mathcal{F} \leq \mathcal{F}$.
2. *Let* $\leq$ *denote the partial order on the lattice of convergences on a given set. Consider* $f : (X, \xi) \to (Y, \tau)$.
   - (a) *Show that*

     $$f(f^-\tau) \geq \tau \text{ and } f(f^-\tau) = \tau \text{ if } f \text{ is surjective}$$

   - (b) *Show that*

     $$f^-(f\xi) \leq \xi \text{ and } f^-(f\xi) = \xi \text{ if } f \text{ is injective.}$$

Hence, symbolically

$$ff^{-} \geq Id_Y \text{ and } ff^{-} = Id_Y \text{ if } f \text{ is surjective} \tag{5.3}$$

$$f^{-}f \leq Id_X \text{ and } f^{-}f = Id_X \text{ if } f \text{ is injective} \tag{5.4}$$

applies to both situations.

Exercise 53

1. *Show that if* $f : (X, \xi) \to (Y, \tau)$ *is continuous then* $f : (X, T\xi) \to (Y, T\tau)$ *is also continuous;*
2. *Show that the converse is false;*
3. *Deduce from* 1) *that the map* T *satisfies*

   $$T(f\xi) \leq f(T\xi) \tag{5.5}$$

   and

   $$T(f^{-}\tau) \geq f^{-}(T\tau) \tag{5.6}$$

   for every map $f : (X, \xi) \to (Y, \tau)$.

5.2.2 Induced convergence

If $A \subset X$ and ξ is a convergence on X, then the *induced convergence* ξ_A of ξ on A is the inital convergence associated to the inclusion map $i : A \to (X, \xi)$. In view of Exercise 51, it means that for $a \in A$ and $\mathcal{F} \in \mathbb{F}A$, $a \in \lim_{\xi_A} \mathcal{F}$ if and only if the filter generated by $\mathcal{F}$ on X converges to a for ξ.

5.2.3 Product convergence

If (X, ξ) and (Y, τ) are two convergence spaces, then the *product convergence* $\xi \times \tau$ on $X \times Y$ is the initial convergence for the projections $p_X : X \times Y \to (X, \xi)$ and $p_Y : X \times Y \to (Y, \tau)$.

Exercise 54 *Let* (X, ξ) *and* (Y, τ) *be two convergence spaces and let* $\mathcal{F} \in \mathbb{F}(X \times Y)$.*Show that the following are equivalent:*

1. $(x, y) \in \lim_{\xi \times \tau} \mathcal{F}$;
2. $x \in \lim_{\xi} p_X(\mathcal{F})$ *and* $y \in \lim_{\tau} p_Y(\mathcal{F})$;

3. *There exist* $\mathcal{L} \in \mathbb{F}X$ *and* $\mathcal{M} \in \mathbb{F}Y$ *such that* $x \in \lim_\xi \mathcal{L}$, $y \in \lim_\tau \mathcal{M}$ *and* $\mathcal{L} \times \mathcal{M} \leq \mathcal{F}$.

More generally, if (X_i, ξ_i) is a family of convergence spaces, then we can define on the set $\prod\limits_{i\in I} X_i$ the *product convergence* $\prod\limits_{i\in I} \xi_i$ as the initial convergence for the family of projections $(p_i : \prod\limits_{i\in I} X_i \to (X_i, \xi_i))_{i\in I}$. If $\mathcal{F} \in\mathbb{F}\left(\prod\limits_{i\in I} X_i\right)$ then

$$(x_i)_{i\in I} \in \lim_{\prod\limits_{i\in I}\xi_i} \mathcal{F} \Longleftrightarrow \forall i \in I, x_i \in \lim_{\xi_i} p_i(\mathcal{F})$$

$$\Longleftrightarrow \forall i \in I, \exists \mathcal{G}_i \in \mathbb{F}X_i : x_i \in \lim_{\xi_i} \mathcal{G}_i \text{ and } \prod_{i\in I} \mathcal{G}_i \leq \mathcal{F}.$$

5.2.4 initial topology versus initial convergence

The description above of the induced convergence and of the product convergence should remind you of the descriptions of the induced topology and product topology in terms of filters. It turns out that the induced convergence associated to a topology is a topology and that the product of a family of topological spaces performed in the realm of convergence spaces turns out to be topological, and more precisely the product topology. These are two instances of a more general phenomenon.

Theorem 55 *Let* $(f_i : X \to (Y_i, \tau_i))_{i\in I}$. *If each* τ_i *is a topology, so is* $\bigvee\limits_{i\in I} f_i^- \tau_i$. *In other words, initial convergence and initial topology coincide.*

Proof. In view of Proposition 48, it is enough to show that given a map $f : X \to (Y, \tau)$, the initial convergence $f^-\tau$ is a topology whenever τ is. Since $\tau \leq T\tau$, we have $f^-\tau \leq f^-(T\tau)$. In view of (5.6) , $f^-\tau \leq T(f^-\tau)$. In other words, $f^-\tau$ is topological. ■

It is interesting to note that we did not use any internal description of topologies, but only the properties of the map T. We will provide a thorough investigation of this phenomenon later.

5.2.5 Direct sum

If (X_i, ξ_i) is a family of convergence spaces, the disjoint sum $\coprod\limits_{i\in I} X_i$ of the underlying sets X_i can be endowed with the *direct sum convergence* $\bigoplus\limits_{i\in I} \xi_i$

defined as the final convergence for the family of inclusions $(f_i : (X_i, \xi_i) \to \coprod_{i\in I} X_i)_{i\in I}$. In other words, $\mathcal{F} \in \mathbb{F}(\coprod_{i\in I} X_i)$ converges to $x \in X_{i_0} \subset \coprod_{i\in I} X_i$ in $\bigoplus_{i\in I} \xi_i$ if and only if $\mathcal{F}|_{X_{i_0}}$ converges to x in X_{i_0}.

5.2.6 Quotient convergence

(X, ξ) is a convergence space and $\sim$ is an equivalence relation on X, the quotient set $Y = X/\sim$ can be endowed with the *quotient convergence*, that is, the finest convergence on Y making the canonical surjection $q : X \to Y$ continuous. By definition the quotient convergence is $q\xi$.

More generally, if $f : (X, \xi) \to (Y, \sigma)$ is a surjection, we say that f is a *convergence quotient map* if $\sigma = f\xi$. It means that Y is the quotient of X by the equivalence relation determined by the partition of X into fibers of f, endowed with the quotient convergence.

Exercise 56 (Quotients are productive) *Let $(f_i : (X_i, \xi_i) \to Y_i)$ be a family of maps. Show that*

$$\left(\prod_{i\in I} f_i\right)\left(\prod_{i\in I} \xi_i\right) = \left(\prod_{i\in I} f_i \xi_i\right).$$

Deduce that an arbitrary product of convergence quotient maps is convergence quotient.

Exercise 57 (Quotients are hereditary) *Let $f : (X, \xi) \to (Y, \tau)$ be a convergence quotient map. Let $A \subset Y$. Show that $f_{|f^{-1}A} : (f^{-1}(A), \xi_{f^{-1}(A)}) \to (A, \tau_A)$ is convergence quotient.*

5.2.7 Final topology versus final convergence

Unlike for initial structures, the final convergence associated to a topology does not need to be a topology. Otherwise, the quotient convergence associated to a topology would be the quotient topology. In view of Exercise 56, a product of quotient maps in the realm of topology would be a quotient map, which we have seen to be false. More concretely:

Example 58 *Consider the subset $X = \bigcup_{i=1}^{\infty} \{i\} \times [0, 1]$ of $\mathbb{R}^2$ with the induced topology. Consider the quotient of X obtained by identifying points of $\{(i, 0) :$*

$i \in \mathbb{N}^*\}$ to a single point ∞. Let $q : X \to X/\{(i,0) : i \in \mathbb{N}^*\}$ deonte the canonical surjection. The neighborhood filter $\mathcal{N}$ of ∞ in the quotient topology (=final topology) has a base formed by sets

$$\left\{\left\{\bigcup_{i=1}^{\infty}[\infty,\varepsilon_i)_i\right\} : (\varepsilon_i)_{i\in I} \in (0,1)^I\right\}$$

where $[\infty,\varepsilon)_i = q(\{i\}\times[0,\varepsilon))$. The filter $\mathcal{N}$ does not converge to ∞ in the convergence $q\nu$ if ν denotes the natural topology of $\mathbb{R}^2$ induced on X. Indeed, if $\mathcal{G} \in \mathbb{F}X$ such that $(i,0) \in \lim_\nu \mathcal{G}$ then $\mathcal{G}\#(\{i\}\times[0,1])$ and does not mesh with any other branch. Hence $q(\mathcal{G})$ is strictly finer than $\mathcal{N}$.

Example 59 *Let (X,τ) be a Hausdorff topological space and let $D \subset X$ be a dense subset. Then the quotient topology on X/D is antidiscrete. Indeed, if $U \subset X/D$ is non-empty and does not contain the equivalence class ∞ of D, then $q^{-1}U$ does not intersect D and therefore cannot be open. If U does contain ∞ then $q^{-1}U$ contains D. If moreover $q^{-1}U$ is open then $q^{-1}U = X$. Indeed, if there is $x \notin q^{-1}U$, there exists an open set $O \in \mathcal{N}_X(x)$ disjoint from $q^{-1}U$, hence disjoint from D; a contradiction.*

In contrast, any filter $\mathcal{F}$ on X that converges to a point x of D^c and contains D^c leads to a filter $f(\mathcal{F})$ that converges to $f(x)$ but not to ∞ in the quotient convergence on X/D.

Let $f : (X,\xi) \to Y$ be a surjection, where ξ is a topology. The quotient topology σ on Y is the finest *topology* making f continuous. The quotient convergence $f\xi$ is the finest *convergence* making f continuous. Hence $\sigma \le f\xi$. Moreover, this inequality characterizes the continuity of $f : (X,\xi) \to (Y,\sigma)$. Hence σ is the finest topology coarser than $f\xi$, that is,

$$\text{the quotient topology is } \sigma = T(f\xi).$$

5.3 Base for a convergence

A *base* for a convergence space (X,ξ) is a family $\mathbb{B}$ of filters on X such that

$$x \in \lim\nolimits_\xi \mathcal{F} \Longrightarrow \exists \mathcal{B} \in \mathbb{B} : \mathcal{B} \le \mathcal{F}, x \in \lim\nolimits_\xi \mathcal{B}.$$

If $\mathbb{B}$ is a base for a convergence space, the convergence is called $\mathbb{B}$-*based*. For instance a convergence is *sequentially based* if every convergent filter is finer than a sequence converging to the same point. To a convergence ξ on X, we can associate a sequentially based convergence $Seq\xi$ on X by

$$x \in \lim_{Seq\xi} \mathcal{F} \Longleftrightarrow \exists (x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \leq \mathcal{F}, x \in \lim_{\xi} (x_n)_{n\in\mathbb{N}}.$$

Exercise 60

1. *Show that $Seq\xi$ is the coarsest sequentially based convergence on X that is finer than ξ.*
2. *Show that the map Seq associating to a convergence its sequentially based modification is monotone (i.e., $\xi \leq \tau \Longrightarrow Seq\xi \leq Seq\tau$), expansive (i.e., $\xi \leq Seq\xi$ for every convergence ξ) and idempotent (i.e., $Seq(Seq\xi) = Seq\xi$ for every convergence ξ).*
3. *Show that if $f : (X, \xi) \to (Y, \tau)$ is continuous, then $f : (X, Seq\xi) \to (Y, Seq\tau)$ is continuous.*
4. *Deduce that*

   $$Seq(f\xi) \leq f(Seq\xi)$$

   and

   $$Seq(f^{-}\tau) \geq f^{-}(Seq\tau)$$

   for every map $f : (X, \xi) \to (Y, \tau)$.
5. *Deduce that if $(f_i : (X_i, \xi_i) \to Y)_{i\in I}$ is a family of maps with sequentially based domains (X_i, ξ_i), then the associated final convergence on Y is sequentially based.*

Example 61 *Let $(\mathbb{R}, \nu)$ be the real line endowed with its natural topology. The convergence $Seq\nu$ is not a topology. Indeed,*

$$\mathcal{N}_\nu(0) = \bigwedge \{(x_n)_{n\in\mathbb{N}} : 0 \in \lim_\nu (x_n)_n = \lim_{Seq\nu} (x_n)_n\}$$

but there is no sequence coarser than $\mathcal{N}_\nu(0)$ converging to 0. Hence $0 \notin \lim_{Seq\nu} \mathcal{N}_\nu(0)$.

Example 62 *Recall that a topology is* sequential *if every sequentially closed set is closed. Let (X, τ) be a topological space. A subset is sequentially closed if and only if it is closed for $Seq\tau$. Hence, sequentially closed and closed subsets coincide if $T\tau = TSeq\tau$, that is, if and only if*

$$\tau \geq TSeq\tau.$$

More generally, we will use blackboard fonts for *classes of filters*. For instance $\mathbb{F}$ denotes the class of all filters, $\mathbb{F}_\omega$ denotes the class of countably based filters and $\mathbb{E}$ denotes the class of filters generated by sequences, with unspecified underlying set. In turn, $\mathbb{F}X$ denotes the set of filters on X, $\mathbb{F}_\omega X$ is the set of countably based filters on X and so on.

Given a class $\mathbb{D}$ of filters containing fixed ultrafilters, and a convergence space (X,ξ), we can define the *$\mathbb{D}$-based modification $B_\mathbb{D}\xi$ of ξ* by

$$x \in \lim_{B_\mathbb{D}\xi} \mathcal{F} \iff \exists \mathcal{D} \in \mathbb{D} : \mathcal{D} \leq \mathcal{F}, x \in \lim_\xi \mathcal{D}.$$

Proposition 63 1. *The convergence $B_\mathbb{D}\xi$ is the coarsest $\mathbb{D}$-based convergence that is finer than ξ.*

2. *The map $B_\mathbb{D}$ is monotone, expansive and idempotent.*

3. *If $f(\mathcal{D}) \in \mathbb{D}(Y)$ whenever $\mathcal{D} \in \mathbb{D}(X)$ and $f : X \to Y$, then $f : (X, B_\mathbb{D}\xi) \to (Y, B_\mathbb{D}\tau)$ is continuous whenever $f : (X,\xi) \to (Y,\tau)$ is. Equivalently:*

$$B_\mathbb{D}(f\xi) \leq f(B_\mathbb{D}\xi)$$

for every map $f : (X,\xi) \to Y$. Equivalently,

$$B_\mathbb{D}(f^-\tau) \geq f^-(B_\mathbb{D}\tau)$$

for every map $f : X \to (Y,\tau)$.

4. *if $(f_i : (X_i,\xi_i) \to Y)_{i\in I}$ is a family of maps with $\mathbb{D}$-based domains (X_i,ξ_i), then the associated final convergence on Y is $\mathbb{D}$-based.*

A convergence is *first-countable* if it is $\mathbb{F}_\omega$-based and *finitely generated* if it is based in the class $\mathbb{F}_1$ of principal filters. Observe that a topology is first-countable in the usual topological sense (i.e., there is a countable base of neighborhoods at each point) if and only if it is first-countable as a convergence. Also, notice that the convergence (Graphs) characterizing the structure of a directed graph is finitely generated.

5.4 Adherence-determined convergences

The *adherence* of a family $\mathcal{H}$ of subsets (in particular of a filter) of a convergence space (X,ξ) is

$$\operatorname{adh}_\xi \mathcal{H} = \bigcup_{\mathcal{F}\#\mathcal{H}} \lim_\xi \mathcal{F}.$$

If $\mathcal{U}$ is an ultrafilter, $\mathcal{U}^{\#} = \mathcal{U}$ so that $\mathrm{adh}\,\mathcal{U} = \lim \mathcal{U}$. Also, no filter is meshing with the generated filter 2^X, so that $\mathrm{adh}\, 2^X = \varnothing$. Notice that

$$\xi \leq \tau \Longrightarrow \forall \mathcal{H} \subset 2^X, \mathrm{adh}_\tau \mathcal{H} \subset \mathrm{adh}_\xi \mathcal{H}.$$

Exercise 64 (Adherence of filters) . *Let $\mathcal{F}$ and $\mathcal{G}$ be two filters on a convergence space (X, ξ).*

1. *Show that*

$$\mathrm{adh}\, \mathcal{F} = \bigcup_{\mathcal{F} \# \mathcal{H}} \lim\nolimits_\xi \mathcal{H} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim\nolimits_\xi \mathcal{U}.$$

2. *Show that*

$$\mathcal{F} \geq \mathcal{G} \Longrightarrow \mathrm{adh}\, \mathcal{F} \subset \mathrm{adh}\, \mathcal{G}.$$

3. *Show that*

$$\mathrm{adh}(\mathcal{F} \wedge \mathcal{G}) = \mathrm{adh}\, \mathcal{F} \cup \mathrm{adh}\, \mathcal{G}.$$

Exercise 65 (Adherence for initial and final convergence) 1. *Show that for every $f : X \to (Y, \tau)$ and every $\mathcal{F} \in \mathbb{F}X$*

$$\mathrm{adh}_{f^- \tau} \mathcal{F} = f^{-1}(\mathrm{adh}_\tau f(\mathcal{F})).$$

2. *Show that for every $f : (X, \xi) \to Y$ and every $\mathcal{G} \in \mathbb{F}Y$*

$$\mathrm{adh}_{f\xi} \mathcal{G} = f(\mathrm{adh}_\xi f^- \mathcal{G}).$$

If A is a subset of a convergence space, we write $\mathrm{adh}\, A$ as a shorthand for $\mathrm{adh}\{A\} = \mathrm{adh}\{A\}^\uparrow$. In view of the above the adherence of sets operator fulfills

$$\mathrm{adh}\, \varnothing = \varnothing; \tag{5.7a}$$

$$\forall A \subset X; A \subset \mathrm{adh}\, A; \tag{5.7b}$$

$$A \subset B \Longrightarrow \mathrm{adh}\, A \subset \mathrm{adh}\, B; \tag{5.7c}$$

$$\mathrm{adh}(A \cup B) = \mathrm{adh}\, A \cup \mathrm{adh}\, B. \tag{5.7d}$$

5.4.1 Pretopologies

Pretopological spaces were originially introduced as sets endowed with a closure operator satisfying the properties (5.7) above. A pretopological closure operator differs from the closure operator for a topology in that it may fail to be idempotent.

Example 66 *Let (X,τ) be the topological space associated to a free convergent bisequence*

$$x_{n,k} \underset{k}{\to} x_n \underset{n}{\to} \infty$$

(1). Consider $A=\{x_{n,k} : n\in\mathbb{N}, k\in\mathbb{N}\}$. Then $\operatorname{adh}_{Seq\tau} A = A\cup\{x_n : n\in\mathbb{N}\}$ and $\operatorname{adh}_{Seq\tau}(\operatorname{adh}_{Seq\tau} A)=X$.

The adherence of sets of a convergence determines a pretopology. We will call a convergence *pretopological* or *a pretopology* if the adherence of sets in turn determines the convergence via

$$\lim \mathcal{F} = \bigcap_{A\#\mathcal{F}} \operatorname{adh} A.$$

Given a convergence space (X,ξ), we define the *vicinity filter of a point* x to be

$$\mathcal{V}_\xi(x) = \bigwedge_{x\in\lim_\xi \mathcal{F}} \mathcal{F}.$$

Exercise 67 *Let (X,ξ) be a convergence space. Show that*

$$x\in \operatorname{adh} A \iff A\#\mathcal{V}(x).$$

Exercise 68 (Characterizations of Pretopological spaces) *Let (X,ξ) be a convergence space. Show that the following are equivalent:*

1. *X is a pretopological space;*

---

[1] X is the set $\{\infty\}\cup\{x_n : n\in\mathbb{N}\}\cup\{x_{n,k} : n\in\mathbb{N}, k\in\mathbb{N}\}$. In the topology of X, every point $x_{n,k}$ is isolated, a base of neighborhoods of a point x_n is formed by $\{\{x_n\}\cup\{x_{n,k} : k\geq p : p\in\mathbb{N}\}$ and a base of neighborhoods of ∞ is formed by

$$\left\{\{\infty\}\cup\{x_n : n\geq p\}\cup\{x_{n,k} : n\geq p, k\geq f(n)\} : p\in\mathbb{N}, f\in\mathbb{N}^{\mathbb{N}}\right\}.$$

2. $x \in \lim \mathcal{F} \iff \mathcal{F} \geq \mathcal{V}(x)$;

3. *For every family* $(\mathcal{F}_i)_{i \in I}$ *of filters on* X

$$\lim \left(\bigwedge_{i \in I} \mathcal{F}_i \right) = \bigcap_{i \in I} \lim \mathcal{F}_i.$$

To a convergence ξ we can associate a pretopology $P\xi$ defined by

$$\lim_{P\xi} \mathcal{F} = \bigcap_{A \# \mathcal{F}} \operatorname{adh}_\xi A. \tag{5.8}$$

Exercise 69 (Pretopological modification) *1. Show that the set of pretopologies on a set* X *is a subset of* $\mathcal{C}X$ *that is closed under supremum.*

2. *Show that* $P\xi$ *is the finest pretopology coarser than* ξ*.*

3. *Show that for every* $A \subset (X, \xi)$

$$\operatorname{adh}_{P\xi} A = \operatorname{adh}_\xi A.$$

4. *Show that the map* P *is monotone, contractive and idempotent.*

5. *Show that if* $f : (X, \xi) \to (Y, \tau)$ *is continuous, then* $f : (X, P\xi) \to (Y, P\tau)$ *is also continuous.*

6. *Deduce that the map* P *satisfies*

$$P(f\xi) \leq f(P\xi)$$

for every map $f : (X, \xi) \to Y$*.*

7. *Use Exercise 65 to show that*

$$P(f^{-}\tau) = f^{-}(P\tau)$$

for every map $f : X \to (Y, \tau)$*.*

Exercise 70 (Quotient in pretopological spaces are hereditary) *1. Let* $f : (X, \xi) \to (Y, \tau)$*. Show that the finest pretopology on* Y *making* f *continuous (from* (X, ξ)*) is* $Pf\xi$*.*

2. *Let* $f:(X,\xi)\to(Y,\tau)$ *be a surjection with* $\xi = P\xi$ *and* $\tau = P\tau$ *and assume that* f *is a pretopological quotient map in the sense that* $\tau = Pf\xi$. *Show that if* $A\subset Y$, *then the restriction* $f:(f^{-1}A,\xi_{|f-A})\to(A,\tau_A)$ *is a pretopological quotient in the sense that* τ_A *is the finest pretopology on* A *making this map continuous.*

5.4.2 Pseudotopologies

A convergence is called a *pseudotopology* if

$$\lim\mathcal{F} = \bigcap_{\mathcal{U}\in\mathbb{U}(\mathcal{F})}\lim\mathcal{U}$$

for every filter $\mathcal{F}$.

To a convergence ξ, we can associate a pseudotopology $S\xi$ called *pseudotopological modification of* ξ and defined by

$$\lim_{S\xi}\mathcal{F} = \bigcap_{\mathcal{U}\in\mathbb{U}(\mathcal{F})}\lim_{\xi}\mathcal{U}.$$

Exercise 71 (Pseudotopological modification) 1. *Show that the set of pseudotopologies on a set* X *is a subset of* $\mathcal{C}X$ *that is closed under supremum.*

2. *Show that* $S\xi$ *is the finest pseudotopology coarser than* ξ.

3. *Show that*

$$\lim_{S\xi}\mathcal{F} = \bigcap_{\mathcal{H}\#\mathcal{F}}\operatorname{adh}_{\xi}\mathcal{H}. \tag{5.9}$$

4. *Show that for every* $\mathcal{H}\in\mathbb{F}X$

$$\operatorname{adh}_{S\xi}\mathcal{H} = \operatorname{adh}_{\xi}\mathcal{H}.$$

5. *Show that the map* S *is monotone, contractive and idempotent.*

6. *Show that if* $f:(X,\xi)\to(Y,\tau)$ *is continuous, then* $f:(X,S\xi)\to(Y,S\tau)$ *is also continuous.*

7. *Deduce that the map S satisfies*

$$S(f\xi) \leq f(S\xi)$$

and

$$S(f^{-}\tau) = f^{-}(S\tau)$$

for every map $f : (X,\xi) \to (Y,\tau)$.

Exercise 72 *Show that every topology is a pretopology and that every pretopology is a pseudotopology. Deduce that*

$$T \leq P \leq S \leq I$$

where I is the identity functor on convergence spaces, and the inequality is defined via the value for each convergence.

Exercise 73

1. *Show that for every family $(\xi_i)_{i\in I}$ of convergences on X, we have*

$$S(\bigvee_{i\in I} \xi_i) = \bigvee_{i\in I} S\xi_i. \tag{5.10}$$

2. *Deduce that for every family (X_i,ξ_i) of convergence spaces*

$$S(\prod_{i\in I} \xi_i) = \prod_{i\in I} S\xi_i. \tag{5.11}$$

Exercise 74 (Continuous convergence is pseudotopological) *Let (X,ξ) be a convergence space and let (Y,σ) be a pseudotopological space. Denote by $[\xi,\sigma]$ the continuous convergence on the set $C(X,Y)$ of continuous function from (X,ξ) to (Y,σ). Let $ev : X \times C(X,Y) \to Y$ be the evaluation map.*

1. *Show that $[\xi,\sigma]$ is the coarsest among convergences τ on $C(X,Y)$ satisfying*

$$\xi \times \tau \geq ev^{-}\sigma.$$

2. *Deduce from 1. and from the previous exercise that $[\xi,\sigma]$ is a pseudotopology.*

More generally, given a class $\mathbb{D}$ of filters, we say that a convergence is *determined by adherences of* $\mathbb{D}$*-filters* if

$$\lim \mathcal{F} = \bigcap_{\mathbb{D} \ni \mathcal{D} \# \mathcal{F}} \operatorname{adh} \mathcal{D}.$$

If $\mathbb{D}$ is a class of filters that contains fixed ultrafilters and (X, ξ) is a convergence space, consider the associated convergence $\operatorname{Adh}_{\mathbb{D}} \xi$ on X determined by adherences of $\mathbb{D}$-filters:

$$\lim_{\operatorname{Adh}_{\mathbb{D}} \xi} \mathcal{F} = \bigcap_{\mathbb{D} \ni \mathcal{D} \# \mathcal{F}} \operatorname{adh}_{\xi} \mathcal{D}.$$

In view of (5.8) and (5.9) , the pretopological modification is the modification determined by principal filters and pseudotopological modification is the modification determined by adherences of filters. In other words, $\operatorname{Adh}_{\mathbb{F}} = S$ and $\operatorname{Adh}_{\mathbb{F}_1} = P$.

Exercise 75 (The modifier $\operatorname{Adh}_{\mathbb{D}}$**)**

1. *Show that the set of convergences determined by adherences of* $\mathbb{D}$*-filters on a set* X *is a subset of* $\mathcal{C}X$ *that is closed under supremum.*
2. *Show that* $\operatorname{Adh}_{\mathbb{D}} \xi$ *is the finest convergence determined by adherences of* $\mathbb{D}$*-filters coarser than* ξ*.*
3. *Show that for every* $\mathcal{D} \in \mathbb{D}(X)$

   $$\operatorname{adh}_{\operatorname{Adh}_{\mathbb{D}} \xi} \mathcal{D} = \operatorname{adh}_{\xi} \mathcal{D}.$$

4. *Show that the map* $\operatorname{Adh}_{\mathbb{D}}$ *is monotone, contractive and idempotent. From now on, assume that* $f^{-1}(\mathcal{D}) \in \mathbb{D}(X)$ *whenever* $\mathcal{D} \in \mathbb{D}(Y)$ *and* $f : X \to Y$ *is continuous.*
5. *Show that if* $f : (X, \xi) \to (Y, \tau)$ *is continuous, then* $f : (X, \operatorname{Adh}_{\mathbb{D}} \xi) \to (Y, \operatorname{Adh}_{\mathbb{D}} \tau)$ *is also continuous.*
6. *Deduce that the map* $\operatorname{Adh}_{\mathbb{D}}$ *satisfies*

   $$\operatorname{Adh}_{\mathbb{D}}(f\xi) \leq f(\operatorname{Adh}_{\mathbb{D}} \xi)$$

   and

   $$\operatorname{Adh}_{\mathbb{D}}(f^{-}\tau) \geq f^{-}(\operatorname{Adh}_{\mathbb{D}} \tau)$$

   for every map $f : (X, \xi) \to (Y, \tau)$*.*

5.5 Concrete functors

A *category* $\mathbf{C} = (Ob(\mathbf{C}), Hom_{\mathbf{C}}, Id, \circ)$ consists of

- a class of objects $Ob(\mathbf{C})$
- for each pair (A, B) of objects, a set $Hom_{\mathbf{C}}(A, B)$ whose members are called *morphisms* and denoted by arrows

$$A \xrightarrow{f} B$$

- For each object A, a morphism $Id_A : A \to A$ called *identity of A*
- a composition law $\circ$ associating with each morphisms $f : A \to B$ and $g : B \to C$ a morphism $g \circ f : A \to C$

subject to the conditions that

- the composition is associative;
- identity morphisms act as identities with respect to composition;
- the sets $Hom(A, B)$ are pairwise disjoint.

If $\mathbf{A}$ and $\mathbf{B}$ are two categories, a *functor* F from $\mathbf{A}$ to $\mathbf{B}$ is a function that assign to each $\mathbf{A}$-object A a $\mathbf{B}$-object FA and to each $\mathbf{A}$-morphism $f : A \to A'$ a $\mathbf{B}$-morphism $Ff : FA \to FA'$ in such a way that F preserves composition and identity morphisms.

The category **CONV** of convergence spaces and continuous maps admits a forgetful functor

$$|.| : \mathbf{CONV} \to \mathbf{SET}$$

to the category **SET** of sets and maps, that associates to a convergence space its underlying set and to a continuous map its underlying **SET**-map. With these conventions, we will denote a convergence space (a **CONV**-object) by a greek letter, say ξ, and the underlying set by $|\boldsymbol{\xi}|$ whenever we need to explicitely talk about the underlying set.

A functor $F : \mathbf{CONV} \to \mathbf{CONV}$ is called *concrete* if $|.| \circ F = |.|$. In other words, $F\xi$ and ξ have the same underlying set for each **CONV**-object ξ. is a functor sending each object to an object with the same underlying set.

Exercise 76 *Show that the following are equivalent:*

1. $F : \mathbf{CONV} \to \mathbf{CONV}$ *is a concrete functor;*
2. F *is isotone and*
$$F(f\xi) \leq f(F\xi)$$
for every $f : (X, \xi) \to Y$;
3. F *is isotone and*
$$F(f^{-}\tau) \geq f^{-}(F\tau)$$
for every $f : X \to (Y, \tau)$.

Examples are modifiers of the type $\mathrm{Adh}_{\mathbb{D}}$ and of the type $\mathrm{Base}_{\mathbb{D}}$. Concrete functors just modify the convergence structure of a convergence space without changing its underlying set, in a way that preserves continuity of maps.

5.5.1 Reflectors

A subcategory $\mathbf{B}$ of $\mathbf{A}$ is *full of* $Hom_{\mathbf{B}}(X, Y) = Hom_{\mathbf{A}}(X, Y)$ for every pair of $\mathbf{B}$-object (X, Y). A full subcategory $\mathbf{R}$ of $\mathbf{A}$ is *reflective* if for every $\mathbf{A}$-object A, there exists an $\mathbf{A}$-morphism $r_A : A \to rA$ called *reflection of* A on $\mathbf{R}$, where rA is a $\mathbf{R}$-object such that whenever an $\mathbf{A}$-morphism $f : A \to B$ has an $\mathbf{R}$-object for range, there exists a morphism $\overline{f} : rA \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
f : A \arrow[r] \arrow[d, "r_A"'] & B \\
rA \arrow[ur] &
\end{tikzcd}$$

In this case, the functor $R : \mathbf{A} \to \mathbf{B}$ defined on objects by $RA = rA$ and on morphisms by the following procedure is called *reflector from* $\mathbf{A}$ to $\mathbf{B}$: if $f : X \to Y$ is an $\mathbf{A}$-morphism, then $r_Y \circ f : X \to rY$ is also a morphism. Then the map $\overline{r_Y \circ f} : rX \to rY$ is the *reflection* Rf of f.

An example of a reflector from the category of **Tych** Tychonoff topological space to the category **HComp** of hausdorff compact spaces is the functor $\beta : \mathbf{Tych} \to \mathbf{HComp}$ that associates to each Tychonoff space X its Stone-Cech compactification βX. However, in the present context, we are primarily interested in concrete reflectors.

Exercise 77 *Show that a class of convergence spaces is a concretely reflective (i.e., reflection arrows are identity carried) subcategory of* **CONV** *if and only if it is stable by suprema, contains antidiscrete convergences and is preserved by initial convergences.*

Exercise 78 *Let* **R** *be a full subcategory of* **CONV** *and let* $F : \mathbf{CONV} \to \mathbf{R}$ *be a concrete functor. Show that* F *is a reflector if and only if* F *is contractive and idempotent.*

The functors T, P, S and more generally functors of the type $\mathrm{Adh}_{\mathbb{D}}$ are concrete reflectors and the corresponding categories of topological, pretopological and pseudotopological spaces are reflective subcategories of **CONV**.

5.5.2 Coreflectors

The concept of a coreflector is dual to that of a reflector. A subcategory **C** of **A** is *coreflective* if for every **A**-object A, there exists a **A**-morphism $c_A : cA \to A$ called *coreflection of* A on **C**, where cA is a **C**-object such that whenever an **A**-morphism $f : B \to A$ has an **C**-object for domain, there exists a morphism $\overline{f} : A \to cB$ such that the following diagram commutes:

$$\begin{tikzcd} f : B \arrow[rr] & & A \\ & cA \arrow[ul, "\overline{f}"] \arrow[ur, "c_A"'] & \end{tikzcd}$$

In this case, the functor $C : \mathbf{A} \to \mathbf{B}$ defined on objects by $CA = cA$ and on morphisms by the following procedure is called *coreflector from* **A** to **B** : if $f : X \to Y$ is an **A**-morphism, then $f \circ c_X : cX \to Y$ is also a morphism. Then the map $\overline{f \circ c_X} : cX \to cY$ is the *coreflection* Cf of f.

Exercise 79 *Show that a class of convergence spaces is a concretely coreflective subcategory of* **CONV** *if and only if it is stable by suprema, contains discrete convergences and is preserved by final convergences.*

Exercise 80 *Let* **R** *be a subcategory of* **CONV** *and let* $F : \mathbf{CONV} \to \mathbf{R}$ *be a concrete functor. Show that* F *is a coreflector if and only if* F *is expansive and idempotent.*

The functors Seq, $First$ and Fin and more generally funcotrs of the type $\mathrm{Base}_{\mathbb{D}}$ are concrete coreflectors.

Exercise 81 *Let* $F : \mathbf{CONV} \to \mathbf{CONV}$ *be a concrete functor. Show that* $\{\xi : \xi \leq F\xi\}$ *form a concretely reflective subcategory of* $\mathbf{CONV}$*and that* $\{\xi : \xi \geq F\xi\}$ *form a coreflective subcategory of* $\mathbf{CONV}$*.*

5.6 compactness

A subset K of a convergence space (X, ξ) is *compact* if

$$\mathcal{F}\#K \Longrightarrow \mathrm{adh}_{\xi}\,\mathcal{F}\cap K \neq \varnothing.$$

More generally, we call a filter $\mathcal{K}$ on X *compact at* $\mathcal{A} \subset 2^X$ if

$$\mathcal{F}\#\mathcal{K} \Longrightarrow \mathrm{adh}_{\xi}\,\mathcal{F}\#\mathcal{A}.$$

We drop 'at $\mathcal{A}$' if $\mathcal{A} = \{X\}$ and call the filter *selfcompact if* $\mathcal{A} = \mathcal{K}$. A subset K of X is compact if and only if its principal filter is selfcompact.

Exercise 82 *1. Show that* ξ*-compactness and* $S\xi$*-compactness coincide;*

2. Show that $x \in \lim_{S\xi} \mathcal{F}$ *if and only if* $\mathcal{F}$ *is* ξ*-compact at* $\{x\}$*.*

More generally, given a class $\mathbb{D}$ of filters, we call a filter $\mathcal{K}$ on X $\mathbb{D}$-*compact* at $\mathcal{A} \subset 2^X$ if

$$\mathcal{D} \in \mathbb{D}, \mathcal{D}\#\mathcal{K} \Longrightarrow \mathrm{adh}_{\xi}\,\mathcal{D}\#\mathcal{A}.$$

A subset K of a convergence space (X, ξ) is resp. compact, countably compact, Lindelöf if its principal filter $\mathbb{D}$-selfcompact for $\mathbb{D}$ respectively the class $\mathbb{F}$ of all filters, the class $\mathbb{F}_{\omega}$ of countably based filters, the class $\mathbb{F}_{\wedge\omega}$ of countably deep filters ([2]).

While $\mathbb{F}_1$-compactness of a subset is trivial (K is $\mathbb{F}_1$-compact at A if and only if $K \subset A$), it is not for filters.

Exercise 83 *Show that* $\mathcal{F}$ *is* $\mathbb{F}_1$*-compact at* A *if and only if*

$$\mathcal{F} \geq \mathcal{V}(A) = \bigwedge_{a \in A} \mathcal{V}(a)$$

---

[2] A filter $\mathcal{F}$ is *countably deep* if $\cap\mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

and more generally that $\mathcal{F}$ *is* $\mathbb{F}_1$*-compact at* $\mathcal{A}$ *if and only if*

$$\mathcal{F} \geq \mathcal{V}(\mathcal{A}) = \bigvee_{A \in A} \bigwedge_{a \in A} \mathcal{V}(a).$$

Exercise 84 *1. Show that* ξ*-*$\mathbb{D}$*-compactness and* $\mathrm{Adh}_{\mathbb{D}}\,\xi$*-*$\mathbb{D}$*-compactness coincide;*

2. Show that $x \in \lim_{\mathrm{Adh}_{\mathbb{D}}\,\xi} \mathcal{F}$ *if and only if* $\mathcal{F}$ *is* ξ*-*$\mathbb{D}$*-compact at* $\{x\}$*.*

3. Show that $x \in \lim_{T\xi} \mathcal{F}$ *if and only if* $\mathcal{F}$ *is* ξ*-*$\mathbb{F}_1$*-compact at* $\mathcal{N}_\xi(x)$*.*

Exercise 85 (Generalized Tychonoff Theorem) *Let* $(X_i, \xi_i)_{i \in I}$ *be a family of convergence spaces and let* $\mathcal{F} \in \mathbb{F}(\prod_{i \in I} X_i)$ *and let* $\mathcal{A}_i \subset 2^{X_i}$*. Show that* $\mathcal{F}$ *is compact at* $\prod_{i \in I} \mathcal{A}_i$ *if and only if* $p_i(\mathcal{F})$ *is compact at* $\mathcal{A}_i$ *for every* $i \in I$*.*

In particular the Tychonoff theorem extended to convergence spaces and the fact that the pseudotopologizer commutes with arbitrary product are two instances of this simple result.

The fact that compact topologies are minimal among Hausdorff topologies extends to pseudotopologies.

A convergence is *Hausdorff* if every filter converges to at most one point.

Exercise 86 *1. Show that a topological space is Hausdorff in the usual sense if and only if it is Hausdorff as a convergence space*

2. If ξ *and* τ *are pseudotopologies such that* ξ *is compact,* τ *is Hausdorff and* $\tau \leq \xi$*,show that* $\xi = \tau$*.*

3. Deduce that if ξ *is a compact and* T-Hausdorff *(i.e.,* $T\xi$ *is a Hausdorff topology) pseudotopological space, then* ξ *is topological.*

4. Let ξ *be a Hausdorff pseudotopology. Show that if* ξ *is minimal among Hausdorff pseudotopologies (i.e., if* $\tau = S\tau$ *is Hausdorff and* $\tau \leq \xi$ *then* $\tau = \xi$*) then* ξ *is compact.*

Problem 87 *Can* $\mathbb{D}$*-compact convergences be characterized in terms of minimality among a (reflective) subclass of Hausdorff convergences? Can it be done at least for* $\mathbb{D} = \mathbb{F}_\omega$*?*

See [19] for related results.

A convergence is *core-compact* if it has a base of filters $\mathcal{F}$ such that for every $F \in \mathcal{F}$ there exists $K_F \in \mathcal{F}$ such that $\{K_F\}^{\uparrow}$ is compact at F. A convergence is T-*core compact* if whenever $x \in \lim \mathcal{F}$, for every $V \in \mathcal{N}(x)$ there exists $F_V \in \mathcal{F}$ which is compact at V. Evidently, every core-compact convergence is T-core compact. The significance of these notions will be made clear in Section 7.1. However we can already state the following question:

Problem 88 *Is there a T-core compact convergence that is not core-compact?*

A more thorough study of compactness for families and filters in the context of convergence spaces can be found in [10] and also [22],[21].

Chapter 6

Functorial Inequalities and classifications of maps and spaces

The main references for this chapter are [9] and [13].

6.1 Local topological properties

Recall from Example 62 that a topology τ is *sequential* if and only if $\tau \geq TSeq\tau$. More generally, we call a convergence ξ sequential if

$$\xi \geq TSeq\xi.$$

Exercise 89 *Show that ξ is sequential if and only if $\xi \geq TFirst\xi$.*

A topology τ is called *Fréchet* if for every $A \subset |\tau|$ and every $x \in \mathrm{cl}_\tau A$ there exists a sequence $(x_n)_n$ on A that converges to x. The topology τ is called *strongly Fréchet* if for every decreasing sequence $(A_n)_n$ of subsets of $|\tau|$ such that $x \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_\tau A_n$ there exists a sequence $x_n \in A_n$ such that $x \in \lim_\tau (x_n)_n$. A topology τ is called *weakly bisequential* if for every countably deep filter $\mathcal{H}$ such that $x \in \mathrm{adh}_\tau \mathcal{H}$, there exists a countably based filter $\mathcal{G}\#\mathcal{H}$ such that $x \in \lim_\tau \mathcal{G}$. A topology τ is called *bisequential* if for every filter $\mathcal{H}$, if $x \in \mathrm{adh}_\tau \mathcal{H}$ then there exists a countably based filter $\mathcal{G}\#\mathcal{H}$ such that $x \in \lim_\tau \mathcal{G}$.

Exercise 90 *Let τ be a topology.*

1. *Show that the following are equivalent:*
    - *(a) τ is Fréchet;*
    - *(b) $\tau \geq PSeq\tau$;*
    - *(c) $\tau \geq PFirst\tau$.*
2. *Denote by P_ω the reflector $\mathrm{Adh}_{\mathbb{F}_\omega}$ on convergences determined by adherence of countably based filters. Show that the following are equivalent:*
    - *(a) τ is strongly Fréchet;*
    - *(b) $\tau \geq P_\omega Seq\tau$;*
    - *(c) $\tau \geq P_\omega First\tau$.*
3. *Denote by L the reflector $\mathrm{Adh}_{\mathbb{F}_{\wedge\omega}}$ on convergences determined by adherence of countably deep filters. Show that the following are equivalent:*
    - *(a) τ is weakly bisequential;*
    - *(b) $\tau \geq LFirst\tau$.*
4. *Show that the following are equivalent:*
    - *(a) τ is bisequential;*
    - *(b) $\tau \geq SFirst\tau$.*

We extend the definitions of Fréchet, strongly Fréchet, weakly bisequential and bisequential topologies to convergences using the characterizations in terms of functorial inequalities obtained in the previous exercise. All the above properties are of the following type. Given two classes of filters $\mathbb{J}$ and $\mathbb{D}$, a convergence ξ is called $(\mathbb{J}/\mathbb{D})$-*accessible* if

$$\mathcal{J} \in \mathbb{J} \Longrightarrow \mathrm{adh}_\xi \mathcal{J} \subset \mathrm{adh}_{B_{\mathbb{D}}\xi} \mathcal{J}.$$

Exercise 91 1. *Show that ξ is $(\mathbb{J}/\mathbb{D})$-acessible if and only if $\xi \geq \mathrm{Adh}_{\mathbb{J}} B_{\mathbb{D}}\xi$;*

2. *Identify $\mathbb{J}$ and $\mathbb{D}$ for each class of $(\mathbb{J}/\mathbb{D})$-acessible spaces considered in Exercise 90.*

A filter $\mathcal{F}$ is called $\mathbb{J}$ to $\mathbb{D}$ *meshable-refinable,* or a $(\mathbb{J}/\mathbb{D})_{\#\geq}$*-filter* if

$$\mathcal{J} \in \mathbb{J}, \mathcal{J} \# \mathcal{F} \Longrightarrow \exists \mathcal{D} \in \mathbb{D} : \mathcal{D} \# \mathcal{J}, \mathcal{D} \geq \mathcal{F}.$$

Exercise 92 *1. Show that if a convergence is* $(\mathbb{J}/\mathbb{D})_{\#\geq}$*-based, then it is* $(\mathbb{J}/\mathbb{D})$*-acessible.*

2. Show that if a pretopology (in particular a topology) is $(\mathbb{J}/\mathbb{D})$*-acessible then it is* $(\mathbb{J}/\mathbb{D})_{\#\geq}$*-based.*

Recall that a topology is *locally compact* if every point has a compact neighborhood. More generally, we call a convergence *locally compact* if every convergent filter contains a compact set.

Exercise 93 *Show that the map* K *defined by*

$$\lim_{K\xi} \mathcal{F} = \begin{cases} \lim_{\xi} \mathcal{F} \text{ if } \mathcal{F} \text{ contains a } \xi\text{-compact} \\ \emptyset \text{ otherwise} \end{cases}$$

is a concrete coreflector onto locally compact convergence spaces.

A topology is called a *k-space* if a subset is closed whenever its intersection with each compact subset is closed. A topology is called k' if whenever $x \in \operatorname{cl} A$, there exists a compact set K such that $x \in \operatorname{cl}(A \cap K)$. A topology is called *strongly* k' if whenever $x \in \operatorname{adh} \mathcal{H}$ where $\mathcal{H}$ is a countably based filter, there exists a compact set K such that $x \in \operatorname{adh}(\mathcal{H} \vee K)$.

Exercise 94 *Let* τ *be a topology. Show that*

1. τ *is a k-topology if and only if* $\tau \geq TK\tau$;
2. τ *is a* k'*-topology if and onlyif* $\tau \geq PK\tau$;
3. τ *is a strongly* k' *topology if and only if* $\tau \geq P_{\omega}\tau$;
4. τ *is locally compact if and only if* $\tau = K\tau$ *if and only if* $\tau \geq SK\tau$.

A k-sequence in the sense of [16] is a countable base of a filter $\mathcal{F}$ that is compact at $\mathcal{F}^{\bullet} = \bigcap \mathcal{F}$. A q-sequence in the sense of [16] is a countable base of a filter $\mathcal{F}$ that is countably compact at $\mathcal{F}^{\bullet}$. Following [16], we call a topological space *of pointwise countable type* if each point has a k-sequence of neighborhoods and a *strict* q space if each point has a q-sequence of neighborhoods. More generally, a convergence is called *of pointwise countable* and *strict* q respectively if every convergent filter contains a k-sequence, respectively a q-sequence. A topology is *(quasi) bi-k* if whenever $x \in \operatorname{adh} \mathcal{H}$, there exists a k-sequence (q-sequence) meshing with $\mathcal{H}$. A topology is *(quasi) countably bi-k* if whenever $x \in \operatorname{adh} \mathcal{H}$ and $\mathcal{H}$ is countably based, there exists a k-sequence (q-sequence) meshing with $\mathcal{H}$. A topology is *(quasi) singly bi-k* if $x \in \operatorname{cl} A$ implies that there is a k-sequence (q-sequence) $(K_n)_{n\in\mathbb{N}}$ such that $x \in \operatorname{adh}((K_n)_n \vee A)$.

Exercise 95 *1. Show that the map $First_K$ defined by*

$$\lim_{First_K \xi} \mathcal{F} = \begin{cases} \lim_\xi \mathcal{F} \text{ if } \mathcal{F} \text{ contains a } k\text{-sequence for } \xi \\ \emptyset \text{ otherwise} \end{cases}$$

is a concrete coreflector onto convergence spaces of pointwise countable type.

2. Show that the map $First_{K_\omega}$ defined by

$$\lim_{First_{K\omega} \xi} \mathcal{F} = \begin{cases} \lim_\xi \mathcal{F} \text{ if } \mathcal{F} \text{ contains a } q\text{-sequence for } \xi \\ \emptyset \text{ otherwise} \end{cases}$$

is a concrete coreflector onto strict q convergence spaces.

Exercise 96 *Let τ be a topology. Show that:*

1. *τ is bi-k if and only if $\tau \geq SFirst_K\tau$;*
2. *τ is countably bi-k if and only if $\tau \geq P_\omega First_K\tau$;*
3. *τ is singly bi-k if and only if $\tau \geq PFirst_K\tau$;*
4. *τ is a k-topology if and only if $\tau \geq TFirst_K\tau$;*
5. *τ is quasi bi-k if and only if $\tau \geq SFirst_{K_\omega}\tau$;*
6. *τ is quasi countably bi-k if and only if $\tau \geq P_\omega First_{K_\omega}\tau$;*
7. *τ is quasi singly bi-k if and only if $\tau \geq PFirst_{K_\omega}\tau$;*
8. *τ is a quasi k-topology if and only if $\tau \geq TFirst_{K_\omega}\tau$.*

A topological space (X,τ) is called a P-space if every G_δ-subset (1) of X is open.

Exercise 97 1. *Show that a topological space is a P-space if and only each of its neighborhood filters is in the class $\mathbb{F}_{\wedge\omega}$ of countably deep filters.*

2. *Let $B_{\mathbb{F}_{\wedge\omega}}$ denote the coreflector on $\mathbb{F}_{\wedge\omega}$-based convergence spaces. Show that*

   (a) *If $\xi = P\xi$, then $PB_{\mathbb{F}_{\wedge\omega}}\xi = B_{\mathbb{F}_{\wedge\omega}}\xi$.*

   (b) *If $\xi = T\xi$ then $TB_{\mathbb{F}_{\wedge\omega}}\xi = B_{\mathbb{F}_{\wedge\omega}}\xi$.*

Hence, a large number of local topological properties can be characterized via

$$\xi \geq JE\xi \tag{6.1}$$

where J is a concrete reflector and E is a concrete coreflector. A convergence satisfying (6.1) is called an *upper JE-convergence.*

The following table gathers local topological properties characterized (for a topology) via a functorial inequality of the type (6.1) .

	$First$	K	$First_K$	$First_{K_\omega}$	$B_{\mathbb{F}_{\wedge\omega}}$
I	first-countable $First$	locally compact K	pointwise countable type $First_K$	strict q $First_{K_\omega}$	P-space $B_{\mathbb{F}_{\wedge\omega}}$
S	bisequential $SFirst$	locally compact SK	bi-k $SFirst_K$	bi-quasi-k $SFirst_{K_\omega}$	P-space
P_ω	strongly Fréchet $P_\omega First$	strongly k' $P_\omega K$	countably bi-k $P_\omega First_K$	countably bi-quasi-k $P_\omega First_{K_\omega}$	P-space
L	weakly bisequential $LFirst$	? LK	? $LFirst_K$	? $LFirst_{K_\omega}$	P-space
P	Fréchet $PFirst$	k' PK	singly bi-k $PFirst_K$	singly bi-quasi-k $PFirst_{K_\omega}$	P-space
T	sequential $TFirst$	k TK	k $TFirst_K$	quasi-k $TFirst_{K_\omega}$	P-space $B_{\mathbb{F}_{\wedge\omega}}$

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[1] A subset G of a topological space X is a *G_δ-set* if G is the intersection of a countable family of open subsets of X.

Problem 98 *Are topological* $LK, LFirst_K$ *and* $LFirst_{K_\omega}$ *convergences exactly topological spaces satisfying any known topological property?*

6.2 Classes of quotient maps

6.3 Modified continuity

Chapter 7

Solving topological problems via convergence spaces: A few examples

7.1 Topological Product problems

This type of problems was thoroughly investigated in the series of papers [12], [18], [20], [17], [14].

7.2 Functional Analysis

See [4].

TO BE PRESENTED AND WRITTEN NEXT SEMESTER

7.3 Pontryiagin Duality

See [4].

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Part IV

Solutions to the exercises

TO BE WRITTEN NEXT SEMESTER

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