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On the representation of three-dimensional elasticity solutions with the aid of complex valued functions

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Abstract. In this paper the representation of three-dimensional displacement fields in linear elasticity in terms of six complex valued functions is considered. The representation includes the complex Muskhelishvili formulation for plane strain as a special case. The completeness of the complex representation for regular solutions is shown and a relationship to the Neuber/Papkovich solution is given.

1. Introduction

In a previous paper [1] the stresses and displacements for 3-dimensional elasticity problems have been derived with the aid of complex valued stress functions. Using the complex variables $\zeta_1 = ix + b_1 y + c_1 z$, $\zeta_2 = a_2 x + iy + c_2 z$, $\zeta_3 = a_3 x + b_3 y + iz$, which involve the real parameters b_1, c_1, a_2, c_2, a_3, b_3, and the complex functions $\phi_i(\zeta_i)$, $\chi_i(\zeta_i)$ $(i = 1, 2, 3)$ the displacements can be written in the following form:

$$
\begin{aligned}
2\mu u = \int \{&\mathrm{Im}[(3-4\nu)\phi_1 + \bar{\zeta}_1\phi_1' + \chi_1'] \\
&+ a_2\,\mathrm{Re}[(3-4\nu)\phi_2 - \bar{\zeta}_2\phi_2' - \chi_2'] \\
&+ a_3\,\mathrm{Re}[(3-4\nu)\phi_3 - \bar{\zeta}_3\phi_3' - \chi_3']\}\,\mathrm{d}t, \\
2\mu v = \int \{&b_1\,\mathrm{Re}[(3-4\nu)\phi_1 - \bar{\zeta}_1\phi_1' - \chi_1'] \\
&+ \mathrm{Im}[(3-4\nu)\phi_2 + \bar{\zeta}_2\phi_2' + \chi_2'] \qquad (1) \\
&+ b_3\,\mathrm{Re}[(3-4\nu)\phi_3 - \bar{\zeta}_3\phi_3' - \chi_3']\}\,\mathrm{d}t,
\end{aligned}
$$

$$2\mu w = \int \{c_1 \operatorname{Re}[(3-4\nu)\phi_1 - \bar{\zeta}_1\phi_1' - \chi_1'] + c_2 \operatorname{Re}[(3-4\nu)\phi_2 - \bar{\zeta}_2\phi_2' - \chi_2'] + \operatorname{Im}[(3-4\nu)\phi_3 + \bar{\zeta}_3\phi_3' + \chi_3']\}\, dt.$$

(′) denotes complex differentiation with respect to the argument of the complex function under consideration. The parameters of the complex variables $\zeta_1, \zeta_2, \zeta_3$ have to satisfy the equations $b_1^2 + c_1^2 = 1$, $a_2^2 + c_2^2 = 1$, $a_3^2 + b_3^2 = 1$ and can be treated as discrete values or parameter functions (i.e. $b_1 = b_1(t)$, $c_1 = c_1(t)$, etc.). The most practical case is that we choose for the parameter functions the trigonometric functions $\sin t$ and $\cos t$ so that we can take $-\pi$ and π as integration limits. If we choose, for example, in the case of discrete parameters $a_2 = 1$, $c_2 = 0$, $\phi_1 = \chi_1' = \phi_3 = \chi_3' = 0$ and take 0 and 1 as lower and upper integration limits, we obtain the plane strain representation of Muskhelishvili [2]:

$$\begin{aligned} 2\mu u &= \operatorname{Re}[(3-4\nu)\phi_2(x+iy) - (x-iy)\phi_2'(x+iy) - \chi_2'(x+iy)], \\ 2\mu v &= \operatorname{Im}[(3-4\nu)\phi_2(x+iy) + (x-iy)\phi_2'(x+iy) + \chi_2'(x+iy)] \end{aligned} \tag{2}$$

or

$$2\mu(u+iv) = (3-4\nu)\phi_2(x+iy) - (x+iy)\overline{\phi_2'(x+iy)} - \overline{\chi_2'(x+iy)}.$$

2. Completeness of the complex representation

The completeness of real representations of displacements and stresses has been treated in [3, 4, 5, 6, 7, 8, 9]. Relations among stress functions have been given by Mindlin [10, 11]. In order to prove the completeness of the presented complex representation of elastic displacement and stress fields, one can look for a relation to a representation for which the completeness has been shown. Indeed we can find a relation to the Neuber/Papkovich representation [12, 13, 14]

$$\begin{aligned} 2\mu u &= -\frac{\partial}{\partial x}[H_0 + xH_1 + yH_2 + zH_3] + 4(1-\nu)H_1, \\ 2\mu v &= -\frac{\partial}{\partial y}[H_0 + xH_1 + yH_2 + zH_3] + 4(1-\nu)H_2, \\ 2\mu w &= -\frac{\partial}{\partial z}[H_0 + xH_1 + yH_2 + zH_3] + 4(1-\nu)H_3, \end{aligned} \tag{3}$$

or

$$2\mu u = -\frac{\partial F}{\partial x} + 4(1-\nu)H_1,$$

$$2\mu v = -\frac{\partial F}{\partial y} + 4(1-\nu)H_2, \tag{4}$$

$$2\mu w = -\frac{\partial F}{\partial x} + 4(1-\nu)H_3,$$

where $F = H_0 + xH_1 + yH_2 + zH_3$ and H_0, H_1, H_2, H_3 are harmonic functions. Often one of the four real harmonic functions may be omitted [3]. By reason of practical considerations the complex representation of elastic stress and displacement fields has been given with the aid of six complex functions instead of three. Since the "general" solution of the three-dimensional Laplace-equation $\Delta H_j = 0$ has the forms [15, 16]

$$H_j(x, y, z) = \int_{-\pi}^{\pi} f_{1j}(z \pm ix\cos t \pm iy\sin t, t)\, \mathrm{d}t \quad (\text{where } j = 0, 1, 2, 3)$$

and

$$H_j(x, y, z) = \int_{-\pi}^{\pi} f_{2j}(y \pm ix\sin t \pm iz\cos t, t)\, \mathrm{d}t \tag{5}$$

and

$$H_j(x, y, z) = \int_{-\pi}^{\pi} f_{3j}(x \pm iy\cos t \pm iz\sin t, t)\, \mathrm{d}t,$$

it is possible to express every regular harmonic function in one of the given forms involving functions of the introduced complex variables ζ_1, ζ_2, ζ_3. The representation (5) of harmonic functions was introduced by Whittaker [17]. By choosing different classes of analytic functions f_{1j}, f_{2j}, f_{3j} one can obtain different classes of three-dimensional harmonic functions in a systematic manner. Details concerning the integral representation of space harmonic functions with the aid of complex valued functions and the choice of analytic functions f_{1j}, f_{2j}, f_{3j} for several curvilinear coördinates are to be found in [15, 17–30]. Whittaker proved that every regular harmonic function can be expressed with the aid of his integral representation. But, as he himself pointed out, there are other harmonic functions, which are not regular, and which can also be represented in the same form. Practical examples of singular harmonic functions can be found in the given literature.

We express now the real harmonic functions H_0, H_1, H_2, H_3 with the aid of the complex functions ϕ_1, ϕ_2, ϕ_3, χ_{01}, χ_{02}, χ_{03} in the following manner:

$$
\begin{aligned}
H_0 &= \int \mathrm{Re}\left[\sum_{i=1}^{3} \chi_{0i}\right] \mathrm{d}t, \\
H_1 &= \int 2\, \mathrm{Im}[\phi_1]\, \mathrm{d}t, \\
H_2 &= \int 2\, \mathrm{Im}[\phi_2]\, \mathrm{d}t, \\
H_3 &= \int 2\, \mathrm{Im}[\phi_3]\, \mathrm{d}t.
\end{aligned}
\tag{6}
$$

It should be noticed that for the sake of a simplified notation, we wrote $\phi_i = \phi_i(\zeta_i)$, $\chi_{0i} = \chi_{0i}(\zeta_i)$ $(i = 1, 2, 3)$ so that the parameter variable t did not appear. To be precise and in agreement with the notation in (5) we ought to have written $\phi_i = \phi_i(\zeta_i(t), t)$, $\chi_{0i} = \chi_{0i}(\zeta_i(t), t)$, but in this paper we use the simplified notation omitting the parameter variable t in the argument list. The representation (6) is possible for every regular harmonic function H_j $(j = 0, 1, 2, 3)$.

For our real stress function F we obtain the representation

$$
\begin{aligned}
F &= \int \left\{ \mathrm{Re}\left[\sum_{i=1}^{3} \chi_{0i}\right] + 2x\, \mathrm{Im}[\phi_1] + 2y\, \mathrm{Im}[\phi_2] + 2z\, \mathrm{Im}[\phi_3] \right\} \mathrm{d}t \\
&= \int \frac{1}{2} \left\{ \sum_{i=1}^{3} [\chi_{0i} + \bar{\chi}_{0i}] - 2ix[\phi_1 - \bar{\phi}_1] - 2iy[\phi_2 - \bar{\phi}_2] - 2iz[\phi_3 - \bar{\phi}_3] \right\} \mathrm{d}t,
\end{aligned}
\tag{7}
$$

which has the following partial derivatives:

$$
\begin{aligned}
F_x &= \int \frac{1}{2} \Big\{ -2i[\phi_1 - \bar{\phi}_1] + 2x[\phi_1' + \bar{\phi}_1'] - 2iya_2[\phi_2' - \bar{\phi}_2'] - 2iz[\phi_3' - \bar{\phi}_3'] \\
&\qquad + i[\chi_{01}' - \bar{\chi}_{01}'] + a_2[\chi_{02}' + \bar{\chi}_{02}'] + a_3[\chi_{03}' + \bar{\chi}_{03}'] \Big\} \mathrm{d}t, \\
F_y &= \int \frac{1}{2} \Big\{ -2i[\phi_2 - \bar{\phi}_2] - 2ixb_1[\phi_1' - \bar{\phi}_1'] + 2y[\phi_2' + \bar{\phi}_2'] \\
&\qquad - 2izb_3[\phi_3' - \bar{\phi}_3'] + b_1[\chi_{01}' + \bar{\chi}_{01}'] + i[\chi_{02}' - \bar{\chi}_{02}'] + b_3[\chi_{03}' + \bar{\chi}_{03}'] \Big\} \mathrm{d}t,
\end{aligned}
\tag{8}
$$

$$F_z = \int \frac{1}{2}\bigg\{ -2i[\phi_3 - \bar{\phi}_3] - 2ixc_1[\phi_1' - \bar{\phi}_1'] - 2iyc_2[\phi_2' - \bar{\phi}_2'] + 2z[\phi_3' + \bar{\phi}_3']$$

$$+ c_1[\chi_{01}' + \bar{\chi}_{01}'] + c_2[\chi_{02}' + \bar{\chi}_{02}'] + i[\chi_{03}' - \bar{\chi}_{03}']\bigg\} \,\mathrm{d}t,$$

or

$$F_x = \int \{\mathrm{Im}[-\chi_{01}' + 2ix\phi_1' + 2\phi_1]$$

$$+ a_2 \,\mathrm{Re}[\chi_{02}' - 2iy\phi_2']$$

$$+ a_3 \,\mathrm{Re}[\chi_{03}' - 2iz\phi_3']\} \,\mathrm{d}t,$$

$$F_y = \int \{b_1 \,\mathrm{Re}[\chi_{01}' - 2ix\phi_1']$$

$$+ \mathrm{Im}[-\chi_{02}' + 2iy\phi_2' + 2\phi_2]$$

$$+ b_3 \,\mathrm{Re}[\chi_{03}' - 2iz\phi_3']\} \,\mathrm{d}t, \tag{9}$$

$$F_z = \int \{c_1 \,\mathrm{Re}[\chi_{01}' - 2ix\phi_1']$$

$$+ c_2 \,\mathrm{Re}[\chi_{02}' - 2iy\phi_2']$$

$$+ \mathrm{Im}[-\chi_{03}' + 2iz\phi_3' + 2\phi_3]\} \,\mathrm{d}t.$$

Substitution of (6) and (9) into (4) yields

$$2\mu u = -\int \{\mathrm{Im}[-\chi_{01}' + 2ix\phi_1' + 2\phi_1 - 8(1-\nu)\phi_1]$$

$$+ a_2 \,\mathrm{Re}[\chi_{02}' - 2iy\phi_2']$$

$$+ a_3 \,\mathrm{Re}[\chi_{03}' - 2iz\phi_3']\} \,\mathrm{d}t,$$

$$2\mu v = -\int \{b_1 \,\mathrm{Re}[\chi_{01}' - 2ix\phi_1']$$

$$+ \mathrm{Im}[-\chi_{02}' + 2iy\phi_2' + 2\phi_2 - 8(1-\nu)\phi_2]$$

$$+ b_3 \,\mathrm{Re}[\chi_{03}' - 2iz\phi_3']\} \,\mathrm{d}t, \tag{10}$$

$$2\mu w = -\int \{c_1 \operatorname{Re}[\chi'_{01} - 2ix\phi'_1]$$

$$+ c_2 \operatorname{Re}[\chi'_{02} - 2iy\phi'_2]$$

$$+ \operatorname{Im}[-\chi'_{03} + 2iz\phi'_3 + 2\phi_3 - 8(1-\nu)\phi_3]\}\, \mathrm{d}t.$$

This form of displacement relationships involving complex functions of the complex variables ζ_1, ζ_2, ζ_3 can be a good starting point for practical applications. Another form involving only one complex variable will be given below by formula (16).

Noting that

$$-2ix\phi'_1 = \bar{\zeta}_1\phi'_1 - \zeta_1\phi'_1,$$

$$-2iy\phi'_2 = \bar{\zeta}_2\phi'_2 - \zeta_2\phi'_2, \tag{11}$$

$$-2iz\phi'_3 = \bar{\zeta}_3\phi'_3 - \zeta_3\phi'_3,$$

we can rewrite the representation (10) to obtain

$$2\mu u = -\int \{\operatorname{Im}[-\chi'_{01} - \bar{\zeta}_1\phi'_1 + \zeta_1\phi'_1 - 2(3-4\nu)\phi_1]$$

$$+ a_2 \operatorname{Re}[\chi'_{02} + \bar{\zeta}_2\phi'_2 - \zeta_2\phi'_2]$$

$$+ a_3 \operatorname{Re}[\chi'_{03} + \bar{\zeta}_3\phi'_3 - \zeta_3\phi'_3]\}\, \mathrm{d}t,$$

$$2\mu v = -\int \{b_1 \operatorname{Re}[\chi'_{01} + \bar{\zeta}_1\phi'_1 - \zeta_1\phi'_1]$$

$$+ \operatorname{Im}[-\chi'_{02} - \bar{\zeta}_2\phi'_2 + \zeta_2\phi'_2 - 2(3-4\nu)\phi_2] \tag{12}$$

$$+ b_3 \operatorname{Re}[\chi'_{03} + \bar{\zeta}_3\phi'_3 - \zeta_3\phi'_3]\}\, \mathrm{d}t,$$

$$2\mu w = -\int \{c_1 \operatorname{Re}[\chi'_{01} + \bar{\zeta}_1\phi'_1 - \zeta_1\phi'_1]$$

$$+ c_2 \operatorname{Re}[\chi'_{02} + \bar{\zeta}_2\phi'_2 - \zeta_2\phi'_2]$$

$$+ \operatorname{Im}[-\chi'_{03} - \bar{\zeta}_3\phi'_3 + \zeta_3\phi'_3 - 2(3-4\nu)\phi_3]\}\, \mathrm{d}t.$$

If we use the substitutions

$$\chi'_{0i} = \chi'_i - (3-4\nu)\phi_i + \zeta_i \phi'_i \quad (\text{where } i = 1, 2, 3), \tag{13}$$

we obtain the representation (1). So every regular three-dimensional elasticity solution can be represented with the aid of complex valued functions.

3. Complex representation involving only one complex variable

For the treatment of concrete problems, it may be advantageous to have a representation which involves complex functions of one complex variable instead of three different complex variables. In order to obtain a representation for the displacements involving only the variable $\zeta_3 = a_3x + b_3y + iz$, we can use the following substitutions:

$$\begin{aligned}
&\int \mathrm{Im}[\phi_1(ix + b_1y + c_1z)]\,\mathrm{d}t = \int \mathrm{Im}[\hat{\phi}_1(a_3x + b_3y + iz)]\,\mathrm{d}t,\\
&\int \mathrm{Im}[\phi_2(a_2x + iy + c_2z)]\,\mathrm{d}t = \int \mathrm{Im}[\hat{\phi}_2(a_3x + b_3y + iz)]\,\mathrm{d}t,\\
&\phi_3(\zeta_3) = \hat{\phi}_3(\zeta_3),\\
&\int \mathrm{Re}[\chi_{01}(ix + b_1y + c_1z)]\,\mathrm{d}t = \int \mathrm{Re}[\hat{\chi}_{01}(a_3x + b_3y + iz)]\,\mathrm{d}t,\\
&\int \mathrm{Re}[\chi_{02}(a_2x + iy + c_2z)]\,\mathrm{d}t = \int \mathrm{Re}[\hat{\chi}_{02}(a_3x + b_3y + iz)]\,\mathrm{d}t,\\
&\chi_{03}(\zeta_3) = \hat{\chi}_{03}(\zeta_3).
\end{aligned} \tag{14}$$

From the partial derivatives of these substitutions with respect to x, y, z, we obtain the following relationships:

$$\begin{aligned}
&\int \mathrm{Im}[i\phi'_1]\,\mathrm{d}t = -\int a_3\,\mathrm{Re}[i\hat{\phi}'_1]\,\mathrm{d}t,\\
&\int b_1\,\mathrm{Re}[i\phi'_1]\,\mathrm{d}t = \int b_3\,\mathrm{Re}[i\hat{\phi}'_1]\,\mathrm{d}t,\\
&-\int c_1\,\mathrm{Re}[i\phi'_1]\,\mathrm{d}t = \int \mathrm{Im}[i\hat{\phi}'_1]\,\mathrm{d}t,
\end{aligned}$$

$$\int a_2 \operatorname{Re}[i\phi_2'] \, dt = \int a_3 \operatorname{Re}[i\hat{\phi}_2'] \, dt,$$

$$\int \operatorname{Im}[i\phi_2'] \, dt = -\int b_3 \operatorname{Re}[i\hat{\phi}_2'] \, dt,$$

$$-\int c_2 \operatorname{Re}[i\phi_2'] \, dt = \int \operatorname{Im}[i\hat{\phi}_2'] \, dt, \qquad (15)$$

$$\int \operatorname{Im}[\chi_{01}'] \, dt = -\int a_3 \operatorname{Re}[\hat{\chi}_{01}'] \, dt,$$

$$\int b_1 \operatorname{Re}[\chi_{01}'] \, dt = \int b_3 \operatorname{Re}[\hat{\chi}_{01}'] \, dt, \text{ etc.}$$

Substitution of (15) in (10) yields

$$\begin{aligned} 2\mu u = \int \{ & -a_3 \operatorname{Re}[\hat{\chi}_{01}' - 2ix\hat{\phi}_1'] + 2(3-4\nu) \operatorname{Im}[\hat{\phi}_1] \\ & -a_3 \operatorname{Re}[\hat{\chi}_{02}' - 2iy\hat{\phi}_2'] \\ & -a_3 \operatorname{Re}[\hat{\chi}_{03}' - 2iz\hat{\phi}_3'] \} \, dt, \\ 2\mu v = \int \{ & -b_3 \operatorname{Re}[\hat{\chi}_{01}' - 2ix\hat{\phi}_1'] \\ & -b_3 \operatorname{Re}[\hat{\chi}_{02}' - 2iy\hat{\phi}_2'] + 2(3-4\nu) \operatorname{Im}[\hat{\phi}_2] \\ & -b_3 \operatorname{Re}[\hat{\chi}_{03}' - 2iz\hat{\phi}_3'] \} \, dt, \\ 2\mu w = \int \{ & \operatorname{Im}[\hat{\chi}_{01}' - 2ix\hat{\phi}_1'] \\ & +\operatorname{Im}[\hat{\chi}_{02}' - 2iy\hat{\phi}_2'] \\ & +\operatorname{Im}[\hat{\chi}_{03}' - 2iz\hat{\phi}_3'] + 2(3-4\nu) \operatorname{Im}[\hat{\phi}_3] \} \, dt. \end{aligned} \qquad (16)$$

4. Example: the solution of the problem of Boussinesq in terms of complex valued functions

In order to express the solution of the problem of Boussinesq (normal concentrated force P acting on the bounding plane $z = 0$ of a semi-infinite region $z > 0$ in direction of z, see [11] and [31, 32]) in complex notation, we need only the following two complex functions:

$$\phi_3 = -\frac{A}{i\zeta_3} = \frac{A}{\zeta}, \qquad \chi'_{03} = -\frac{B}{i\zeta_3} = \frac{B}{\zeta}, \tag{17}$$

where $\zeta_3 = iz - x\cos t - y\sin t$ and $\zeta = z + ix\cos t + iy\sin t$. The substitution of (17) and $a_3 = -\cos t$, $b_3 = -\sin t$ into (10) yields

$$2\mu u = \int_{-\pi}^{\pi} \mathrm{Re}[(B\cdot\zeta^{-1} + 2zA\cdot\zeta^{-2})\cos t]\,\mathrm{d}t,$$

$$2\mu v = \int_{-\pi}^{\pi} \mathrm{Re}[(B\cdot\zeta^{-1} + 2zA\cdot\zeta^{-2})\sin t]\,\mathrm{d}t, \tag{18}$$

$$2\mu w = \int_{-\pi}^{\pi} \mathrm{Im}[B\cdot\zeta^{-1} + 2zA\cdot\zeta^{-2} + 2(3-4\nu)A\cdot\zeta^{-1}]\,\mathrm{d}t.$$

Taking into account the relationships

$$\int_{-\pi}^{\pi} \frac{1}{\zeta}\,\mathrm{d}t = \frac{2\pi}{r}, \qquad \int_{-\pi}^{\pi} \frac{1}{\zeta^2} = 2\pi\frac{z}{r^3},$$

$$\int_{-\pi}^{\pi} \frac{\cos t}{\zeta}\,\mathrm{d}t = -i\frac{2\pi x}{r(z+r)}, \qquad \int_{-\pi}^{\pi} \frac{\cos t}{\zeta^2}\,\mathrm{d}t = -i\frac{2\pi x}{r^3}, \tag{19}$$

$$\int_{-\pi}^{\pi} \frac{\sin t}{\zeta}\,\mathrm{d}t = -i\frac{2\pi y}{r(z+r)}, \qquad \int_{-\pi}^{\pi} \frac{\sin t}{\zeta^2}\,\mathrm{d}t = -i\frac{2\pi y}{r^3}$$

and choosing $A = iP/(8\pi^2)$ and $B = -i(1-2\nu)P/(4\pi^2)$, we obtain the classical Boussinesq solution

$$2\mu u = \frac{P}{2\pi}\left[\frac{xz}{r^3} - (1-2\nu)\frac{x}{r(r+z)}\right],$$

$$2\mu v = \frac{P}{2\pi}\left[\frac{yz}{r^3} - (1-2\nu)\frac{y}{r(r+z)}\right], \tag{20}$$

$$2\mu w = \frac{P}{2\pi}\left[\frac{z^2}{r^3} + 2(1-\nu)\frac{1}{r}\right].$$

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