COMBINATORIAL CONSEQUENCES OF THE ALGEBRAIC STRUCTURE OF THE SPACE OF ULTRAFILTERS OVER $\mathbb{N}$

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The intent of this talk (and possibly a couple more talks) is to illustrate how combinatorial results with sometimes intricate combinatorial proofs can be obtained as easy by-products of structural results on $\beta\mathbb{N}$, the space of ultrafilters on $\mathbb{N}$, aka the Cečh-Stone compactification of $\mathbb{N}$. We will consider for instance a simple structural proof of the following theorem of Hindman:

Theorem 1. *For any finite coloring of the natural numbers, there is an infinite sequence whose set of finite sums is monochromatic.*

In fact, with this approach, it is not significantly harder to see that we can find two infinite sequences $(x_n)_n$ and $(y_n)_n$ such that the set of finite sums of $(x_n)_n$ and the set of finite products of $(y_n)_n$ are both monochromatic and of the same color.

Addition (and/or multiplication) can be lifted from $\mathbb{N}$ to $\beta\mathbb{N}$, which becomes a compact topological (non-commutative) semigroup. $\beta\mathbb{N}$ is a very complicated object, both topologically and algebraically. Yet, simple observations on its topological/algebraic structure can yield many interesting non-trivial consequences. The intent is to take the time to spell out constructions and proofs without assuming any particular background beyond elementary topology and algebra.

The area is also connected to topological dynamics and ergodic theory, but this side of things will not be touched upon in the first few talks.